Tutorials of the
2014 European Frequency and Time Forum

The Leeson effect

Enrico Rubiola

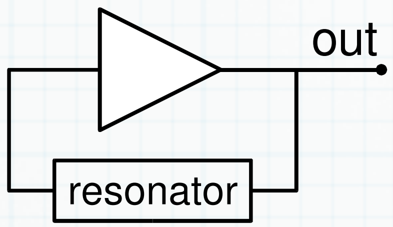
CNRS FEMTO-ST Institute, Besancon, France

Contents

home page http://rubiola.org

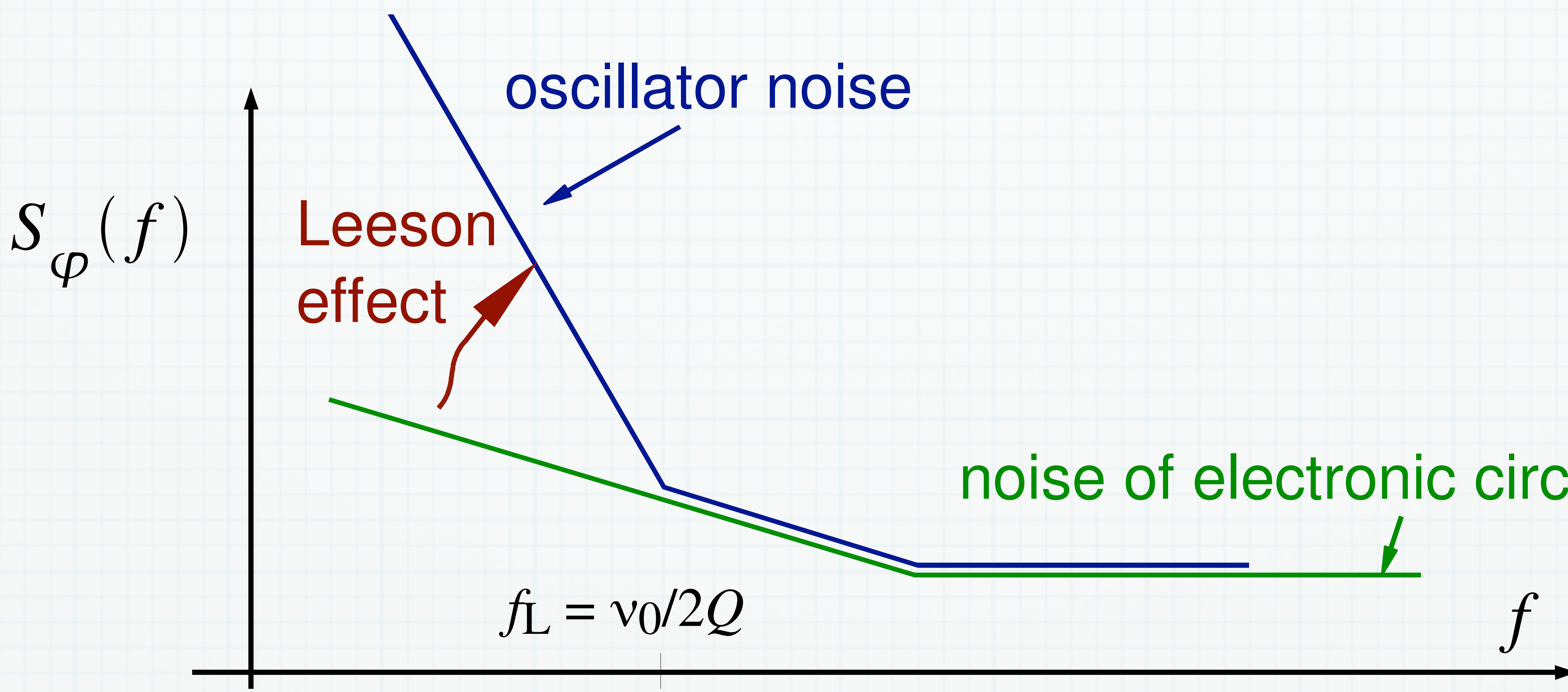
The Leeson effect in a nutshell

David B. Leeson, A simple model for feed back oscillator noise, Proc. IEEE 54(2):329 (Feb 1966)



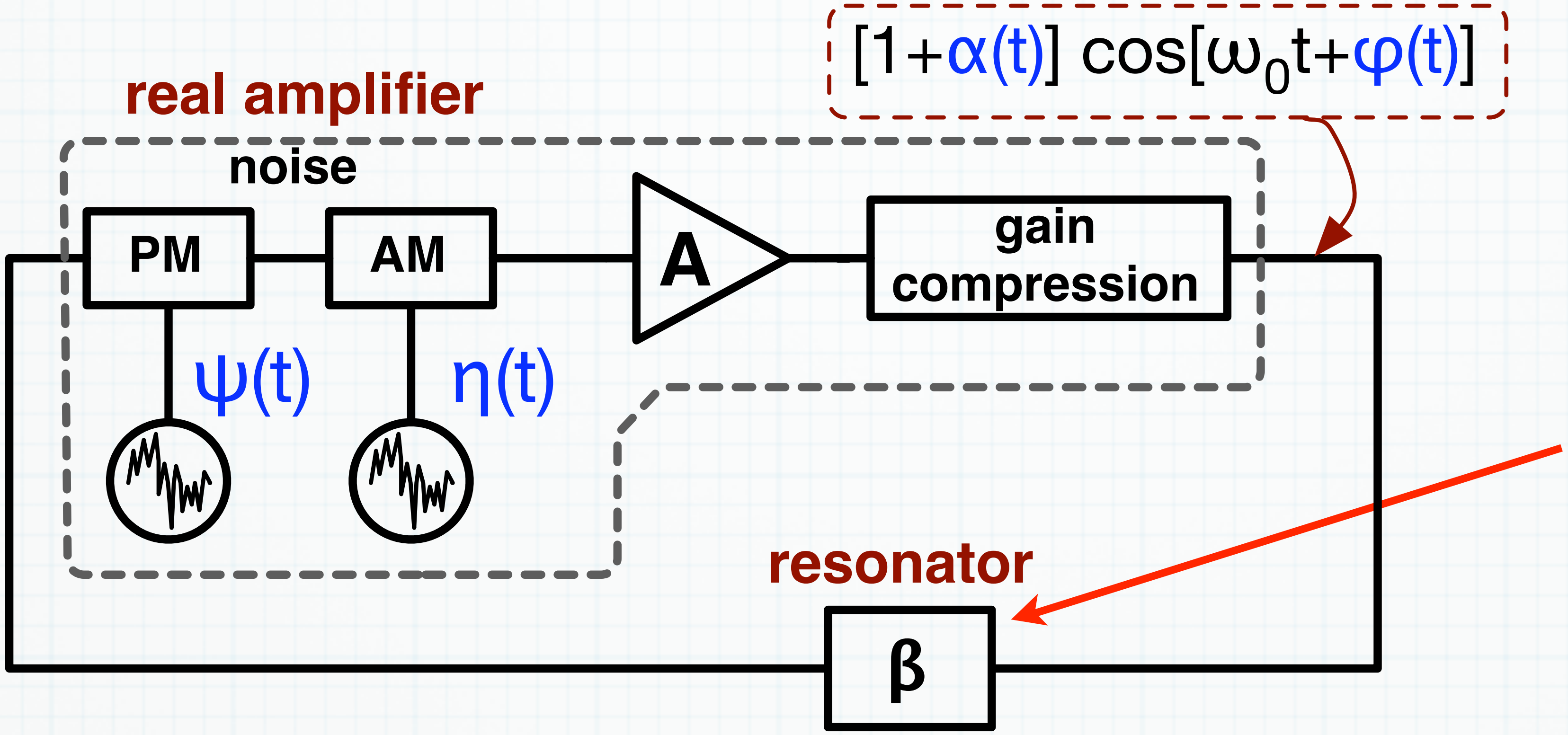
$$S_\varphi(f) = \left[1 + \left(\frac{\nu_0}{2Q}\right)^2 \frac{1}{f^2}\right] S_\psi(f)$$

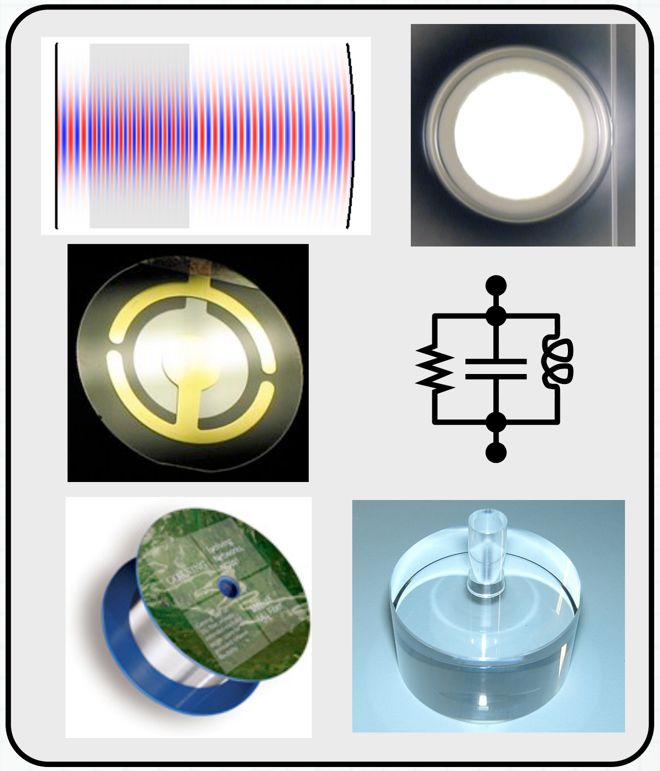
oscillator noise — ampli noise



Heuristic explanation of the Leeson effect

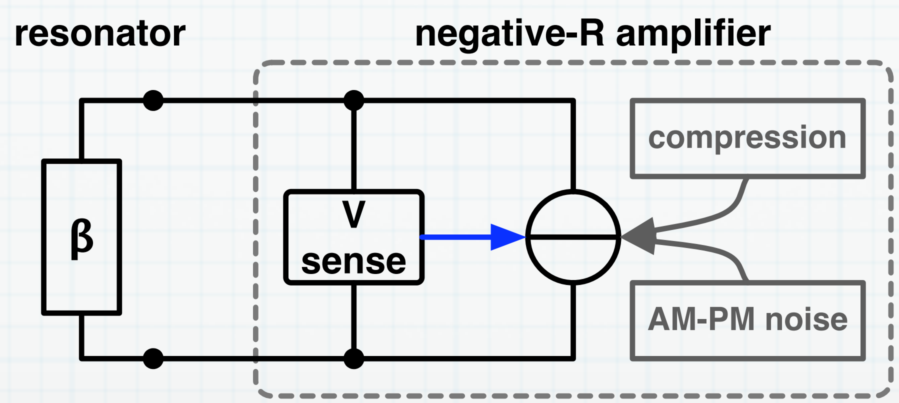
General oscillator model





Barkhausen condition $A\beta = 1$ at ω_0
(phase matching)

The model also describes the negative-R oscillator



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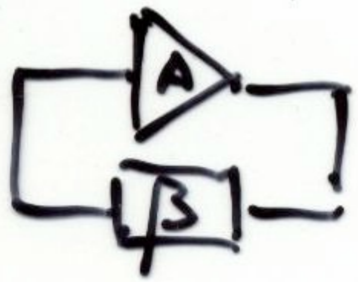
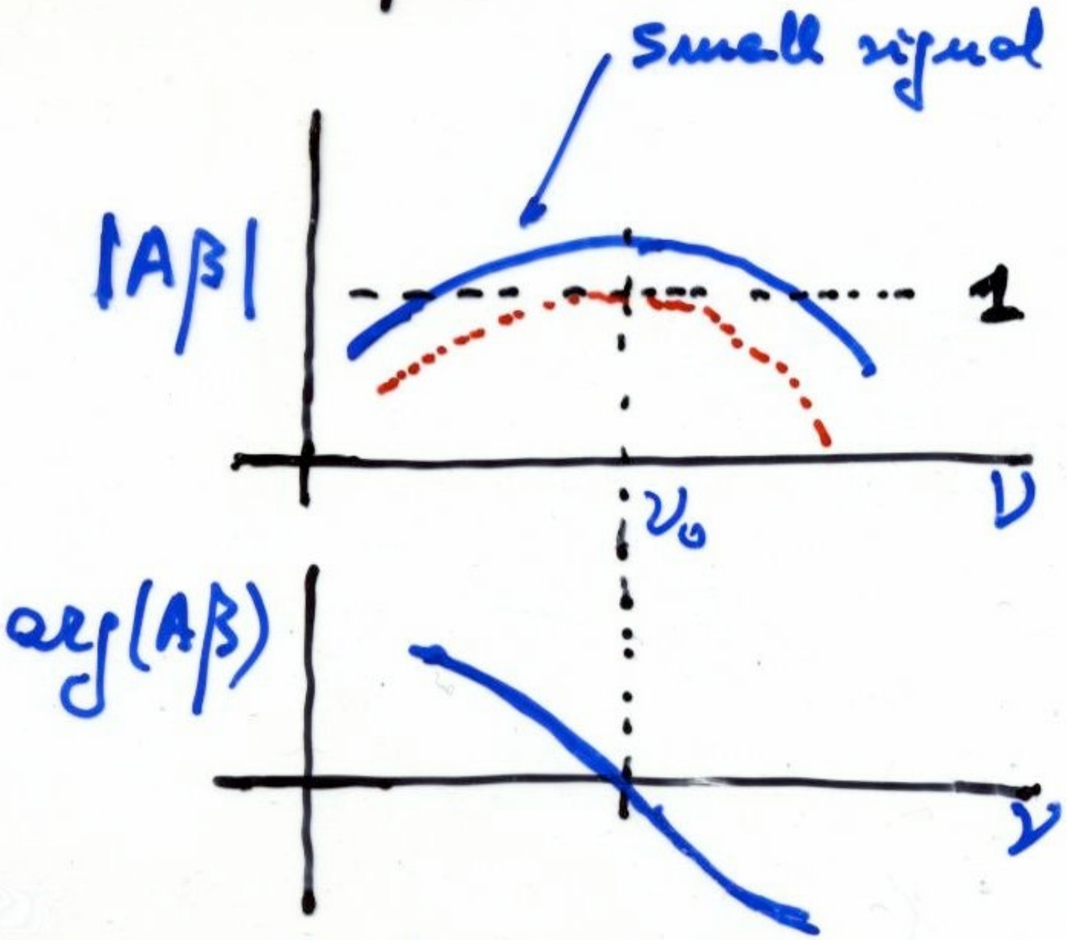
BARKHAUSEN CONDITION



let $A = \text{const}$

β: 2nd order diff. eq resonator



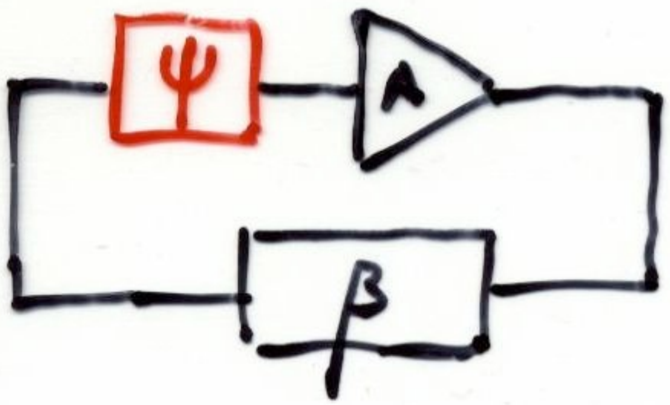
$$\arg(\beta) = -\arctan Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\simeq -2Q\,\frac{\omega - \omega_0}{\omega_0}$$

$$-2Q\,\frac{\Delta\omega}{\omega_0}$$

- $\arg(\beta)$ sets the oscillation frequency
- saturation fixes $|A\beta| = 1$

TUNING AN OSCILLATOR



add a phase ψ

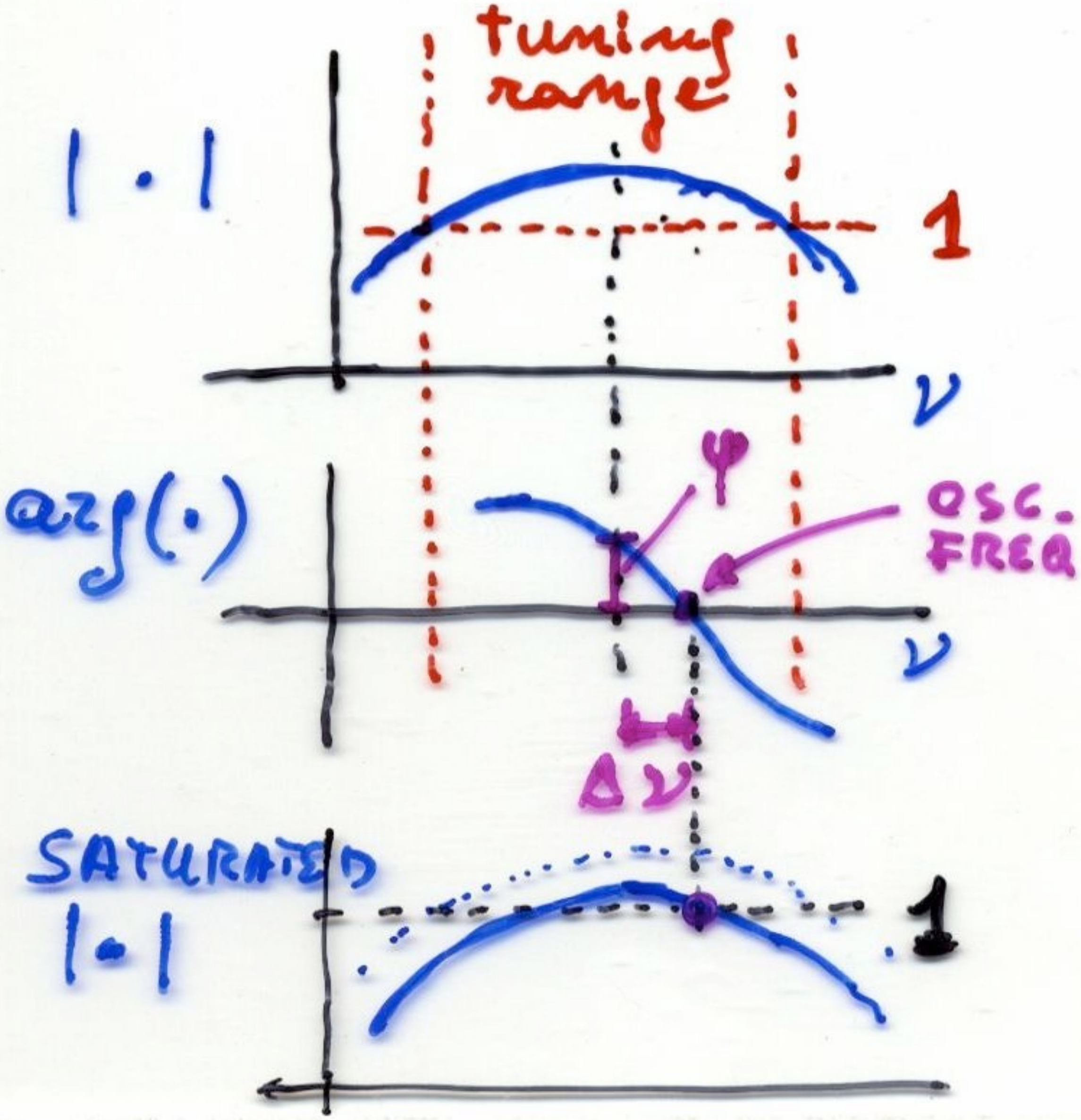
$$+\arg(\beta) + \psi = 0$$

$$+\arg(\beta) = \psi$$

approx:

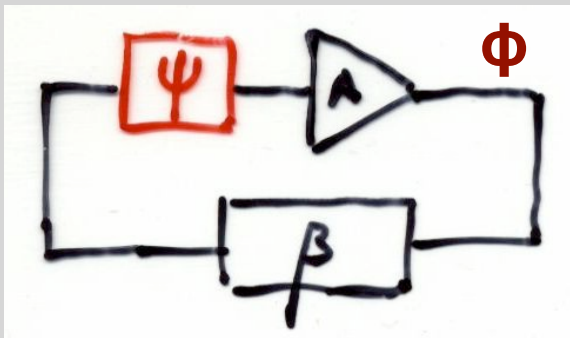
$$2Q\,\frac{\Delta\omega}{\omega_0} = \psi$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta\nu}{\nu_0} = \frac{\psi}{2Q}$$



Heuristic derivation of the Leeson formula

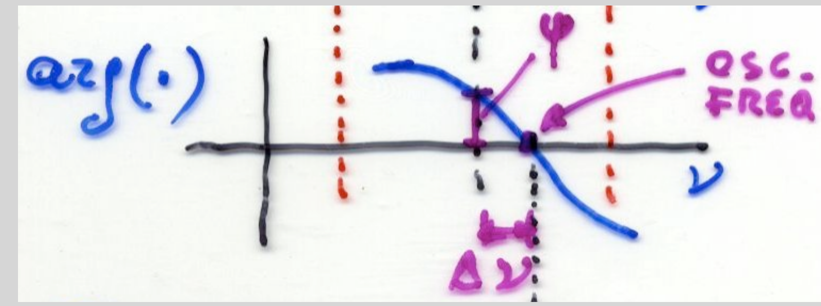
fast fluctuation: no feedback



$$\varphi(t) = \psi(t)$$

$$S_\varphi(f) = S_\psi(f)$$

slow fluctuations: ψ ⇒ Δν conversion



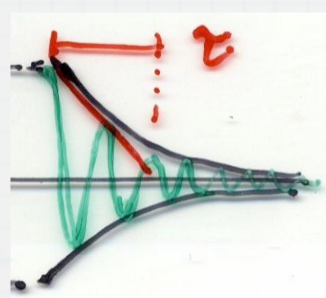
$$\Delta\nu = \frac{\nu_0}{2Q}\,\psi$$ **static**

$$S_{\Delta\nu}(f) = \left(\frac{\nu_0}{2Q}\right)^2 S_\psi(f)$$

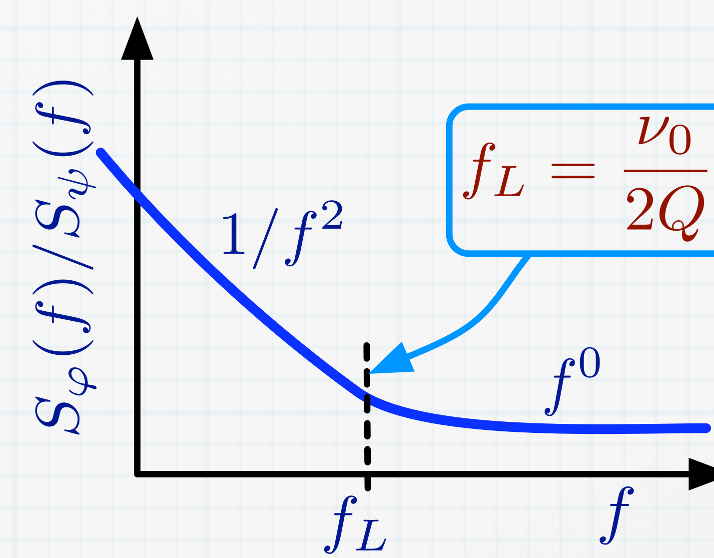
$$S_\varphi(f) = \frac{1}{f^2}\left(\frac{\nu_0}{2Q}\right)^2 S_\psi(f)$$ **integral**

fast or slow?

$$\tau = \frac{Q}{\pi\nu_0}$$

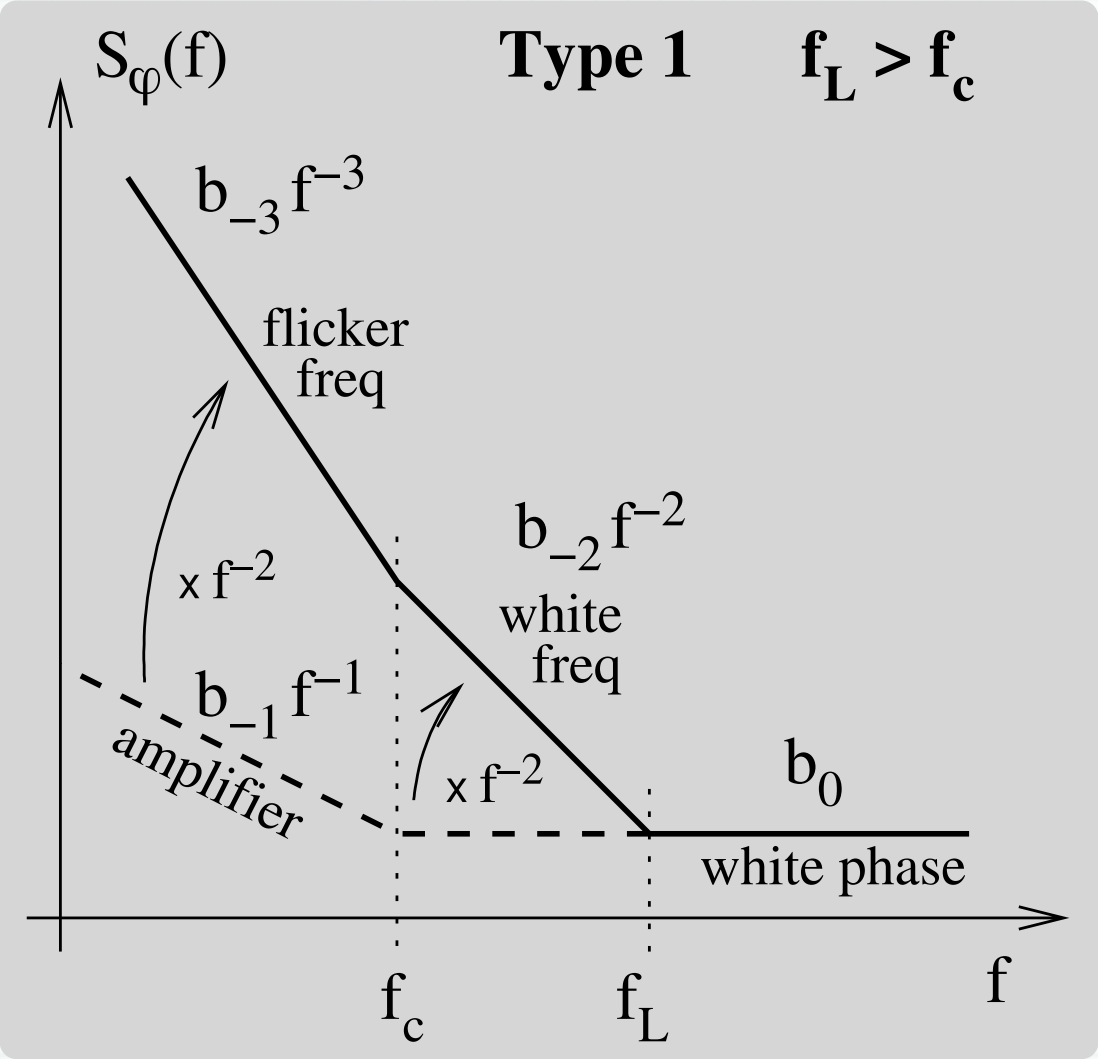


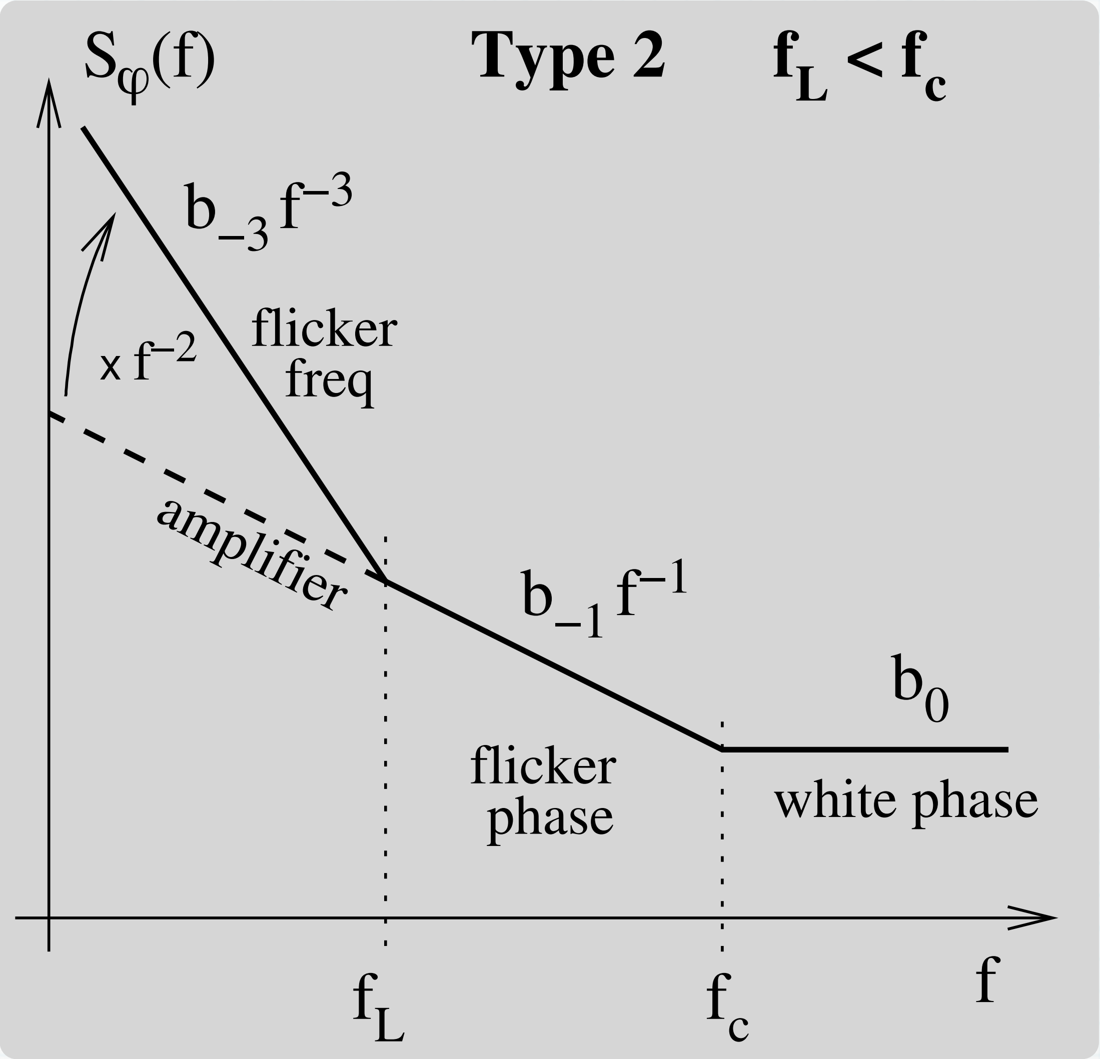
$$S_\varphi(f) = \left[1 + \frac{1}{f^2}\left(\frac{\nu_0}{2Q}\right)^2\right] S_\psi(f)$$



Though obtained with simplifications, this result turns out to be is exact

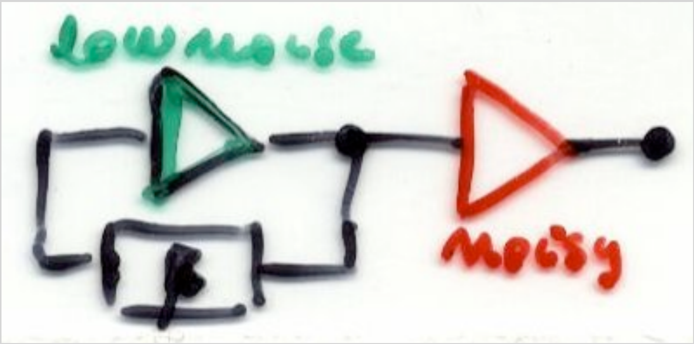
Oscillator noise -- real sustaining amplifier --





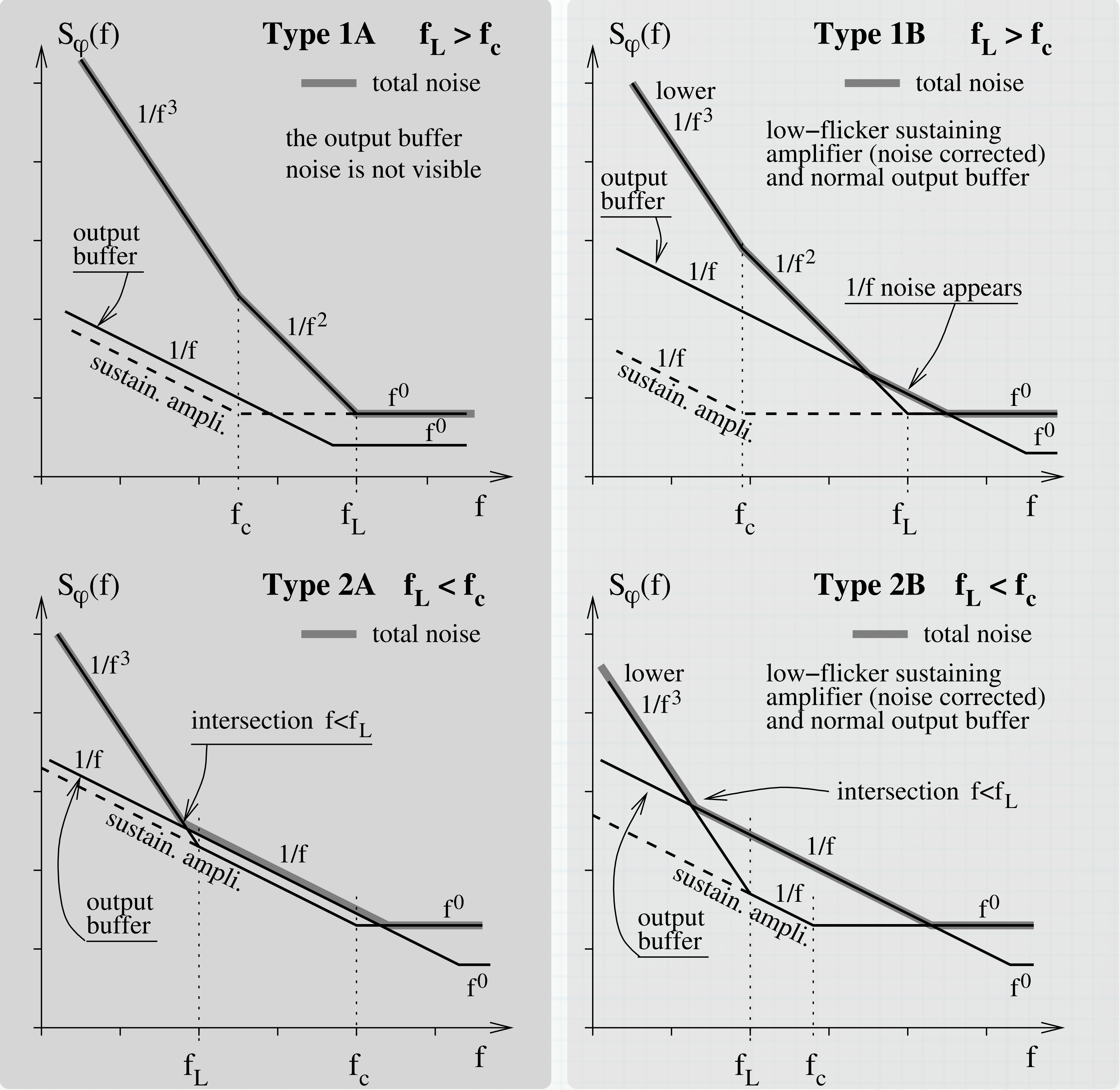
The sustaining-amplifier noise is $S_\varphi(f) = b_0 + b_{-1}/f$ (white and flicker)

The effect of the output buffer



Cascading two amplifiers, flicker noise adds as

$$S_\phi(f) = [S_\phi(f)]_1 + [S_\phi(f)]_2$$



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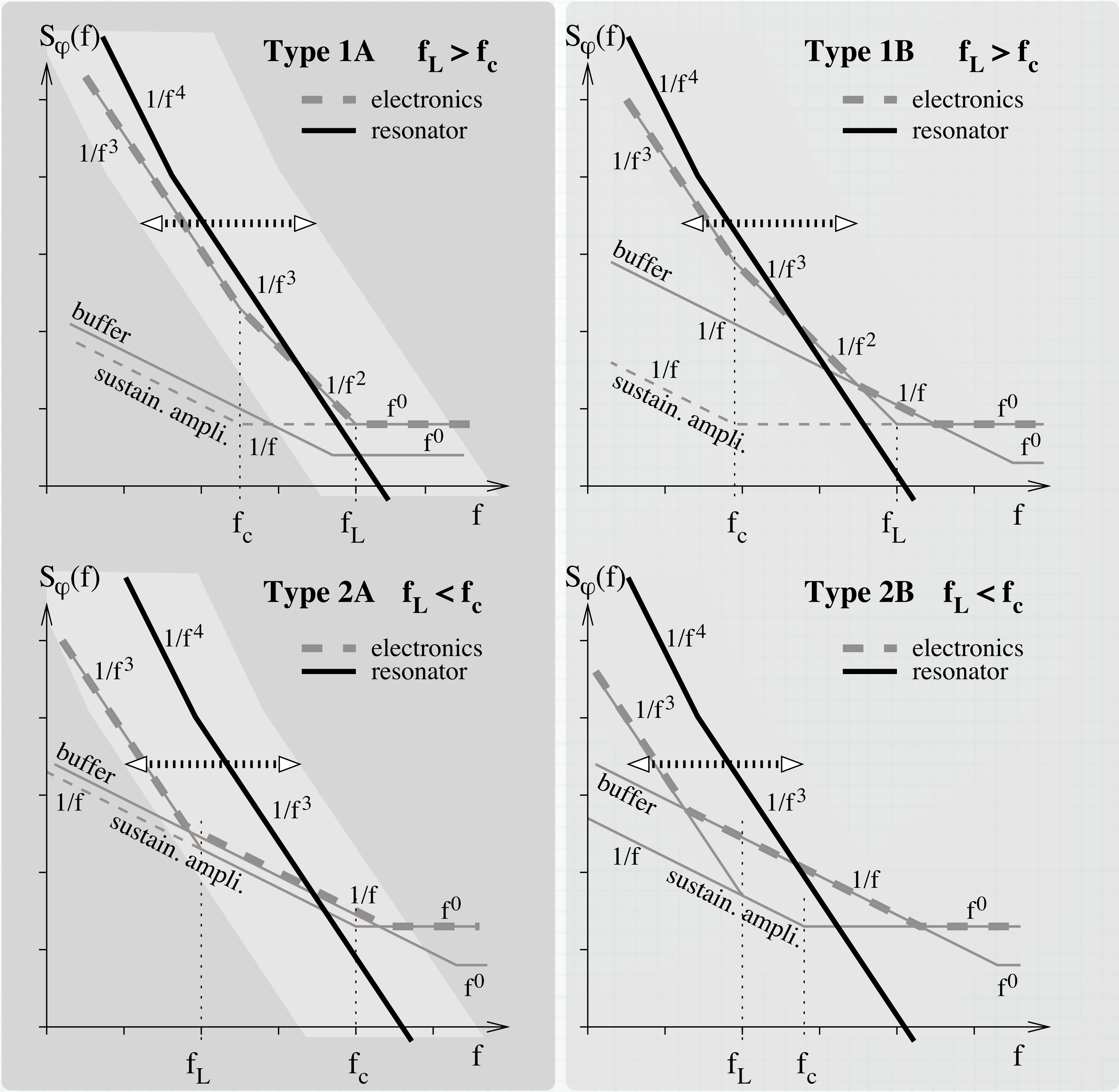
The resonator natural frequency fluctuates

- The oscillator tracks the resonator natural frequency, hence its fluctuations
- The fluctuations of the resonator natural frequency contain **$1/f$ and $1/f^2$** (frequency flicker and random walk), thus **$1/f^3$ and $1/f^4$** of the oscillator phase
- The resonator bandwidth does not apply to the natural-frequency fluctuation. (Tip: an oscillator can be frequency modulated at a rate >> f_L)



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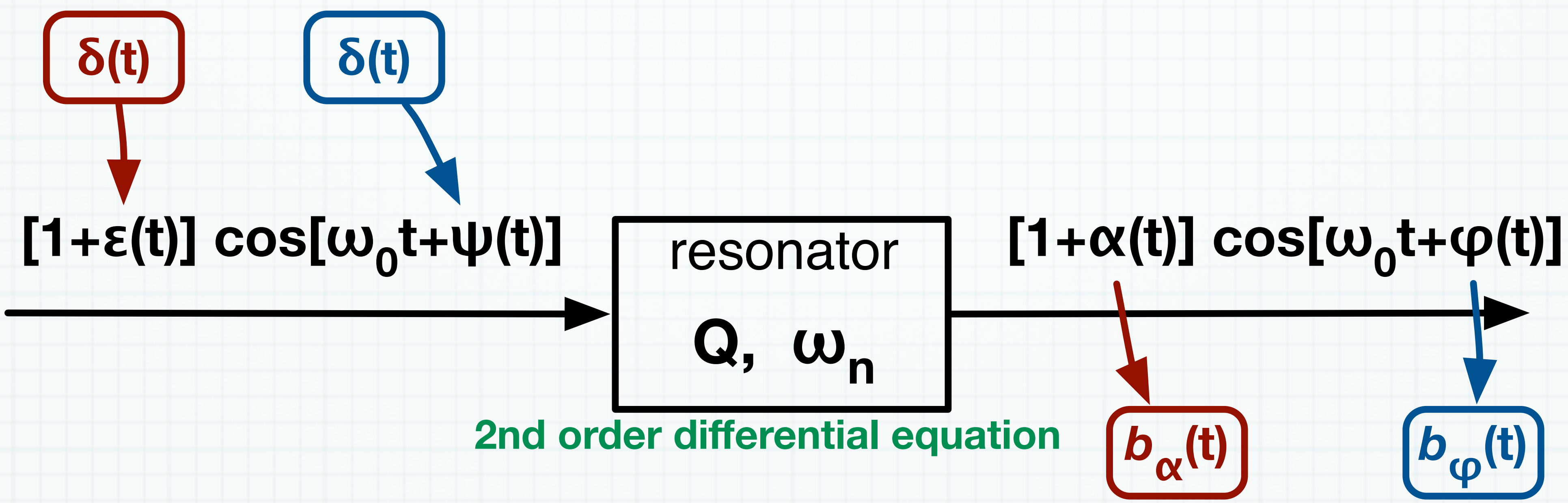
Resonator theory



Linear Time-Invariant system

Impulse response and frequency response in the amplitude-phase space

Resonator – time domain

$$\ddot{x} + \frac{\omega_n}{Q}\dot{x} + \omega_n^2 x = \frac{\omega_n}{Q}\dot{v}(t)$$

ω_n natural frequency

Q quality factor

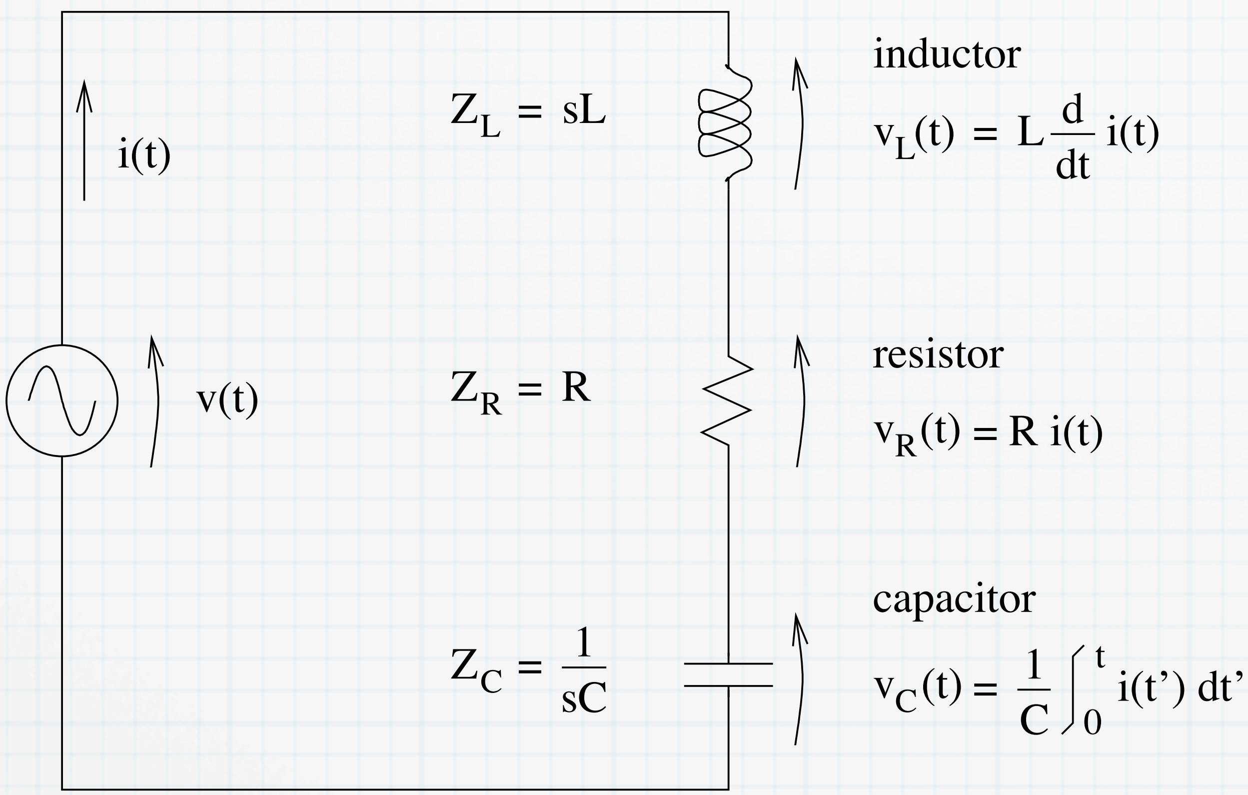
τ relaxation time

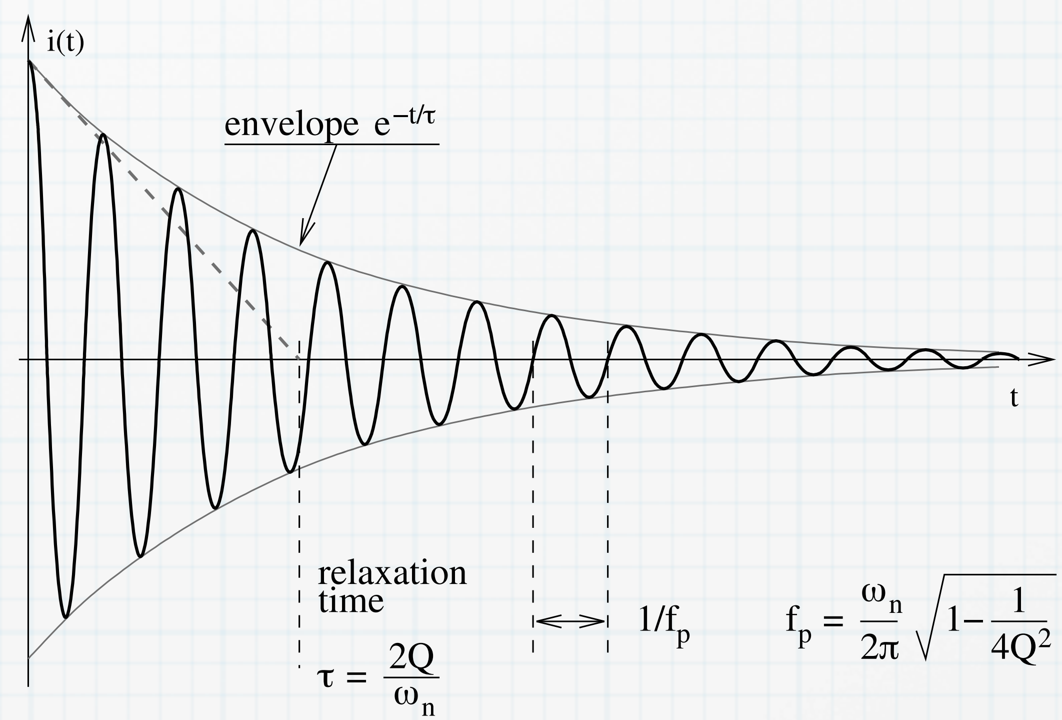
$$\tau = \frac{2Q}{\omega_n}$$

ω_p free-decay pseudofrequency

$$\omega_p = \omega_n\sqrt{1 - 1/4Q^2}$$

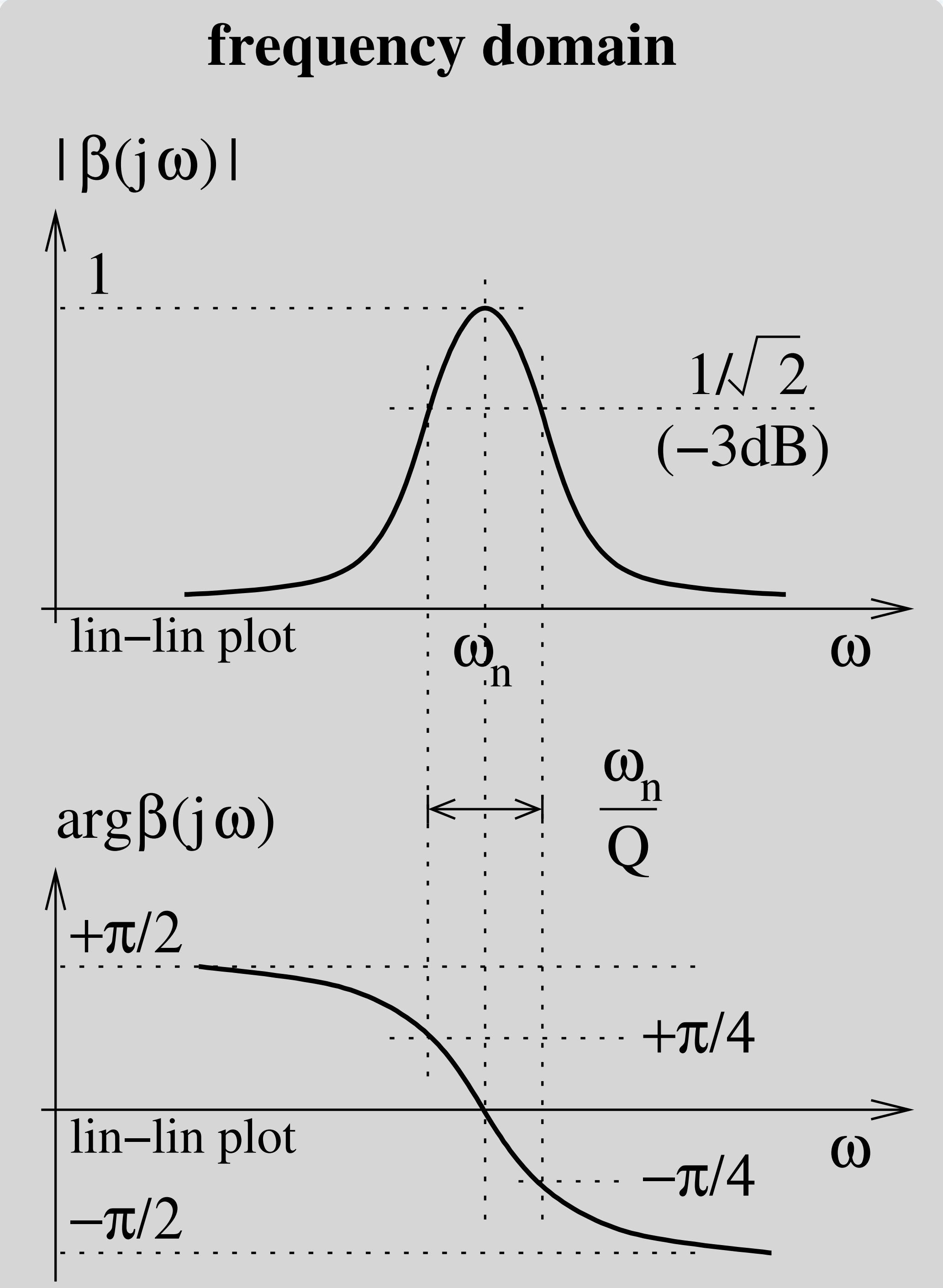
shorthand: f = ω/2π

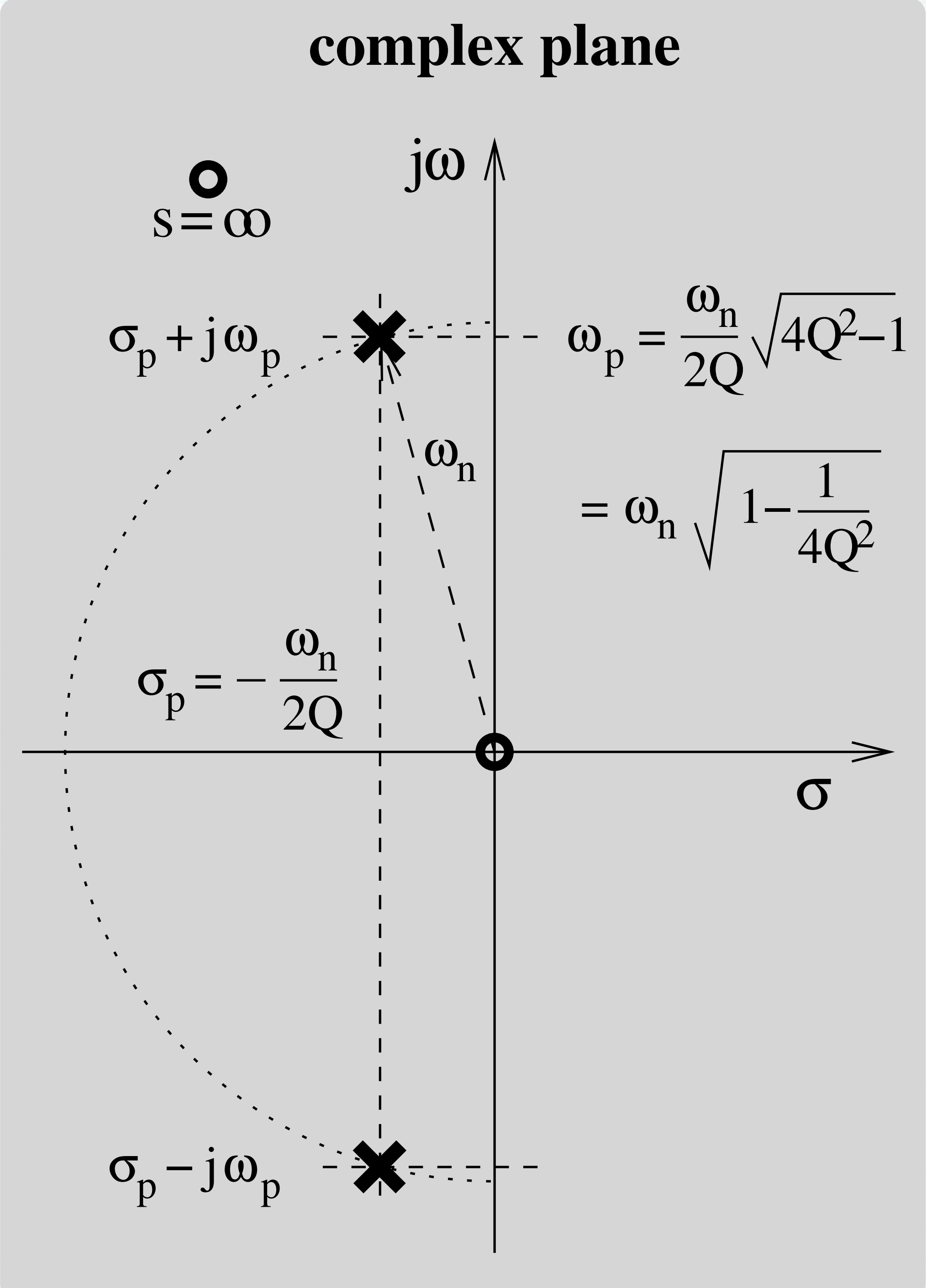


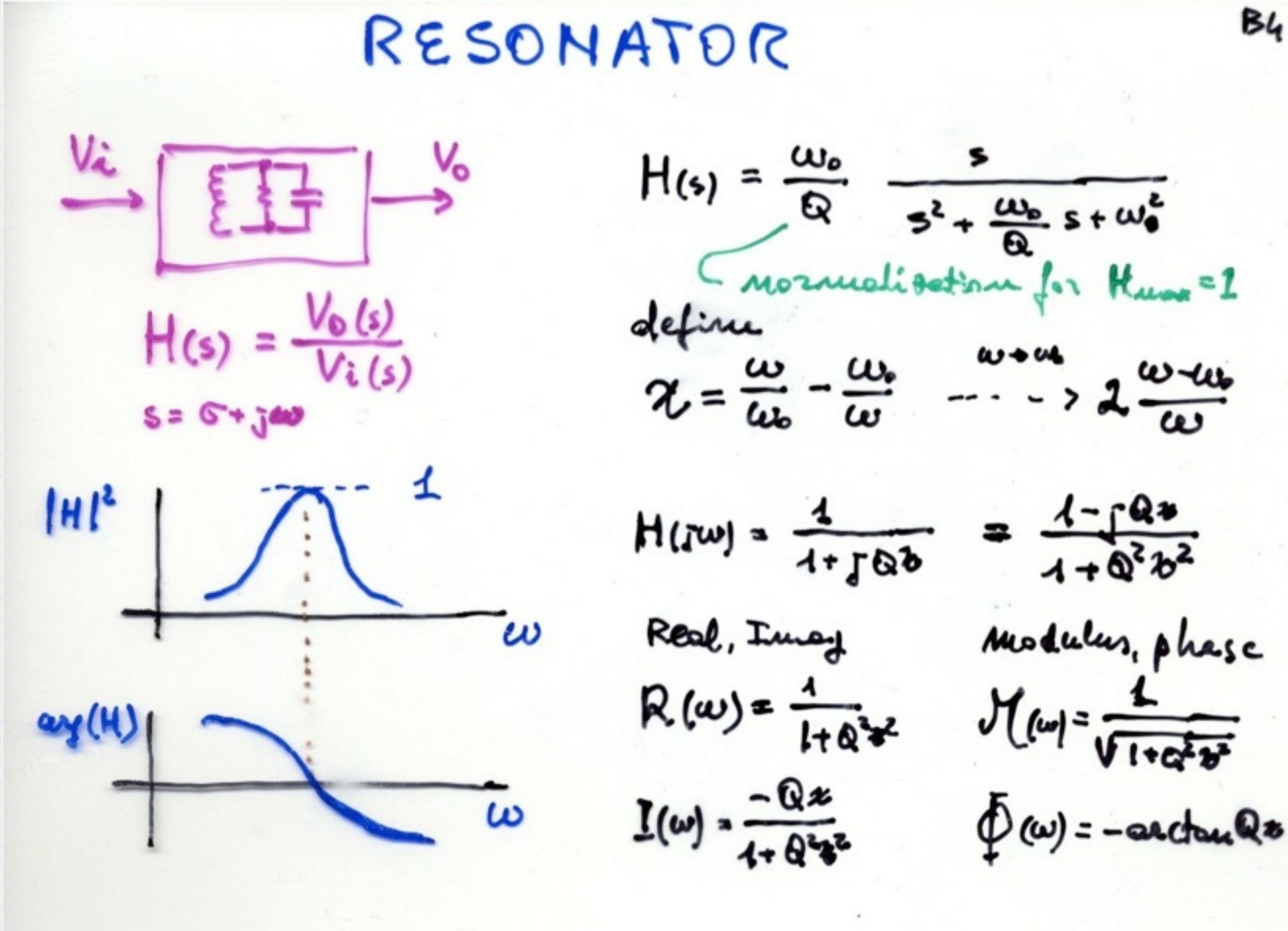


Resonator – frequency domain

$$\beta(s) = \frac{\omega_n}{Q} \frac{s}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2} \qquad s = \sigma + j\omega$$







$$\chi = \frac{\omega}{\omega_n} - \frac{\omega_n}{\omega}$$

$$\beta = \frac{1}{1 + jQ\chi}$$

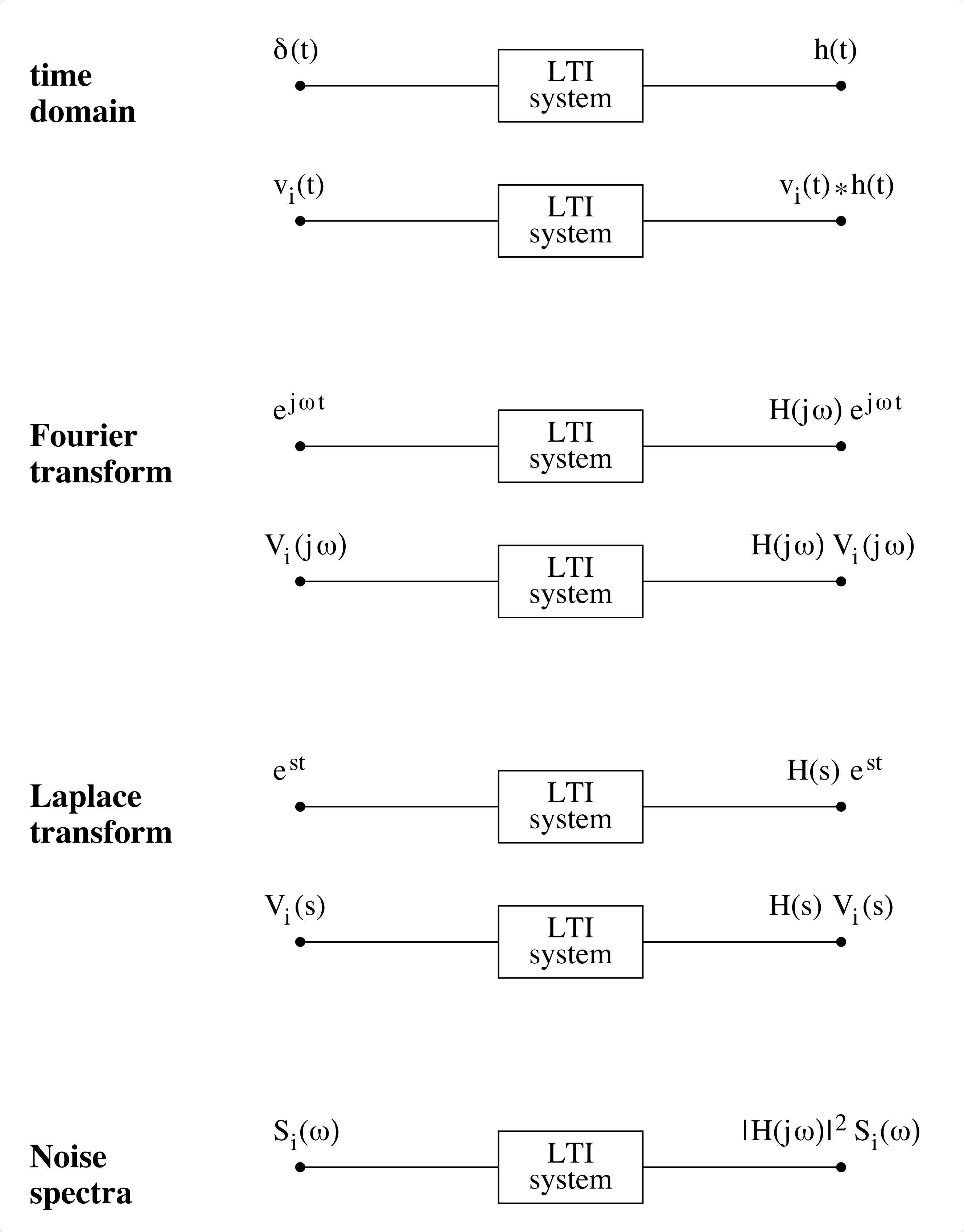
$$\Re\{\beta\} = \frac{1}{1 + Q^2\chi^2}$$

$$\Im\{\beta\} = \frac{-Q\chi}{1 + Q^2\chi^2}$$

$$|\beta|^2 = \frac{1}{1 + Q^2\chi^2}$$

$$\arg(\beta) = -\arctan(Q\chi)$$

Linear time-invariant (LTI) systems



impulse response

response to the generic signal $v_i(t)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

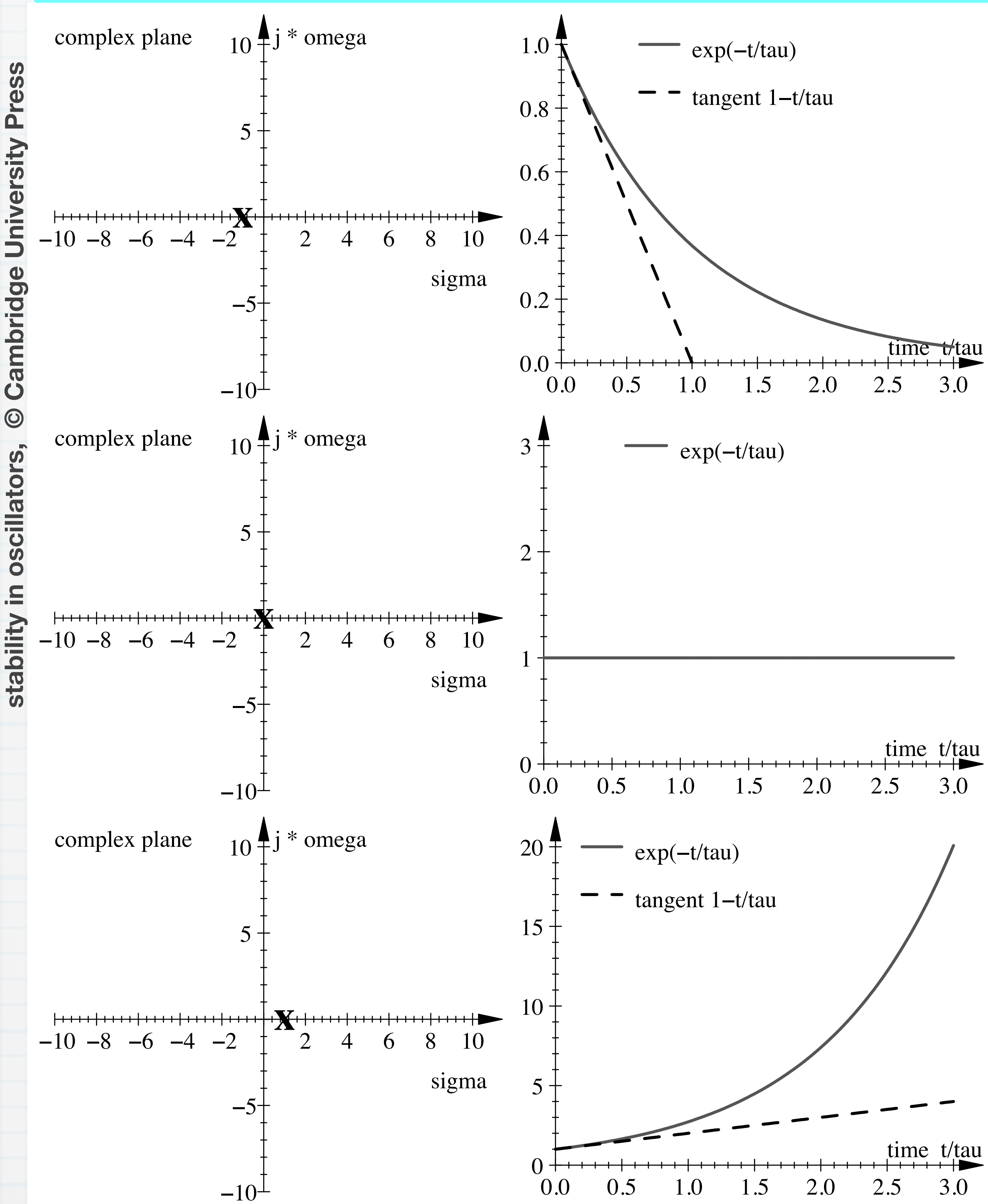
$$H(s) = \int_{0}^{\infty} h(t)\, e^{-st}\, dt$$

H(s), s=σ+jω, is the analytic continuation of H(ω) for causal system, where h(t)=0 for t<0

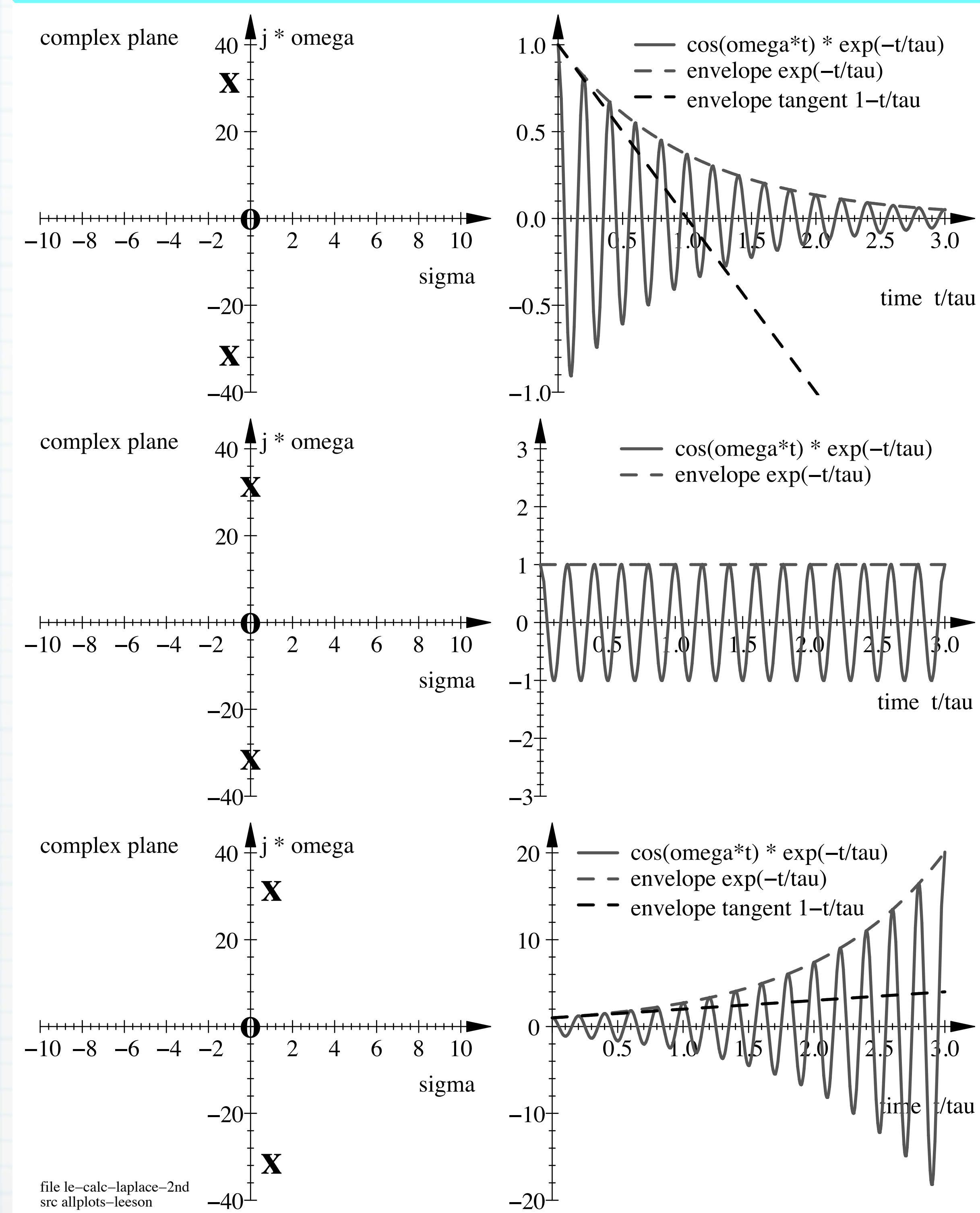
Laplace-transform patterns

Fundamental theorem of complex algebra: F(s) is completely determined by its roots

$$F(s) = \frac{1}{s + 1/\tau}$$



$$F(s) = \frac{s}{s^2 + 2s/\tau + \omega_n^2}$$



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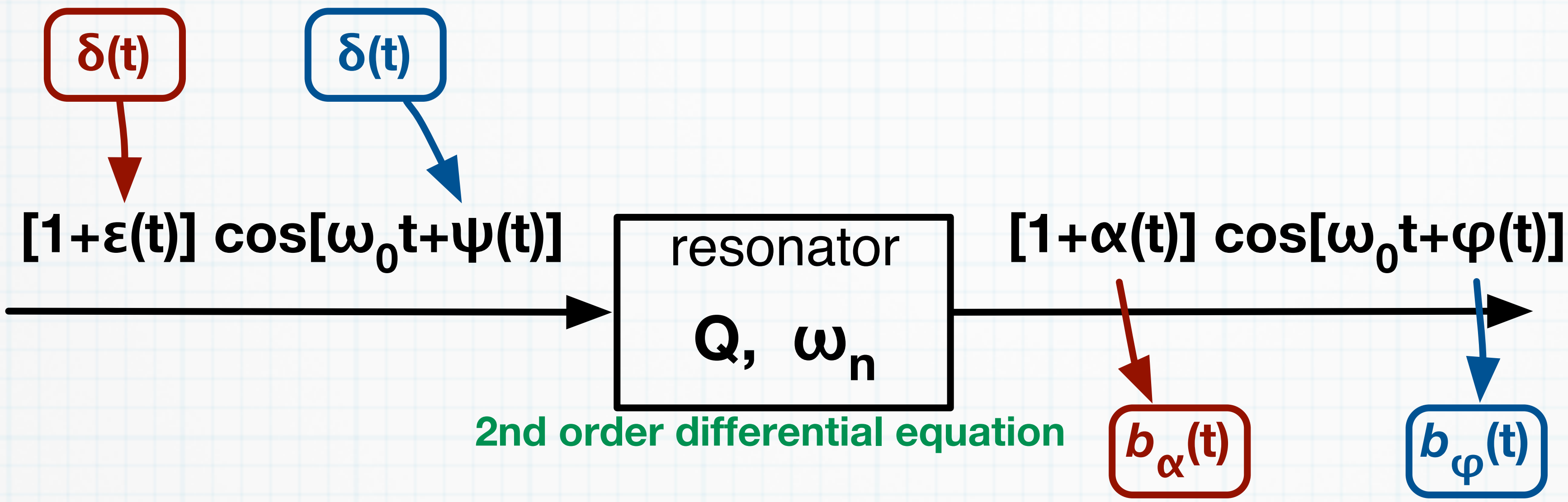
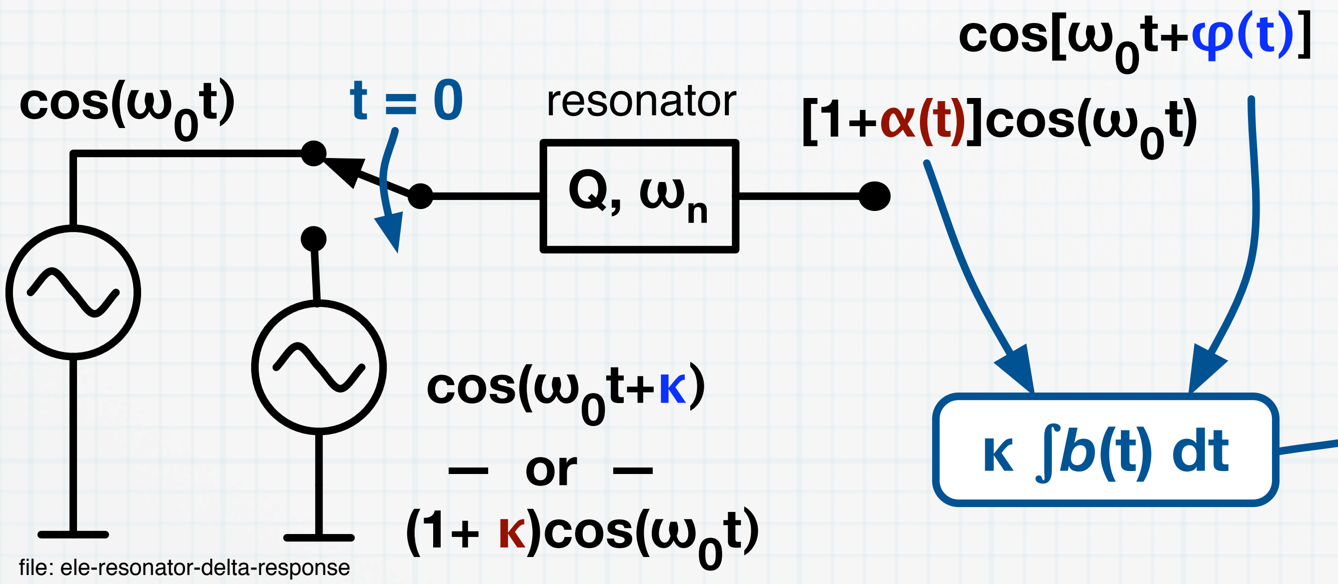
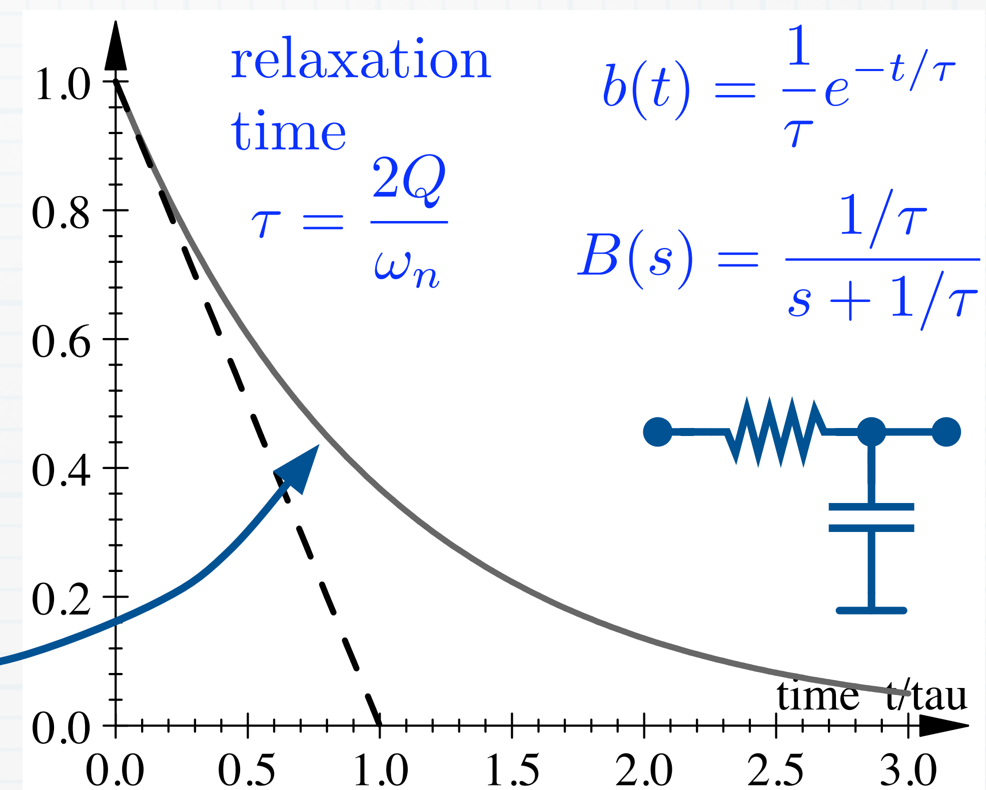
Resonator impulse response



Can't figure out a δ(t) of phase or amplitude? Use Heaviside (step) u(t) and differentiate

set a small phase or amplitude step κ at t=0, and linearize for κ→0

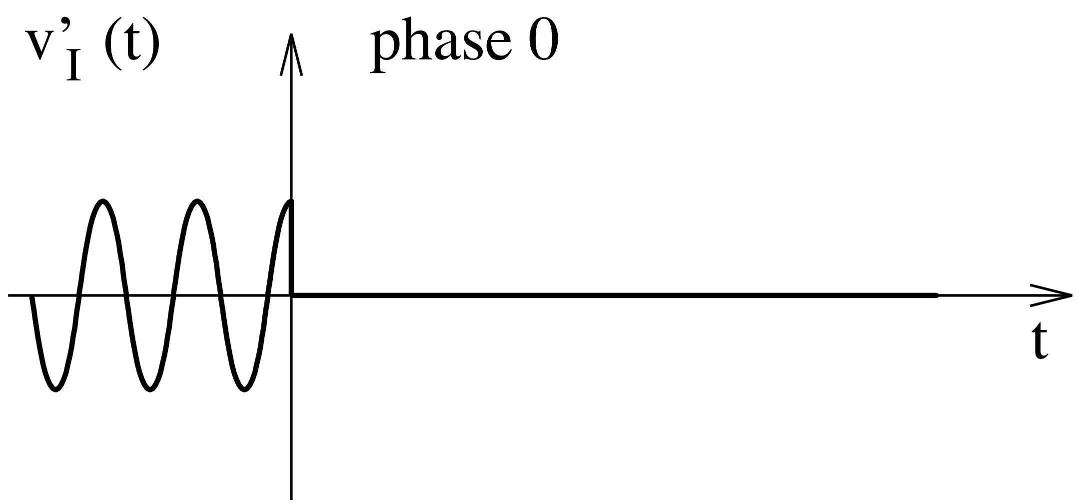




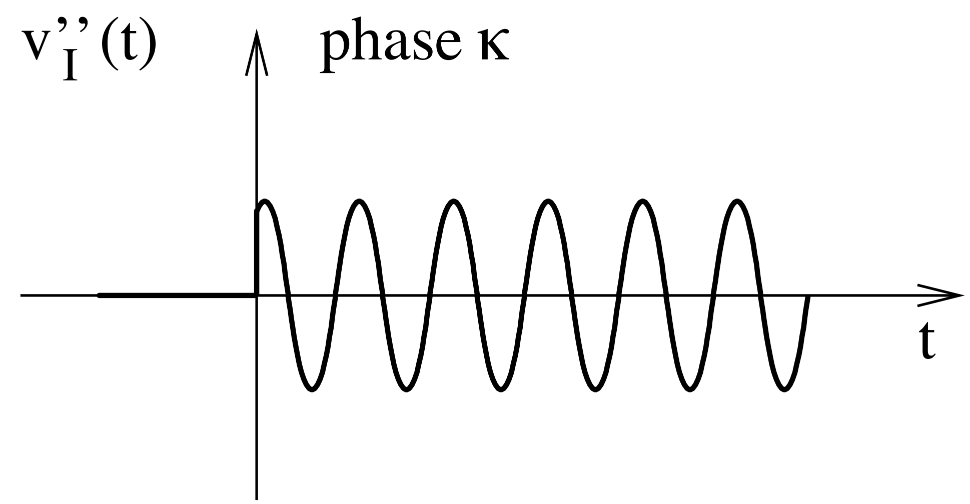
Response to a phase step κ

A phase step is equivalent to switching a sinusoid off at t = 0, and switching a shifted sinusoid on at t=0

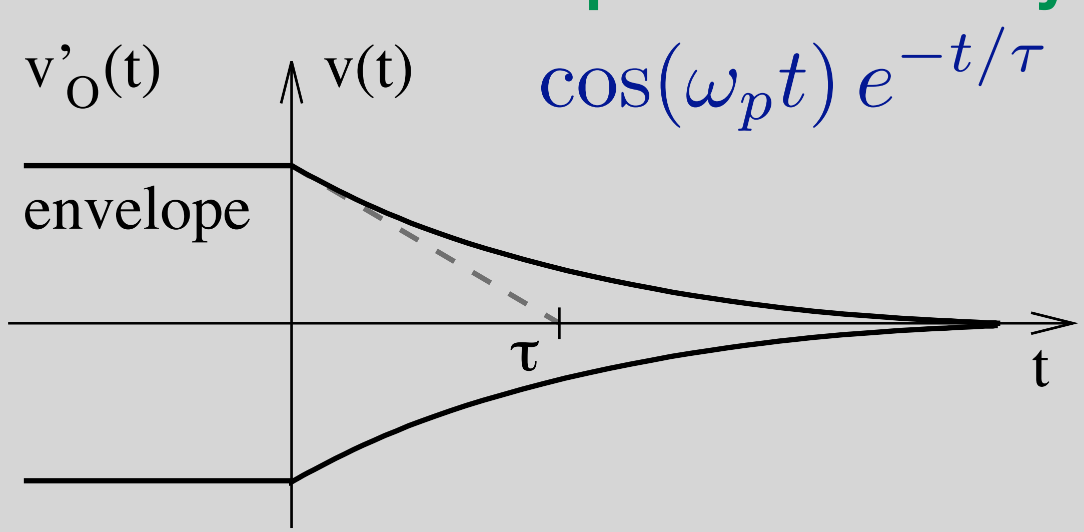
switched off at t = 0



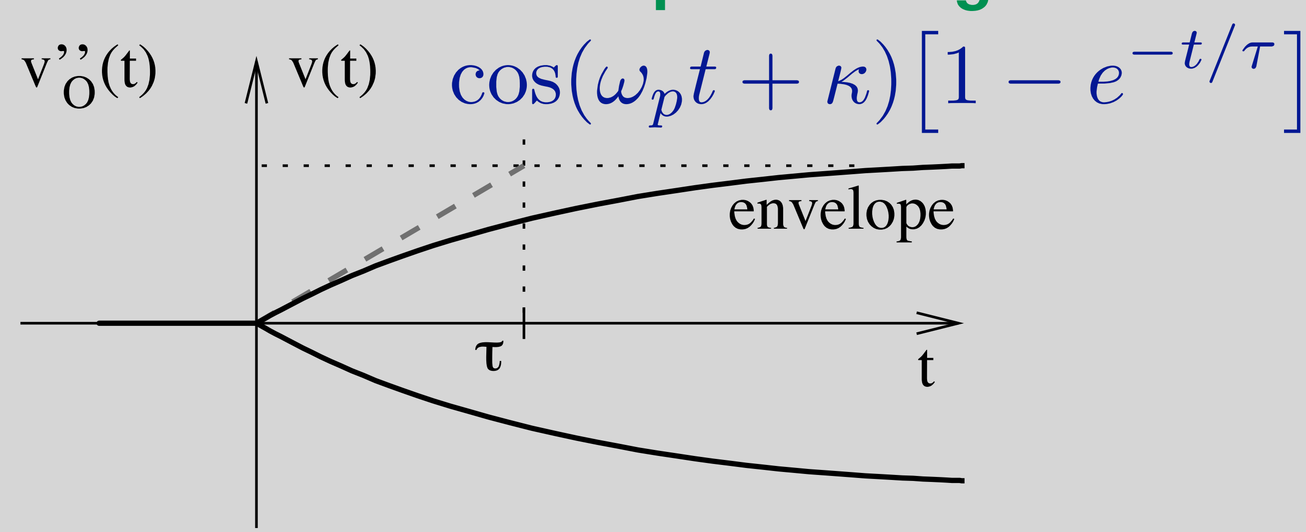
switched on at t = 0



exponential decay



exponential growth



Resonator impulse response ($\omega_0 = \omega_n$)

$$v_i(t) = \underbrace{\cos(\omega_0 t)\,\mathfrak{u}(-t)}_{\text{switched off at } t=0} + \underbrace{\cos(\omega_0 t + \kappa)\,\mathfrak{u}(t)}_{\text{switched on at } t=0}$$

phase step κ at t=0

$$v_o(t) = \cos(\omega_p t)\,e^{-t/\tau} + \cos(\omega_p t + \kappa)\left[1 - e^{-t/\tau}\right] \qquad t > 0$$

output

$$v_o(t) = \cos(\omega_p t) - \kappa \sin(\omega_p t)\left[1 - e^{-t/\tau}\right] \qquad \kappa \to 0$$

linearize

$$v_o(t) = \cos(\omega_0 t) - \kappa \sin(\omega_0 t)\left[1 - e^{-t/\tau}\right] \qquad \omega_p \to \omega_0$$

high Q

$$\mathbf{V_o}(t) = \frac{1}{\sqrt{2}}\left\{1 + j\kappa\left[1 - e^{-t/\tau}\right]\right\}$$

slow-varying phase vector

$$\arctan\left(\frac{\Im\{\mathbf{V_o}(t)\}}{\Re\{\mathbf{V_o}(t)\}}\right) \simeq \kappa\left[1 - e^{-t/\tau}\right]$$

phasor angle

delete κ and differentiate

impulse response

$$\mathrm{b}(t) = \frac{1}{\tau}e^{-s\tau} \quad \leftrightarrow \quad \mathrm{B}(s) = \frac{1/\tau}{s + 1/\tau}$$

Detuned resonator ($\omega_0 \neq \omega_n$)

amplitude
phase

$$\begin{bmatrix}\alpha\\ \varphi\end{bmatrix} = \begin{bmatrix}\mathrm{b}_{\alpha\alpha} & \mathrm{b}_{\alpha\varphi}\\ \mathrm{b}_{\varphi\alpha} & \mathrm{b}_{\varphi\varphi}\end{bmatrix} * \begin{bmatrix}\varepsilon\\ \psi\end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix}\mathcal{A}\\ \Phi\end{bmatrix} = \begin{bmatrix}\mathrm{B}_{\alpha\alpha} & \mathrm{B}_{\alpha\varphi}\\ \mathrm{B}_{\varphi\alpha} & \mathrm{B}_{\varphi\varphi}\end{bmatrix}\begin{bmatrix}\mathcal{E}\\ \Psi\end{bmatrix}$$

$$\Omega = \omega_0 - \omega_n$$ **detuning**

$$\beta_0 = |\beta(j\omega_0)|$$ **modulus**

$$\theta = \arg(\beta(j\omega_0))$$ **phase**

$$v_i(t) = \underbrace{\frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t)}_{\text{switched off at } t=0} + \underbrace{\frac{1}{\beta_0}\cos(\omega_0 t - \theta + \kappa)\,\mathfrak{u}(t)}_{\text{switched on at } t=0}$$

phase step κ at t=0

$$= \frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t) + \frac{1}{\beta_0}\Big[\cos(\omega_0 t - \theta)\cos\kappa - \sin(\omega_0 t - \theta)\sin\kappa\Big]\,\mathfrak{u}(t)$$

$$\simeq \frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t) + \frac{1}{\beta_0}\Big[\cos(\omega_0 t - \theta) - \kappa\sin(\omega_0 t - \theta)\Big]\,\mathfrak{u}(t) \qquad \kappa \ll 1.$$

Detuned resonator (cont.)

$$v_o(t) = \cos(\omega_0 t) - \kappa \sin(\omega_0 t) + \kappa \sin(\omega_n t)\, e^{-t/\tau}$$ **output, large Q (ω_p=ω_n)**

use $\Omega = \omega_0 - \omega_n$

$$v_o(t) = \cos(\omega_0 t)\left[1 - \kappa \sin(\Omega t) e^{-t/\tau}\right] - \kappa \sin(\omega_0 t)\left[1 - \cos(\Omega t) e^{-t/\tau}\right]$$

slow-varying phase vector

$$\mathbf{V_o}(t) = \frac{1}{\sqrt{2}}\left\{1 - \kappa \sin(\Omega t) e^{-t/\tau} + j\kappa\left[1 - \cos(\Omega t) e^{-t/\tau}\right]\right\} \qquad \kappa \ll 1$$

$$\arctan \frac{\Im\{\mathbf{V_o}(t)\}}{\Re\{\mathbf{V_o}(t)\}} = \kappa\left[1 - \cos(\Omega t) e^{-t/\tau}\right]$$ **angle**

$$|\mathbf{V_o}(t)| = |\mathbf{V_o}(0)| - \kappa \sin(\Omega t) e^{-t/\tau}$$ **amplitude**

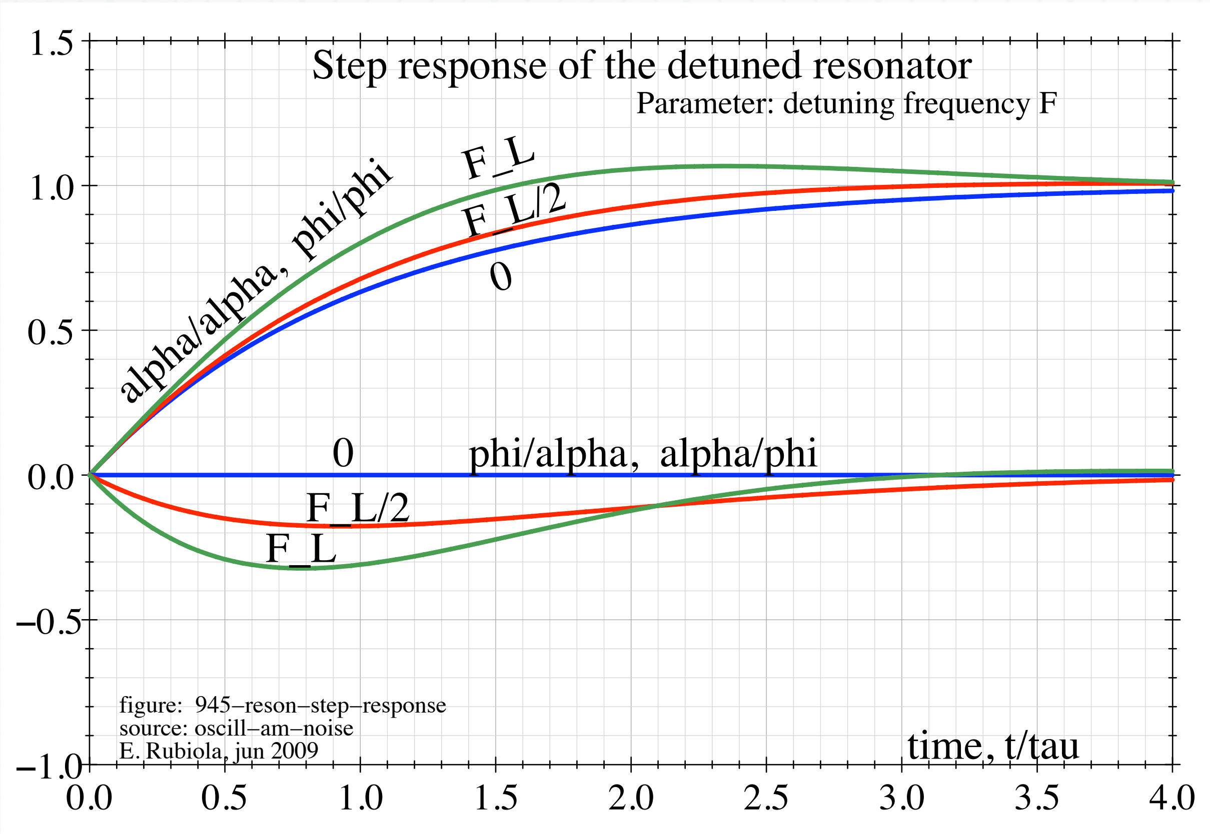
delete κ and differentiate

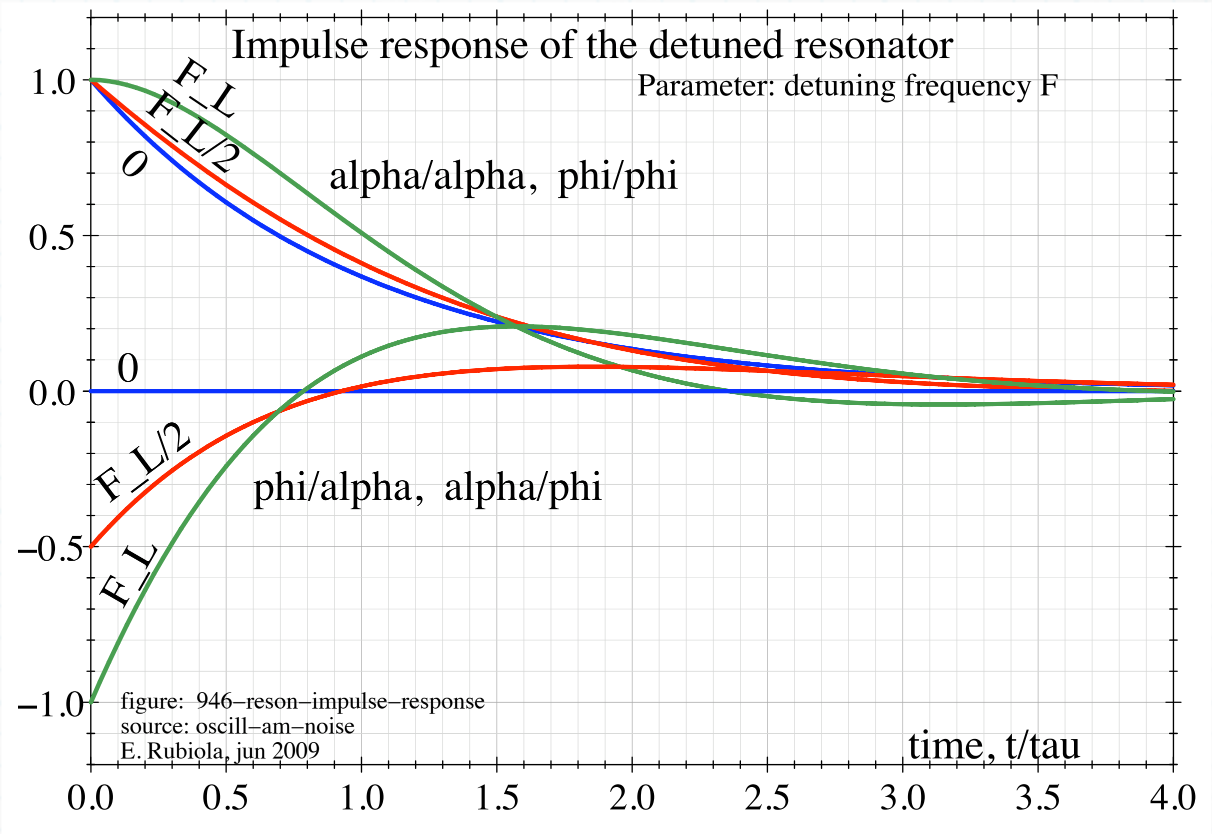
impulse response

$$\mathrm{b}_{\varphi\varphi}(t) = \left[\Omega \sin(\Omega t) + \frac{1}{\tau}\cos(\Omega t)\right] e^{-t/\tau} \qquad \text{phase}$$

$$\mathrm{b}_{\alpha\varphi}(t) = \left[-\Omega \cos(\Omega t) + \frac{1}{\tau}\sin(\Omega t)\right] e^{-t/\tau} \qquad \text{amplitude}$$

Resonator step and impulse response

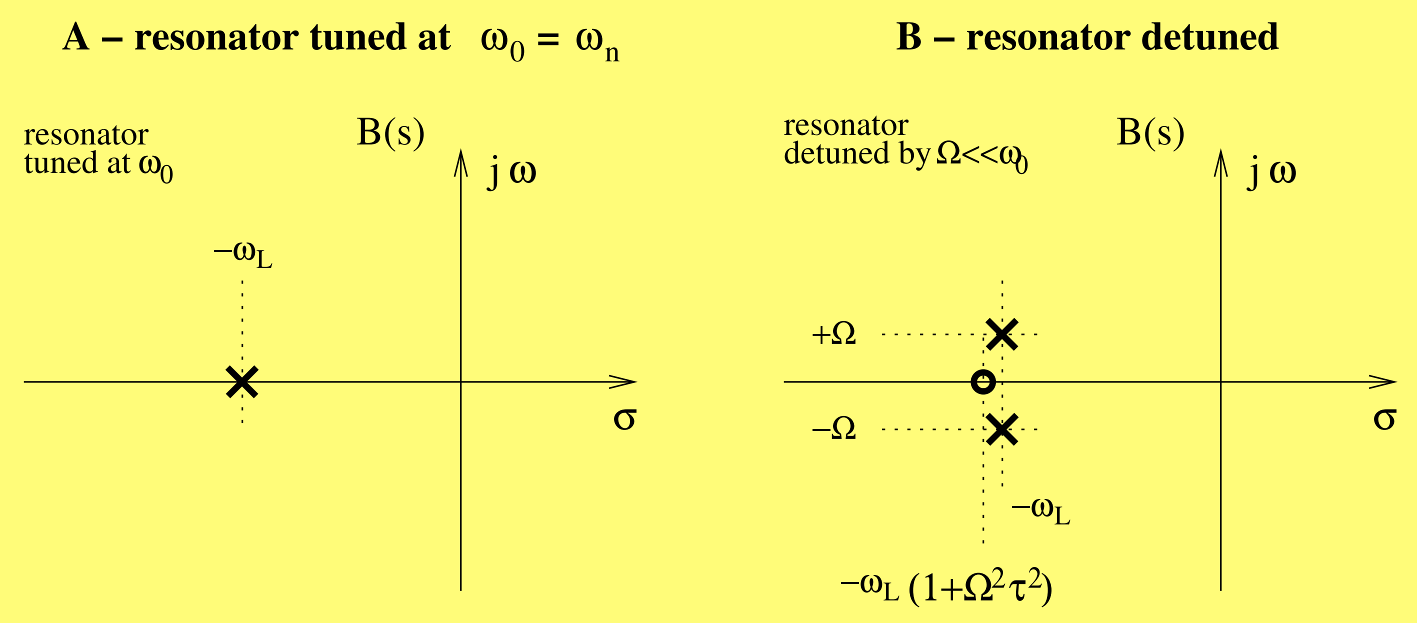




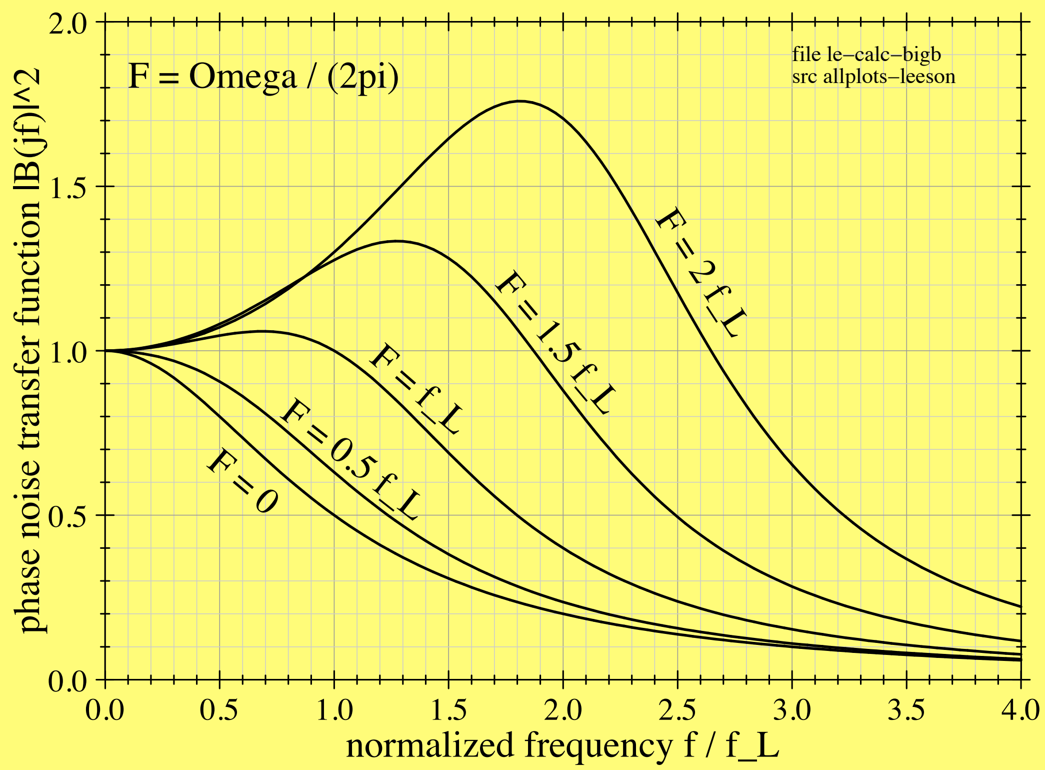
$$[\mathrm{b}](t) = \begin{bmatrix} \left(\Omega \sin \Omega t + \frac{1}{\tau} \cos \Omega t\right) e^{-t/\tau} & \left(-\Omega \cos \Omega t + \frac{1}{\tau} \sin \Omega t\right) e^{-t/\tau} \\ \left(-\Omega \cos \Omega t + \frac{1}{\tau} \sin \Omega t\right) e^{-t/\tau} & \left(\Omega \sin \Omega t + \frac{1}{\tau} \cos \Omega t\right) e^{-t/\tau} \end{bmatrix}$$

check on the sign of b_{21}

Frequency response



diagonal terms



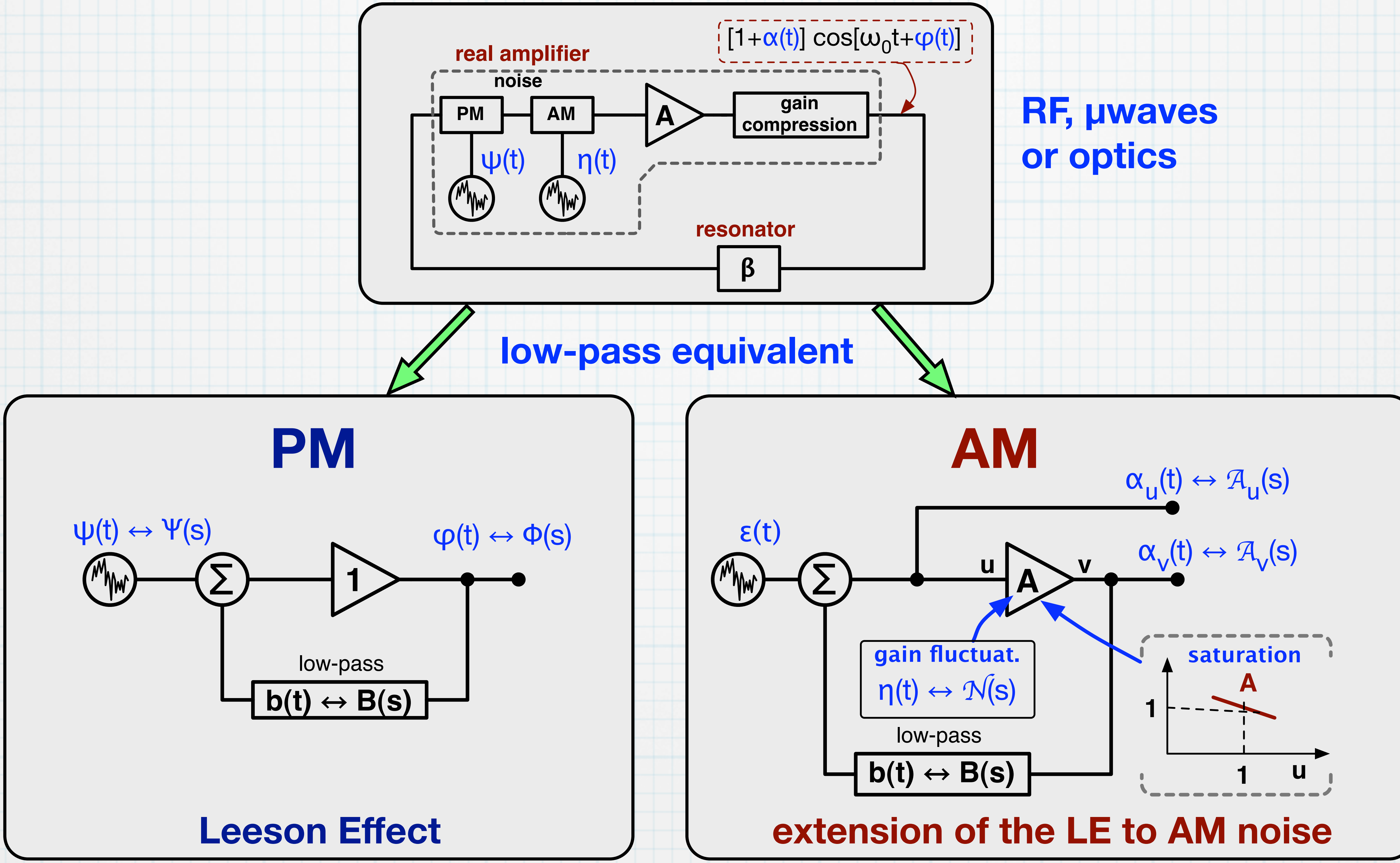
diagonal terms

$$[\mathrm{B}](s) = \begin{bmatrix} \dfrac{1}{\tau}\,\dfrac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \dfrac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \\ \dfrac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \dfrac{1}{\tau}\,\dfrac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \end{bmatrix}$$

check on the sign of B_{21}

Formal proof for the Leeson effect

Low-pass representation of AM-PM noise

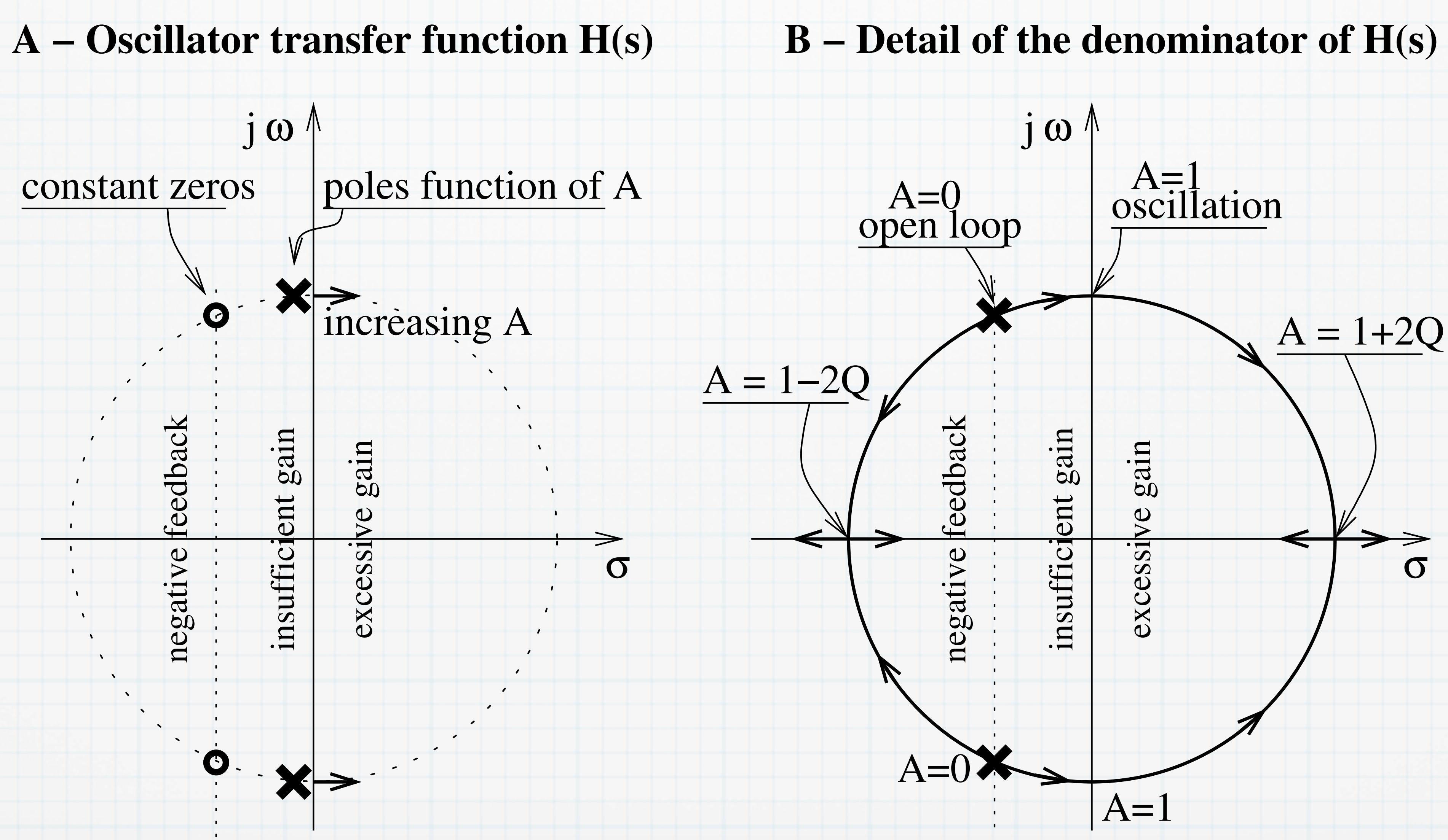


The amplifier
- “copies” the input phase to the out
- adds phase noise

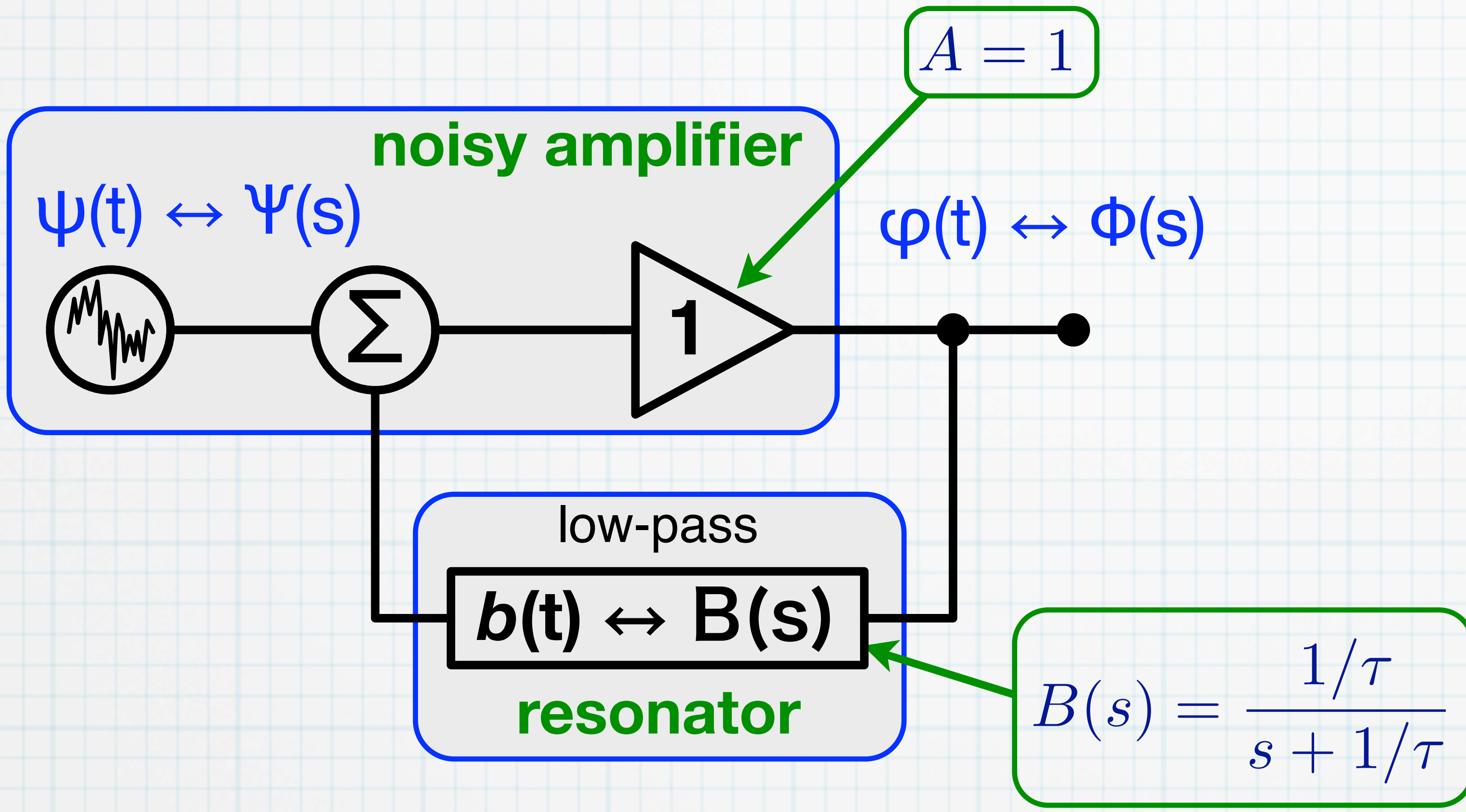
The amplifier
- compresses the amplitude
- adds amplitude noise

Effect of feedback

Oscillator transfer function (RF)



Leeson effect

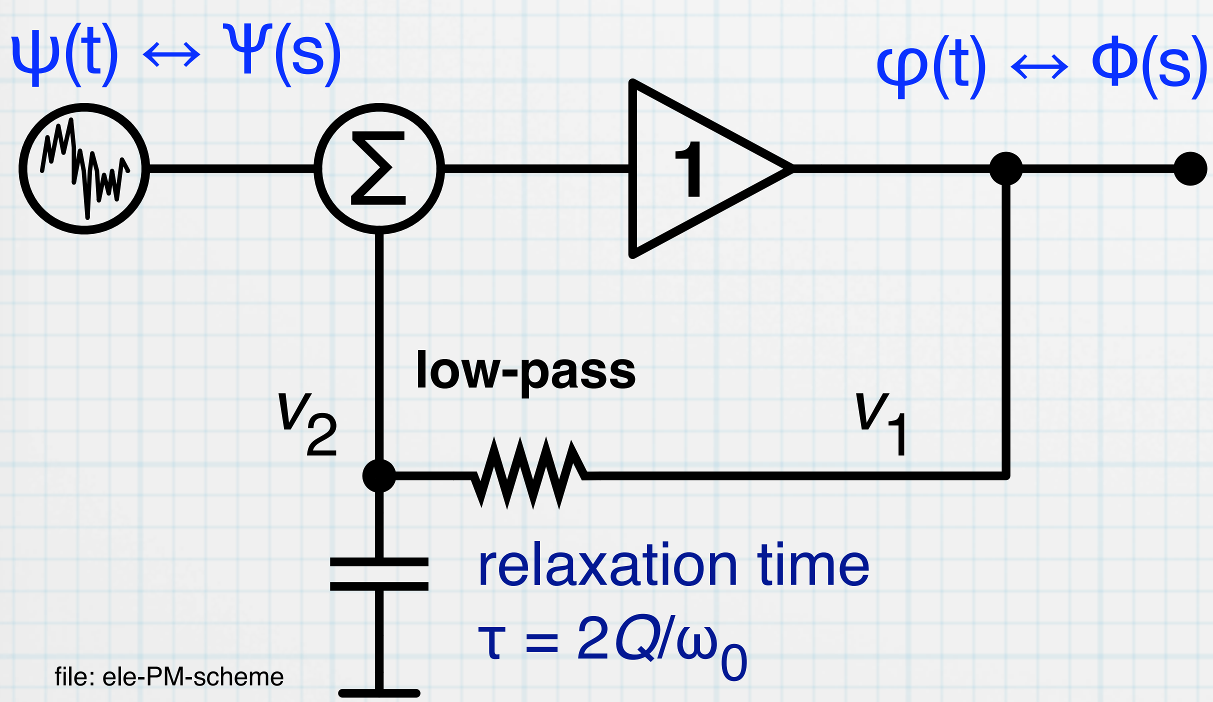


phase-noise transfer function

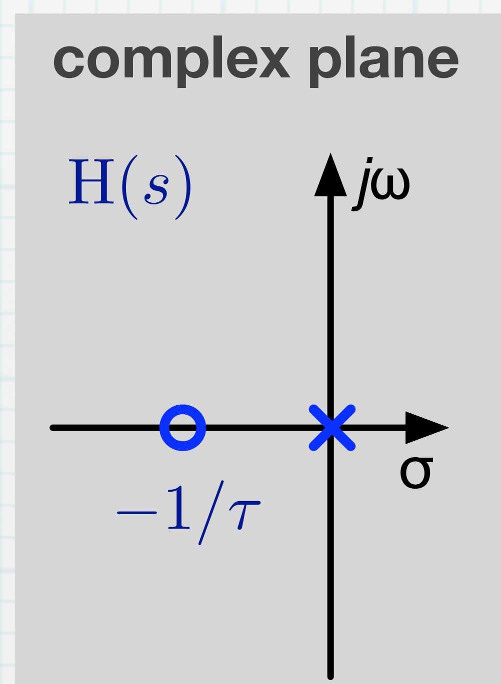
$$H(s) = \frac{\Phi(s)}{\Psi(s)}$$ **definition**

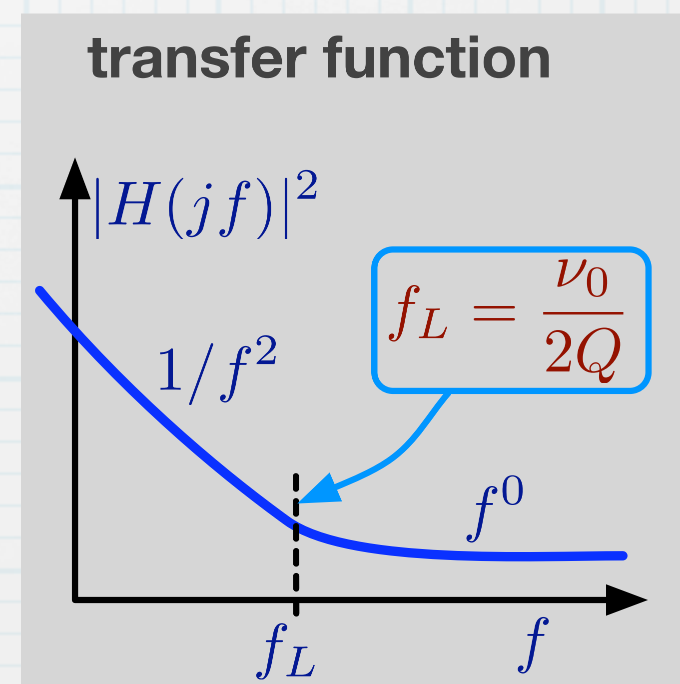
$$H(s) = \frac{1}{1 + AB(s)}$$ **general feedback theory**

$$H(s) = \frac{1 + s\tau}{s\tau}$$ **Leeson effect**

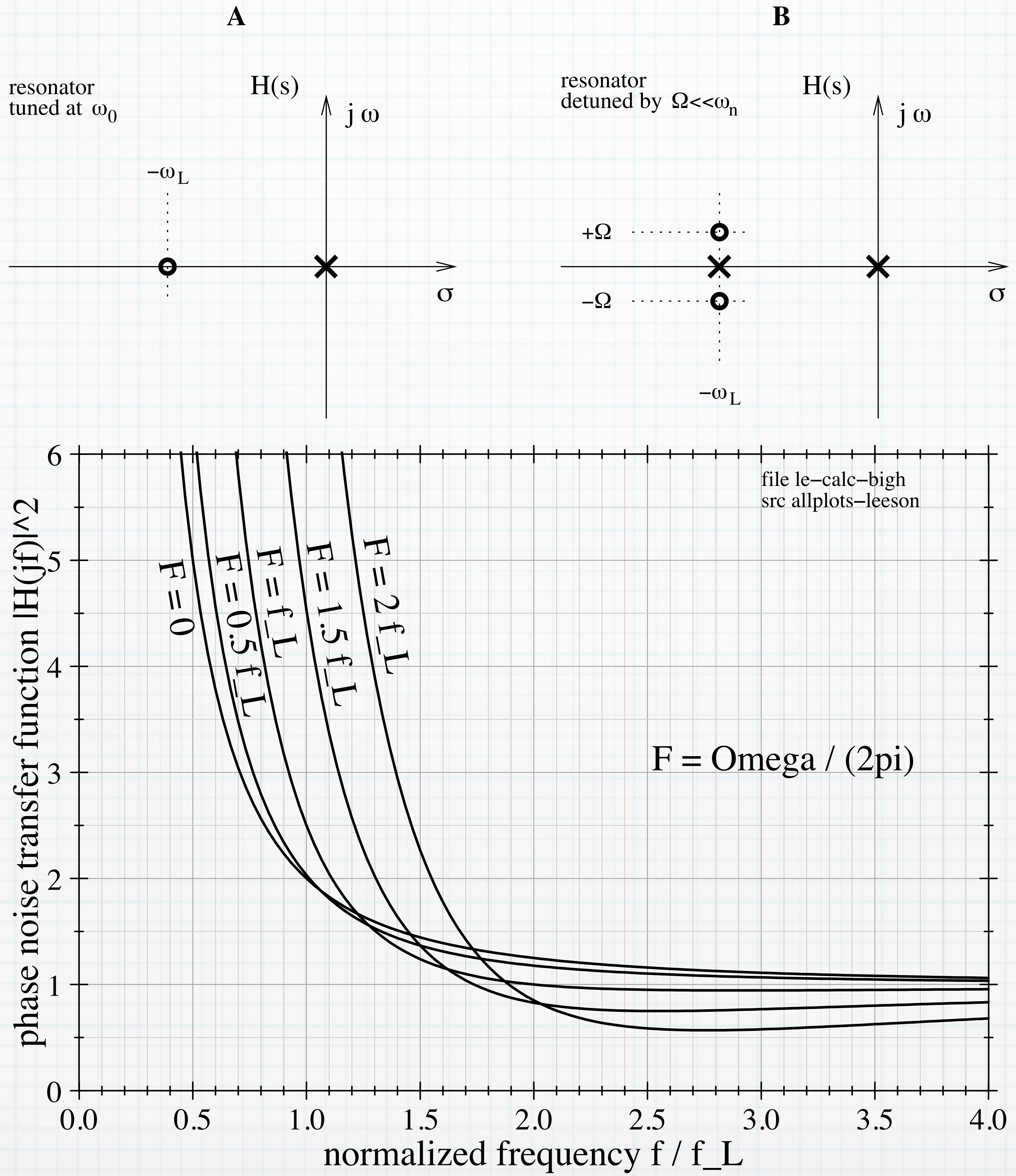


file: ele-PM-scheme



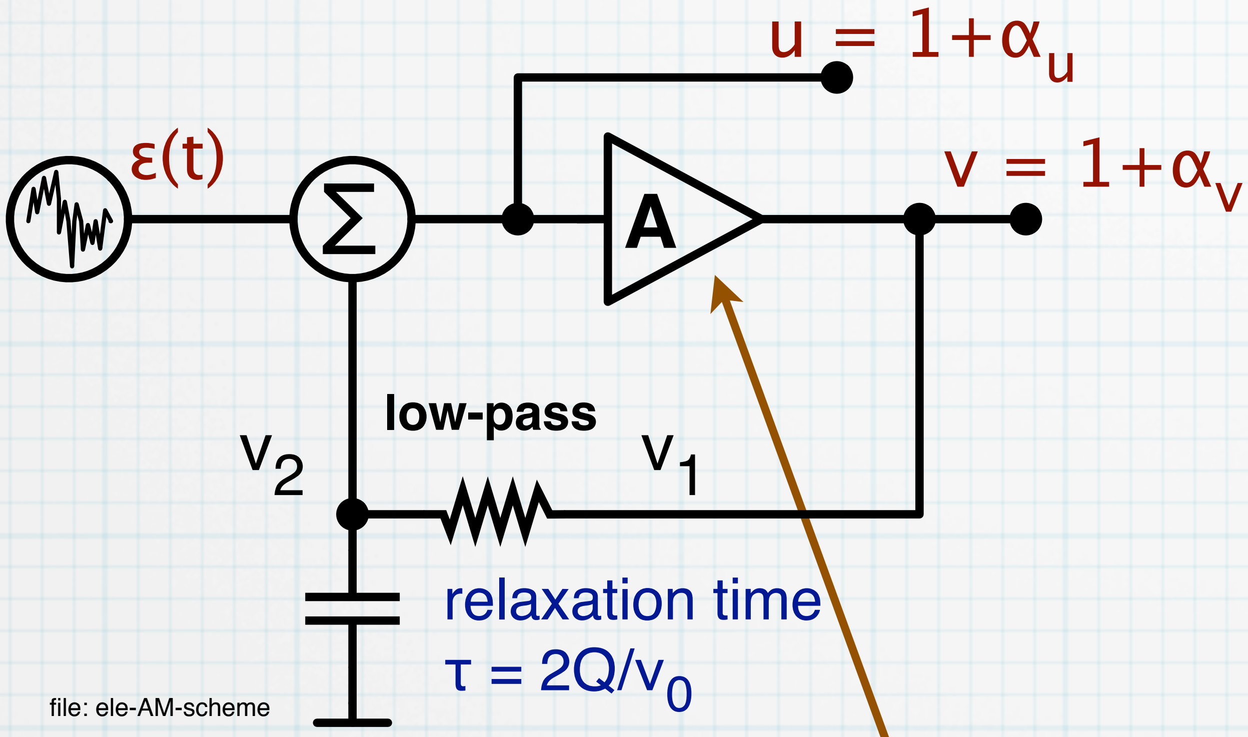


Oscillator with detuned resonator



Low-pass model of amplitude (1)

First we need to relate the system restoring time τ_r to the relaxation time τ



simple feedback theory

$$u = \epsilon + v_2$$

$$v_2 = \frac{1}{\tau}\int (v_1 - v_2)\, dt$$

$$v_2 = u - \epsilon$$

$$v_1 = v = Au$$

$$u = \epsilon + \frac{1}{\tau}\int (A-1)u + \epsilon\, dt$$

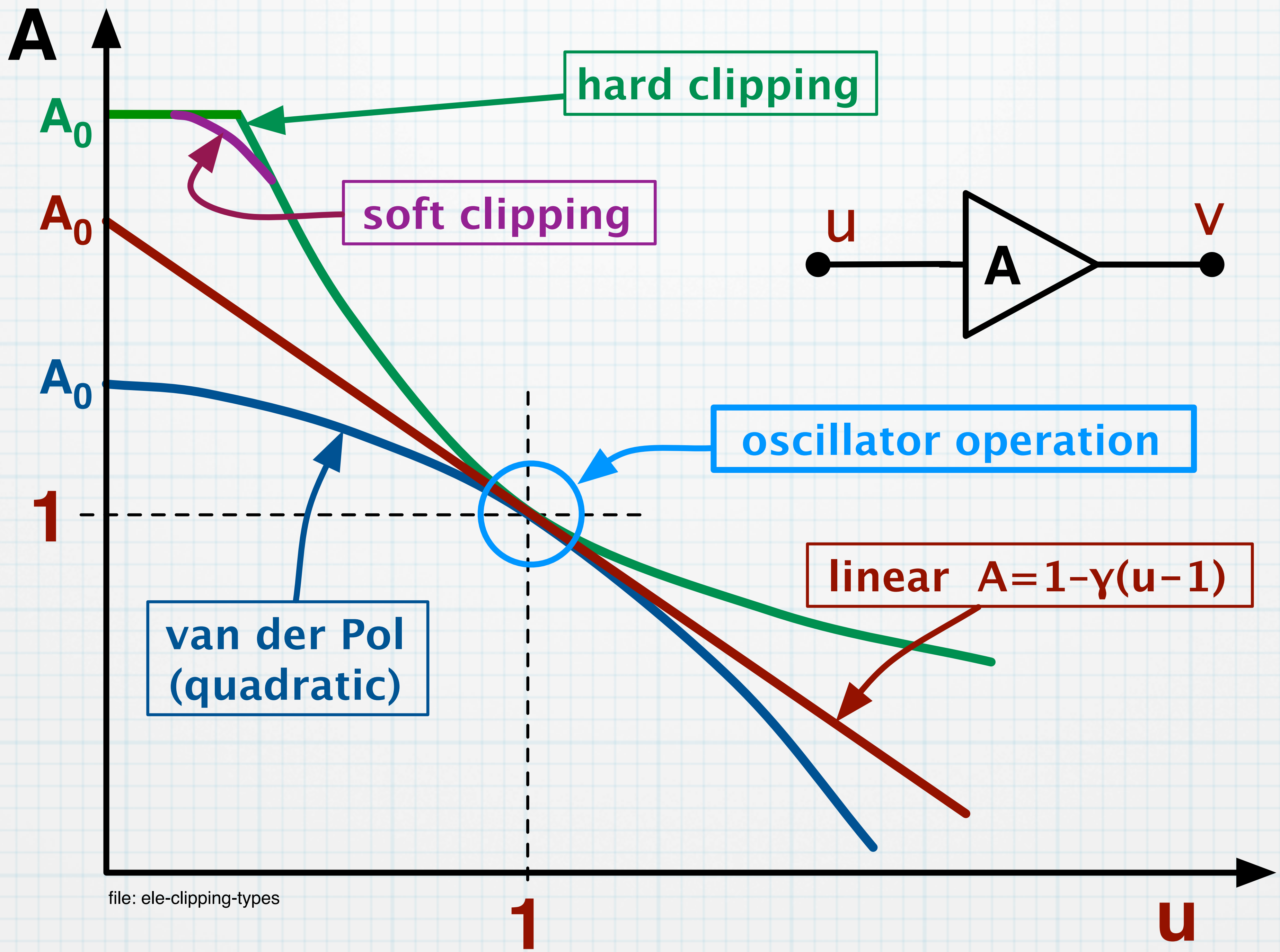
differential equation

$$\dot{u} - \frac{1}{\tau}(A-1)\,u = \frac{1}{\tau}\epsilon + \dot{\epsilon}$$

Gain compression is necessary for the oscillation amplitude to be stable

The Laplace / Heaviside formalism cannot be used because the amplifier is non-linear

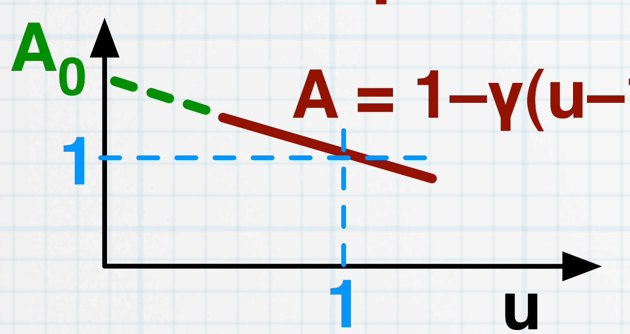
Common types of gain saturation



Gain compression is necessary for the oscillation amplitude to be stable

Low-pass model of amplitude (2)

Gain compression

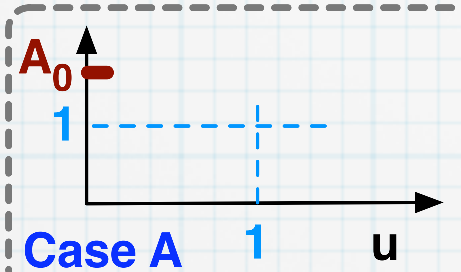


homogeneous differential equation

$$\dot{u} - \frac{1}{\tau}\,(A-1)\,u = 0$$

Three asymptotic cases

At low RF amplitude, let the gain be an arbitrary value denoted with A_0

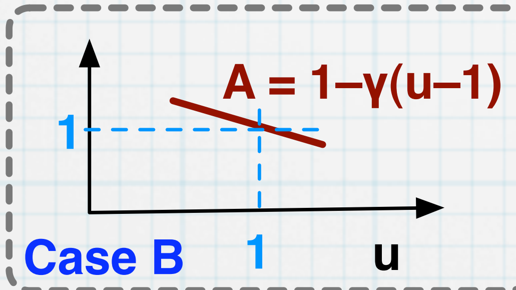


Startup: u→0, A → A_0 > 1

$$\dot{u} - \frac{1}{\tau}\,(A_0-1)\,u = 0 \quad \Rightarrow \quad u = C_1 e^{(A_0-1)\,t/\tau}$$

rising exponential

For small fluctuation of the stationary RF amplitude, the gain amplitude, the gain varies linearly with V

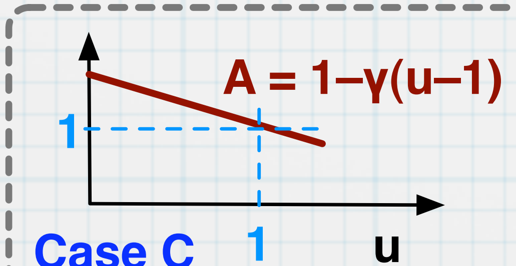


Regime: u→1, A = 1 – γ (u – 1)

$$\dot{u} + \frac{\gamma}{\tau}\,(u-1)\,u = 0 \quad \Rightarrow \quad u = C_2 e^{-\gamma\,t/\tau}$$

restoring time constant $\tau_r = \tau/\gamma$

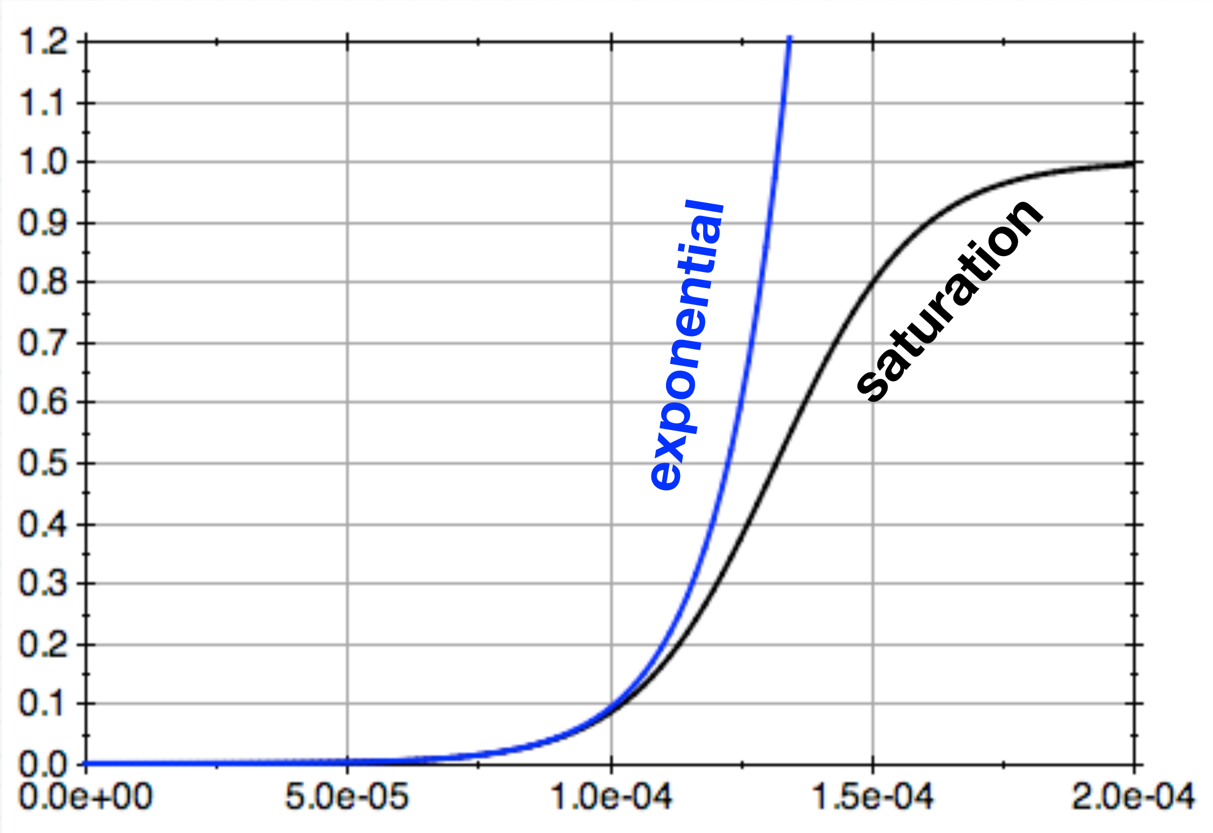
Simplification: the gain varies linearly with V in all the input range



Linear gain: A = 1 – γ (u – 1)

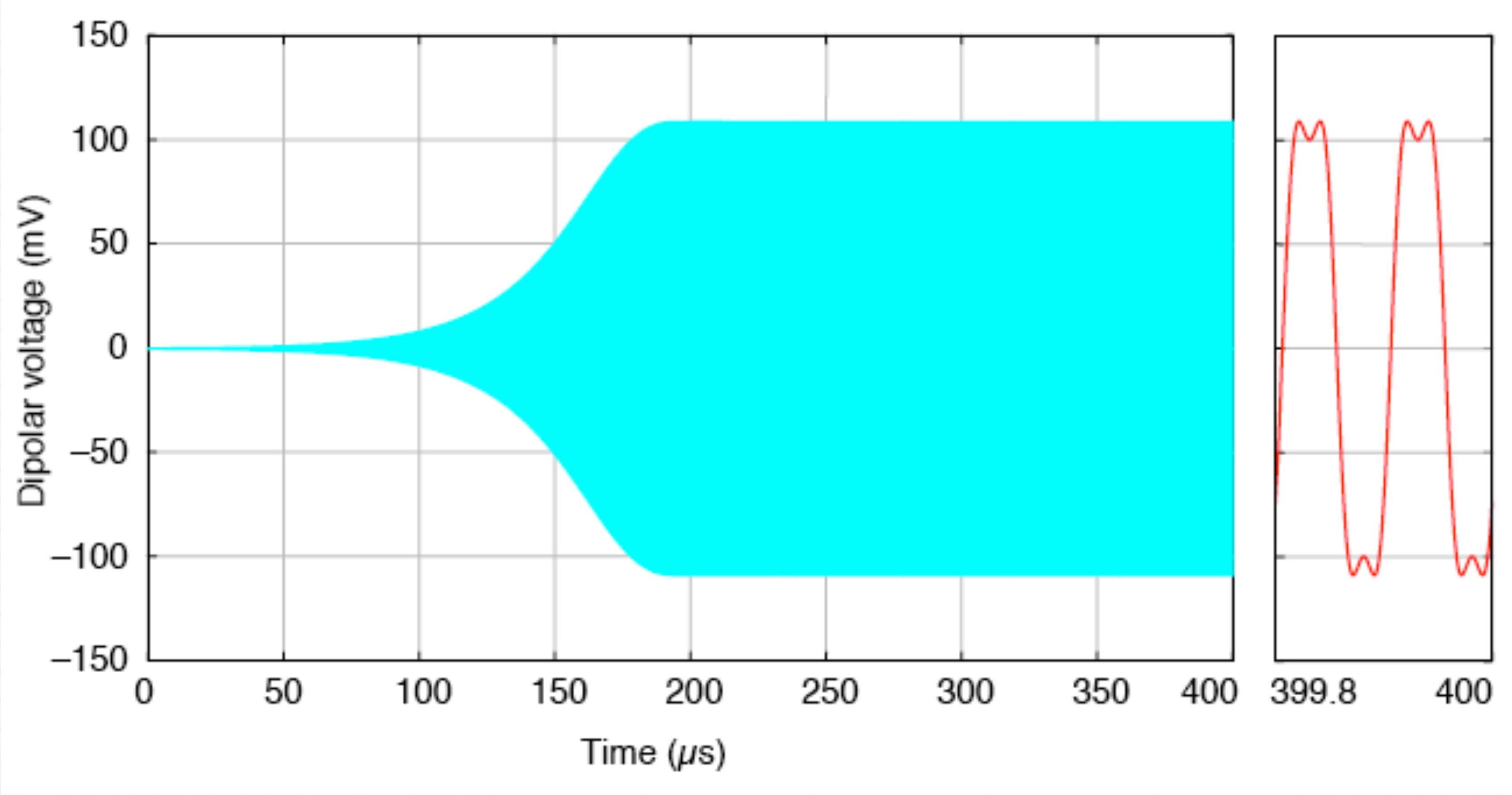
$$u = \frac{1}{\left(\frac{1}{u(0)} - 1\right) e^{-\gamma t/\tau} + 1}$$

Startup – analysis vs. simulation



analytical solution,
A = 1–γ(u–1)

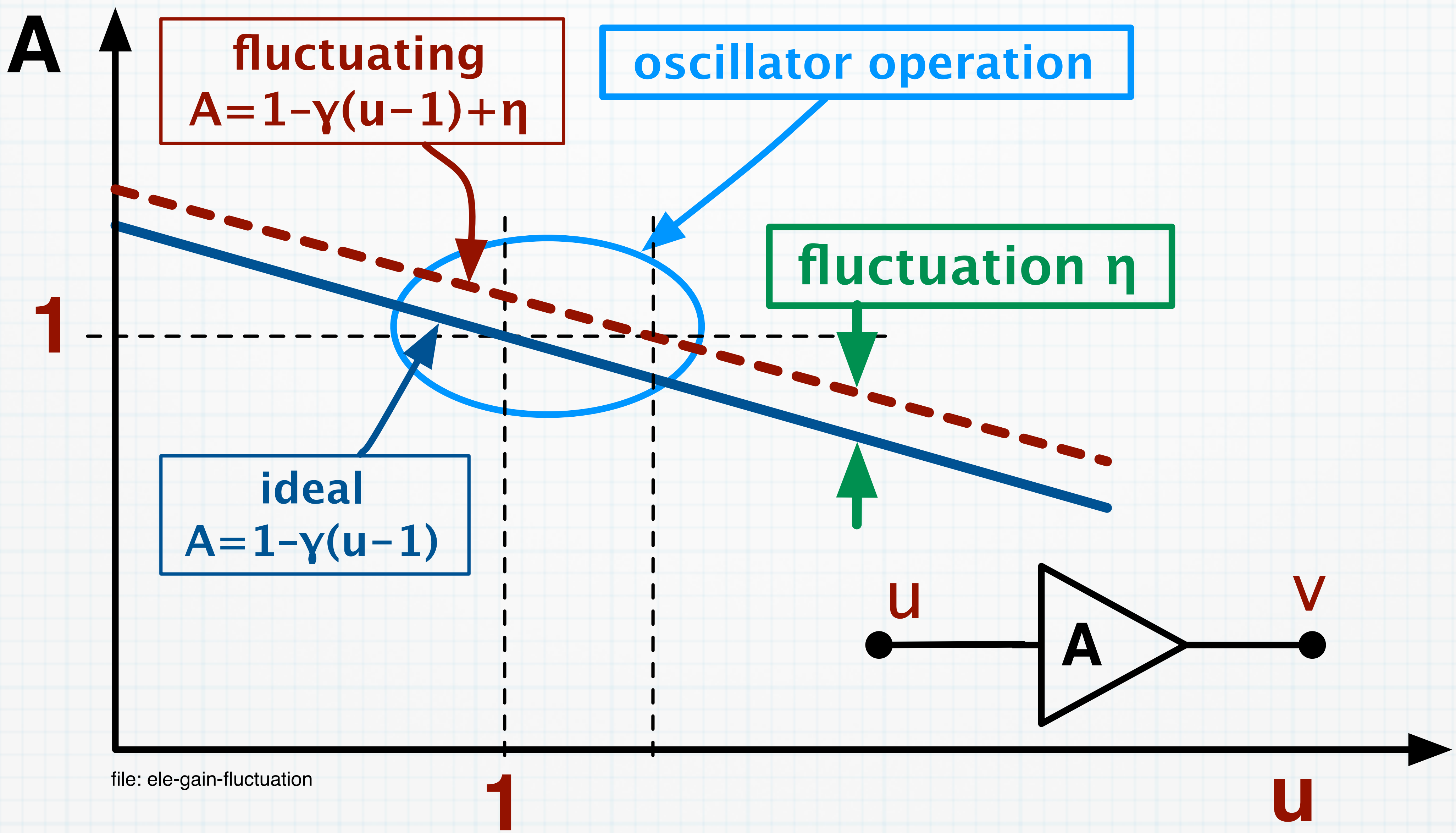
10 MHz oscillator
L = 1 mH
R = 125 Ω
Q ~ 503



van der Pol oscillator
simulated by RB

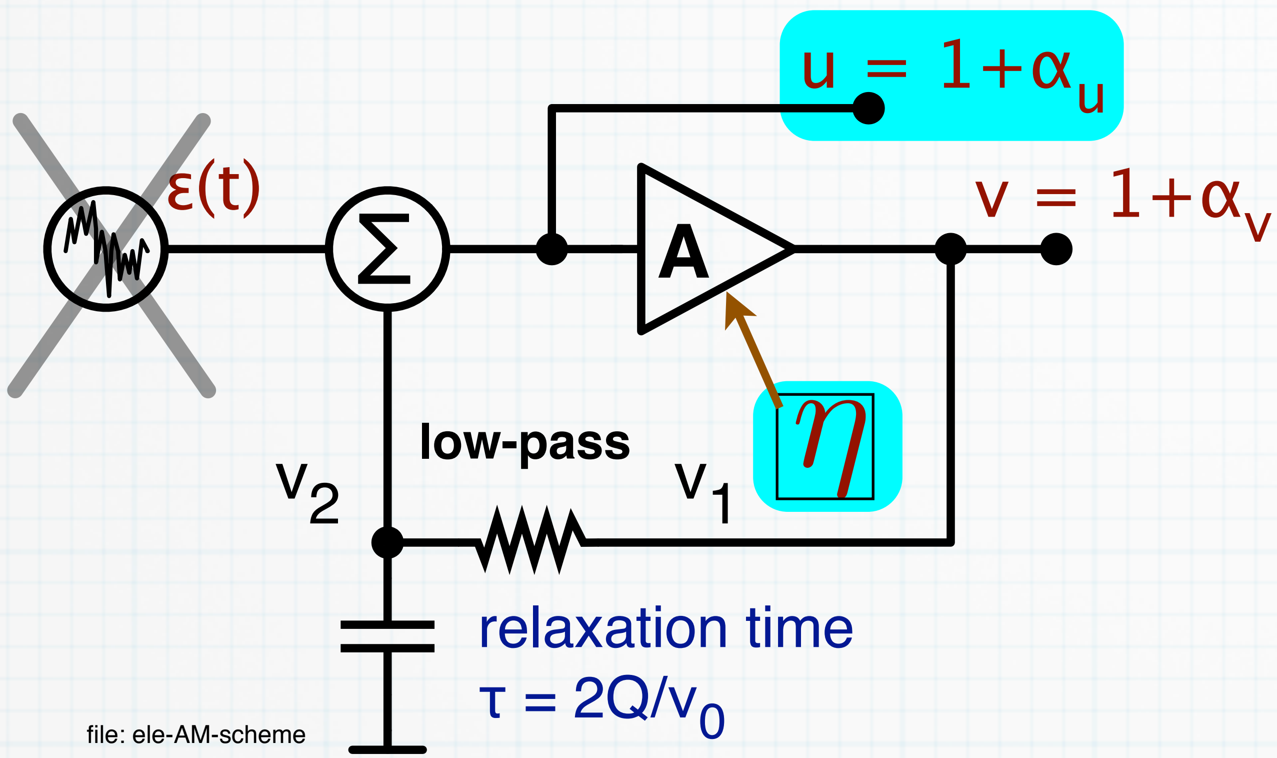
Rising exponential.
We find the same
time constant –τ/γ

Gain fluctuations – definition



Gain compression is necessary for the oscillation amplitude to be stable

Gain fluctuations – output is u



$$\dot{u} = \frac{1}{\tau}(A-1)u$$ **non-linear equation**

$$A = 1 - \gamma(u-1) + \eta$$

$$\dot{u} + \frac{\gamma}{\tau}(u-1)u = \frac{\eta}{\tau}u$$ **linearization for low noise**

($\dot{u} \to \dot{\alpha}_u$, $(u-1) \to \alpha_u$, $u \to 1$, $u \to 1$)

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \frac{1}{\tau}\eta$$ **linearized equation**

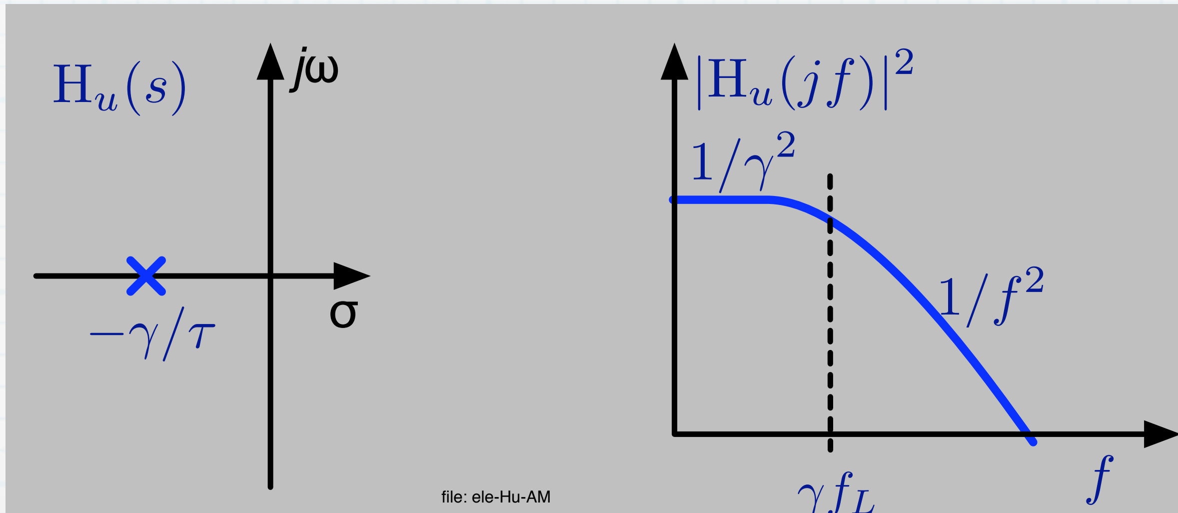
$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \frac{1}{\tau}\mathcal{N}(s)$$ **Laplace transform**

Linearize for low noise and use the Laplace transforms

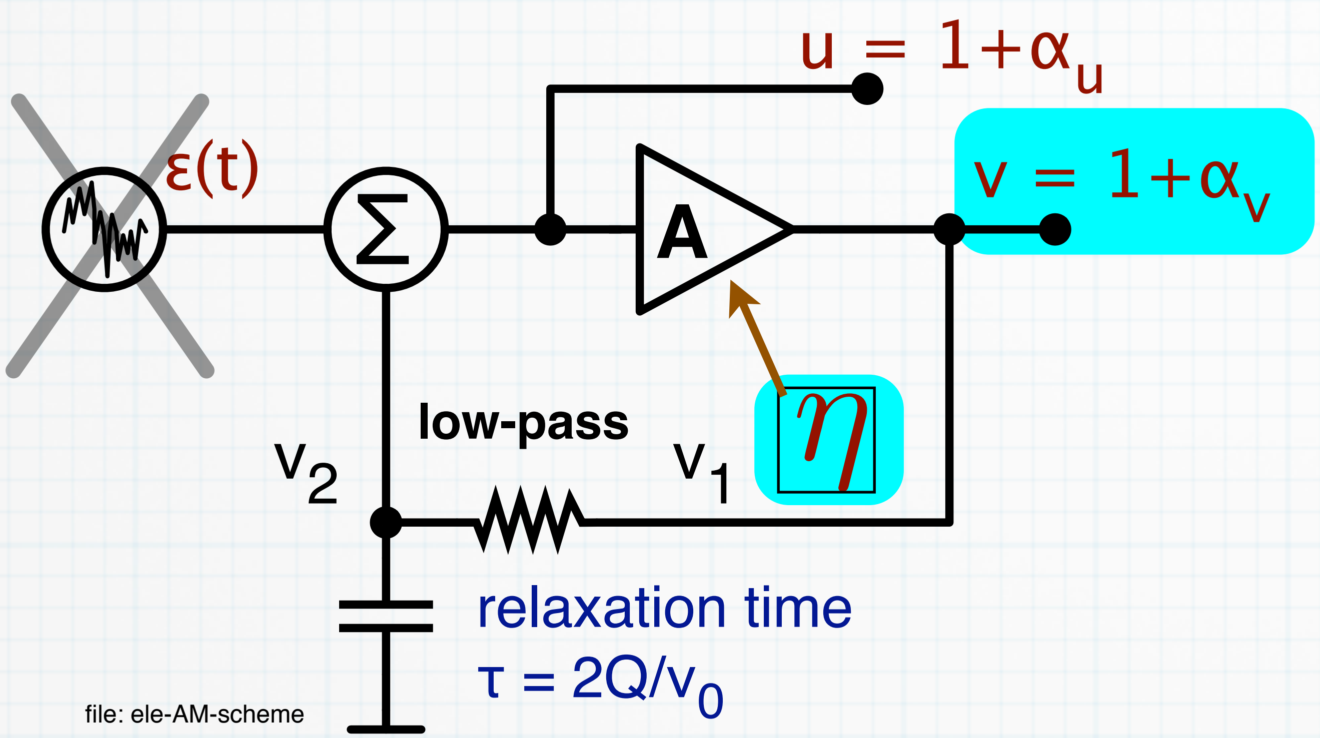
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \eta(t) \leftrightarrow \mathcal{N}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}_u(s) = \frac{1/\tau}{s + \gamma/\tau}$$ **result**



Gain fluctuations – output is v



$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_u(s)=\frac{1}{\tau}\mathcal{N}(s)$$ **starting equation**

$$\mathcal{A}_u(s)=\frac{\mathcal{A}_v(s)-\mathcal{N}(s)}{1-\gamma}$$

$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_v(s)=\left(s+\frac{1}{\tau}\right)\mathcal{N}(s)$$

$$\mathrm{H}(s)=\frac{\mathcal{A}_v(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}(s)=\frac{s+1/\tau}{s+\gamma/\tau}$$ **result**

boring algebra relates α_v to α_u

$$v=Au$$

$$A=-\gamma(u-1)+1+\eta$$

$$v=[-\gamma(u-1)+1+\eta]\,u$$

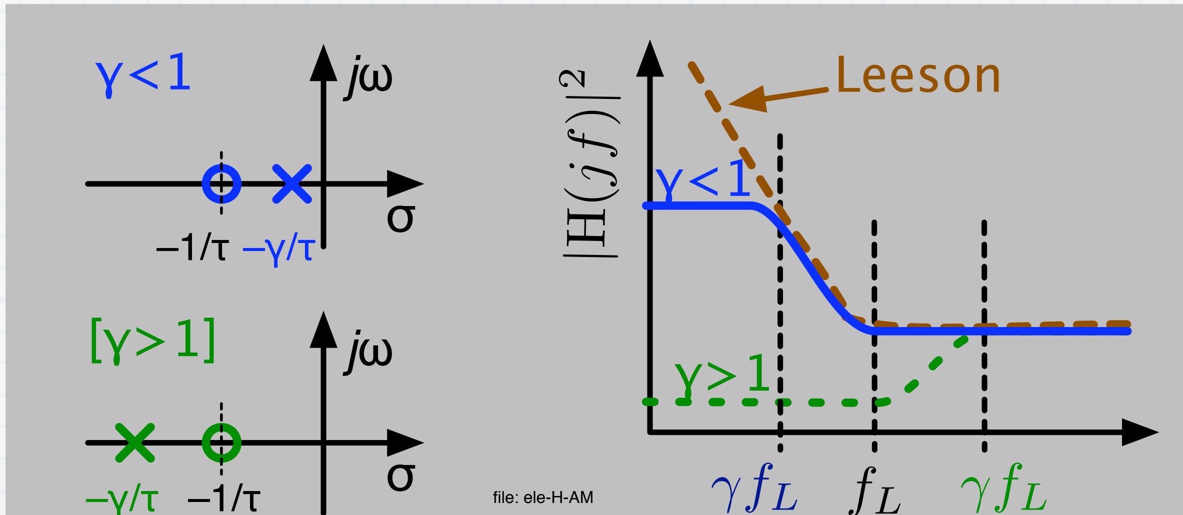
$$v=[-\gamma\alpha_u+1+\eta]\,[1+\alpha_u]$$

$$\cancel{1}+\alpha_v=\cancel{1}+\eta-\gamma\alpha_u+\alpha_u-\cancel{\alpha_u\eta}-\cancel{\gamma\alpha_u^2}$$

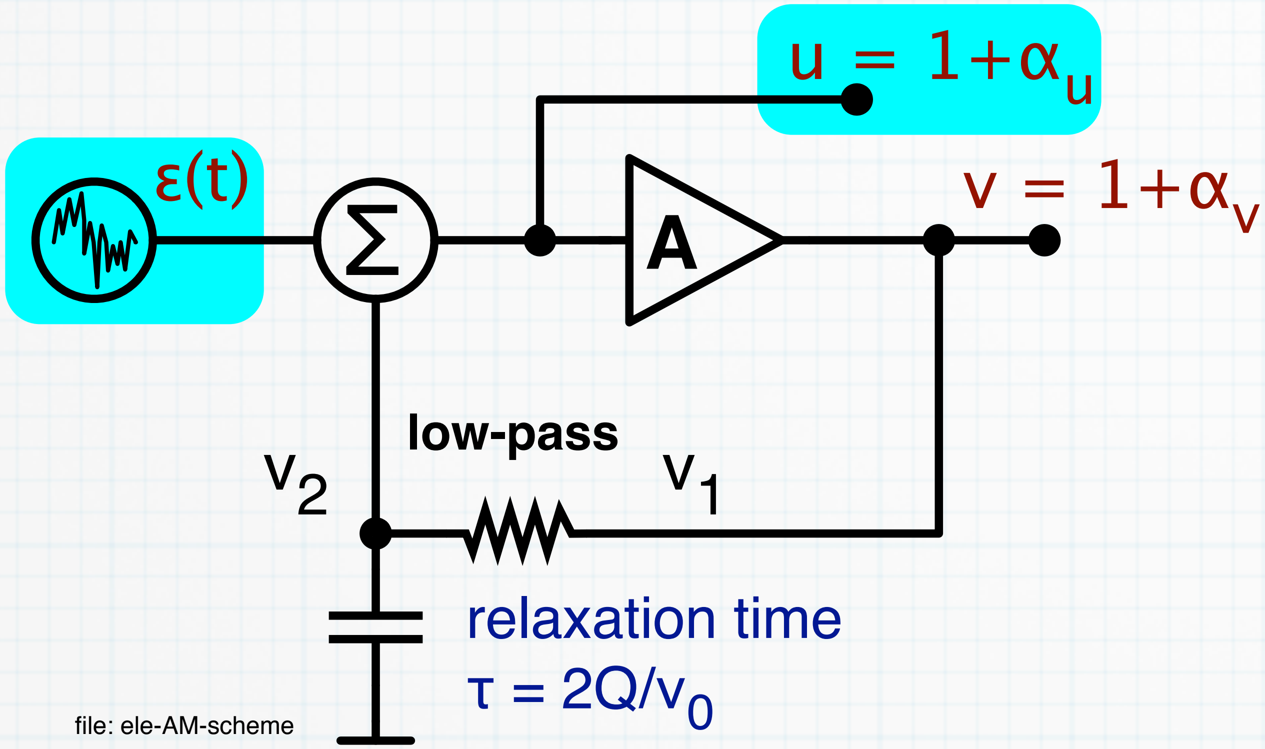
linearization for low noise

$$\alpha_v=(1-\gamma)\alpha_u+\eta$$

$$\alpha_u=\frac{\alpha_v-\eta}{1-\gamma}$$



Additive noise – output is u



$$\dot{u} = \frac{1}{\tau}(A-1)u + \dot{\epsilon} + \frac{1}{\tau}\epsilon$$ **non-linear equation**

$$A = 1 - \gamma(u-1)$$

$$\dot{u} + \frac{\gamma}{\tau}(u-1)u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$ **lineariz. for low noise**

($\dot{u} \to \dot{\alpha}_u$, $(u-1) \to \alpha_u$, $u \to 1$)

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$ **linearized equation**

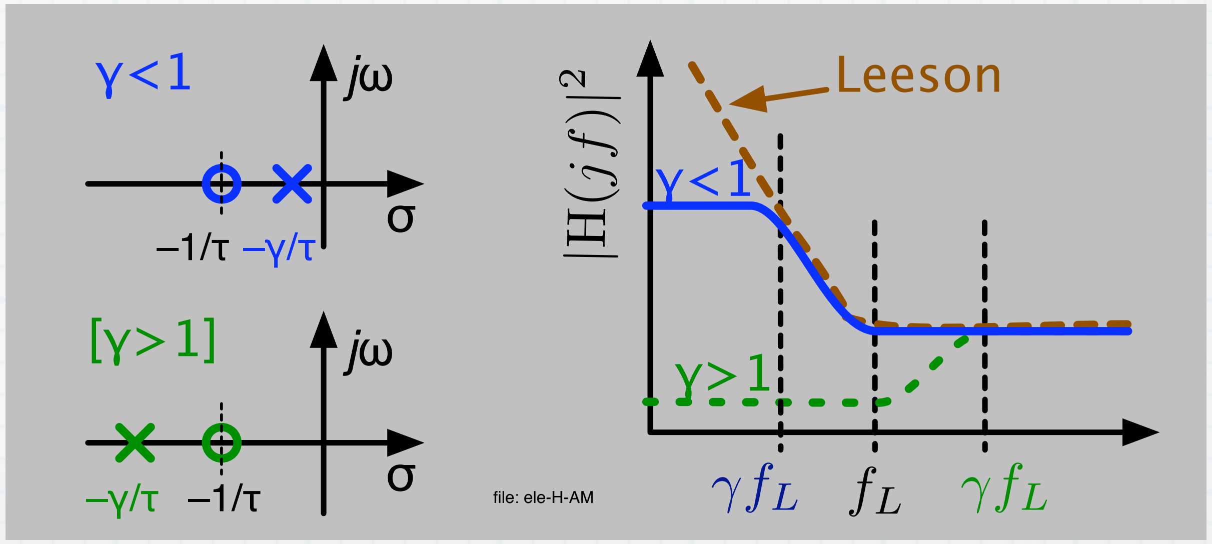
$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \left(s + \frac{1}{\tau}\right)\mathcal{E}(s)$$ **Laplace transform**

Linearize for low noise and use the Laplace transforms

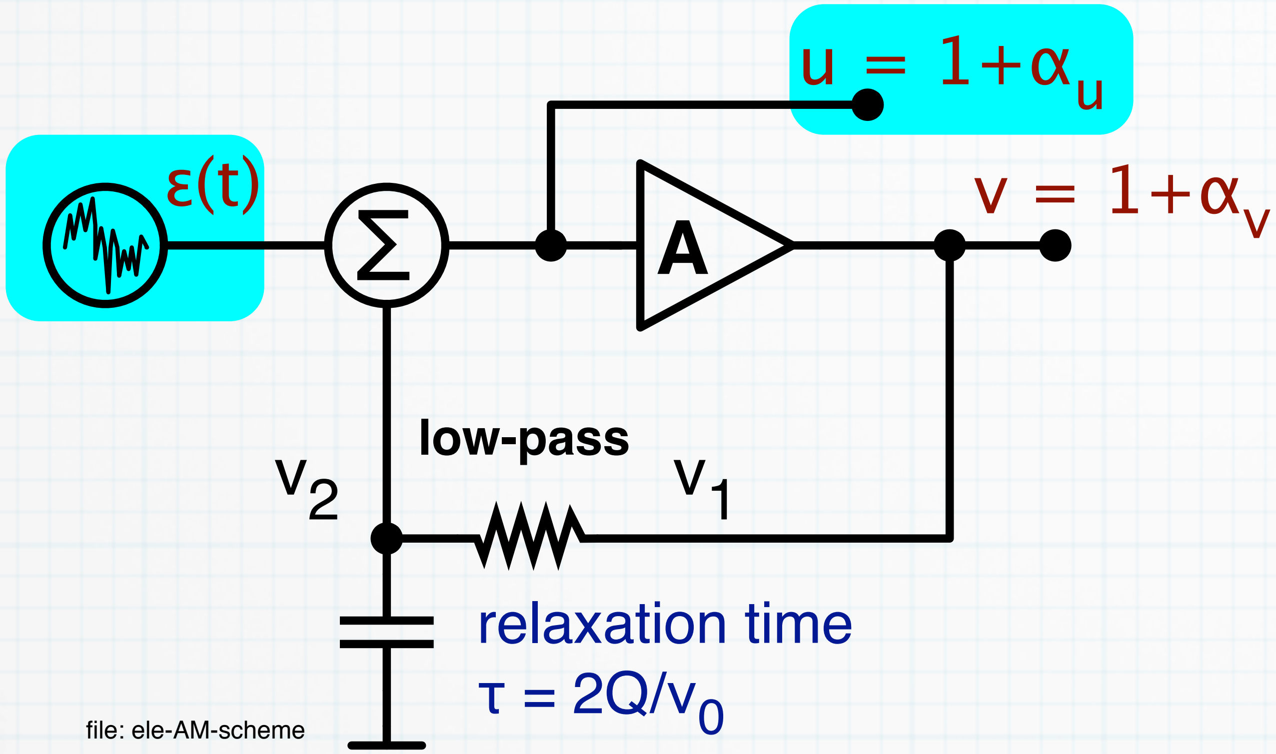
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \epsilon(t) \leftrightarrow \mathcal{E}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{E}(s)}$$ **definition**

$$\mathrm{H}_u(s) = \frac{s + 1/\tau}{s + \gamma/\tau}$$ **result**



Additive noise – output is v



$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$ **linearized equation**

$$\alpha_u = \alpha_v/(1-\gamma)$$

$$\frac{1}{1-\gamma}\left(\dot{\alpha}_v + \frac{\gamma}{\tau}\alpha_v\right) = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

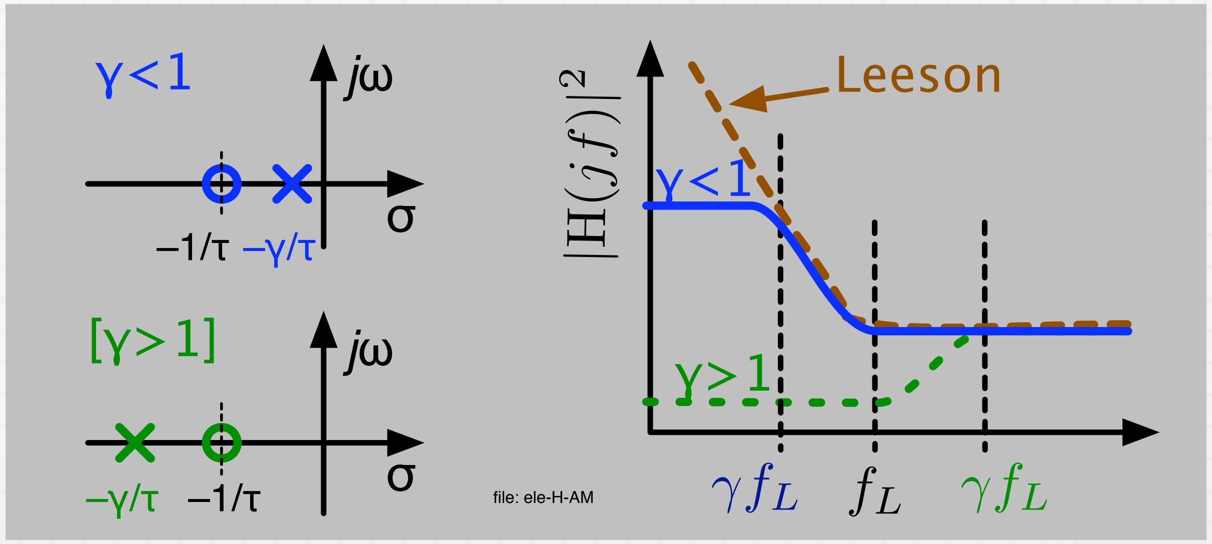
$$\frac{1}{1-\gamma}\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_v(s) = \left(s + \frac{1}{\tau}\right)\mathcal{E}(s)$$ **Laplace transform**

$$\mathrm{H}(s) = \frac{\mathcal{A}_v(s)}{\mathcal{E}(s)}$$ **definition**

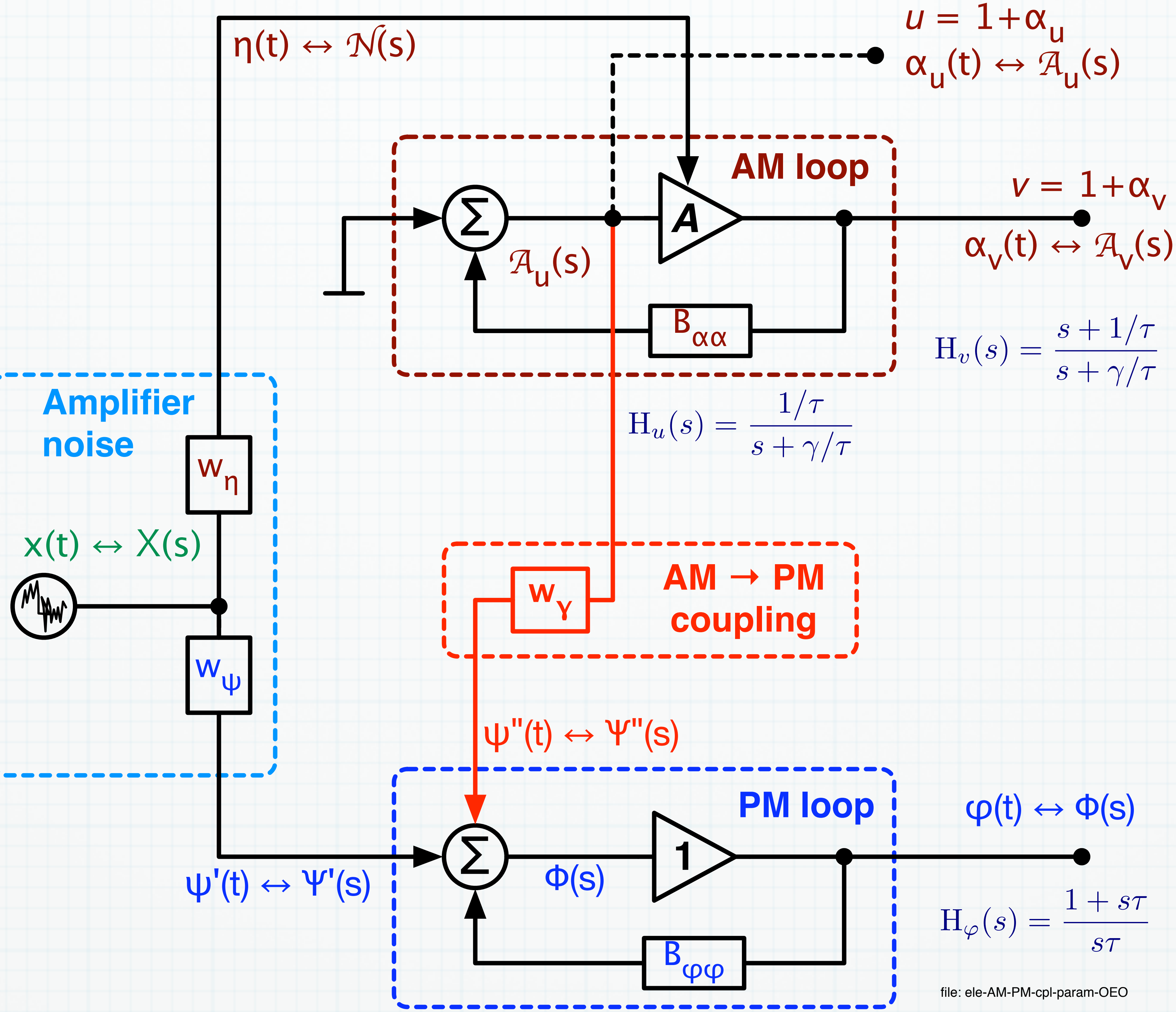
$$\mathrm{H}(s) = (1-\gamma)\,\frac{s+1/\tau}{s+\gamma/\tau}$$ **result**

boring algebra relates α' to α

$$v = Au$$
$$A = 1 - \gamma(u-1)$$
$$v = [1-\gamma(u-1)]\,u$$
$$1+\alpha_v = [1-\gamma\alpha_u]\,[1+\alpha_u]$$
$$\not{1} + \alpha_v = \not{1} + \alpha_u - \gamma\alpha_u - \not{\gamma\alpha_u^2}$$
linearization for low noise
$$\alpha_v = (1-\gamma)\alpha_u$$
$$\alpha_u = \frac{\alpha_v}{1-\gamma}$$

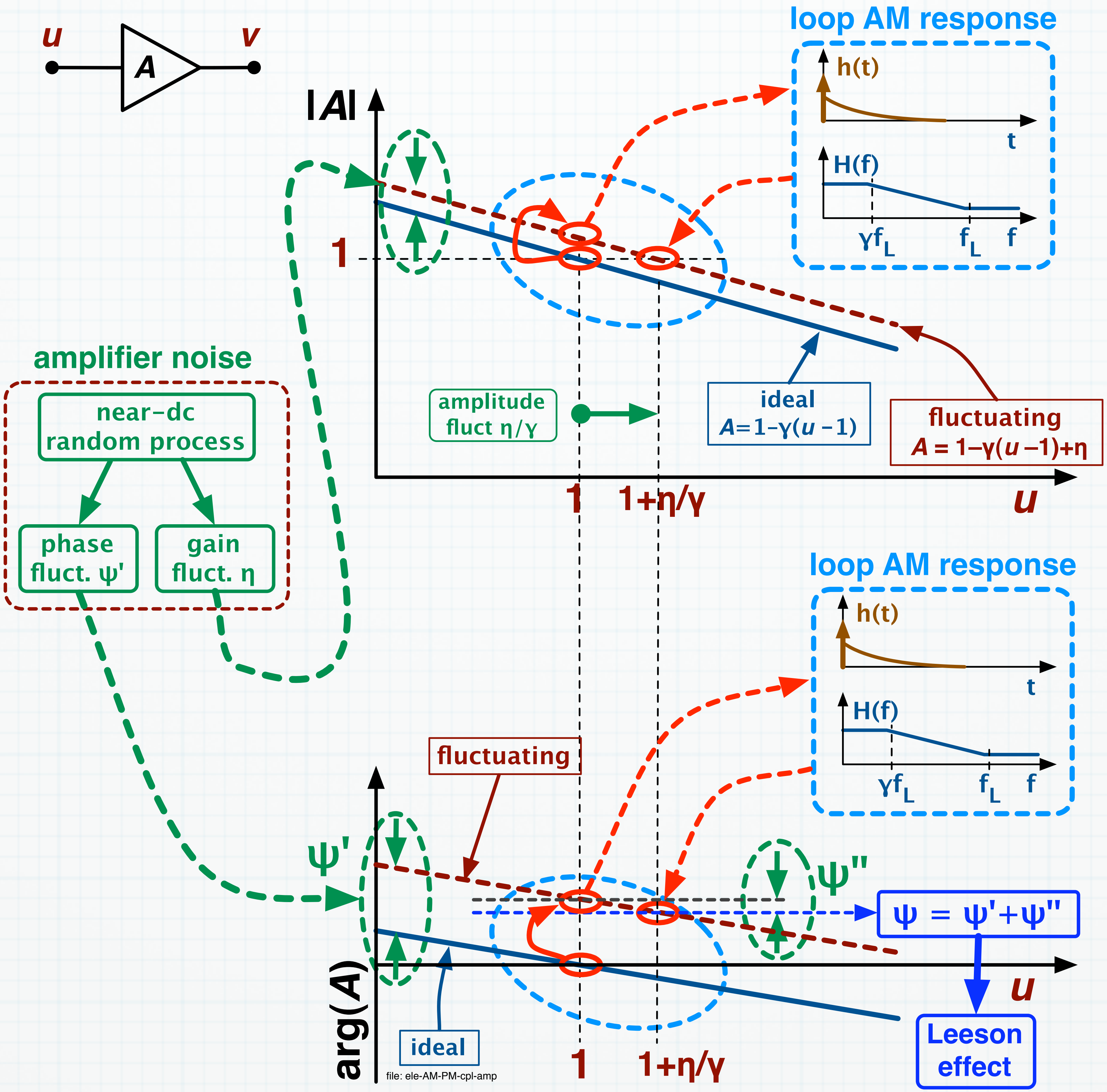


Parametric noise & AM-PM noise coupling



E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

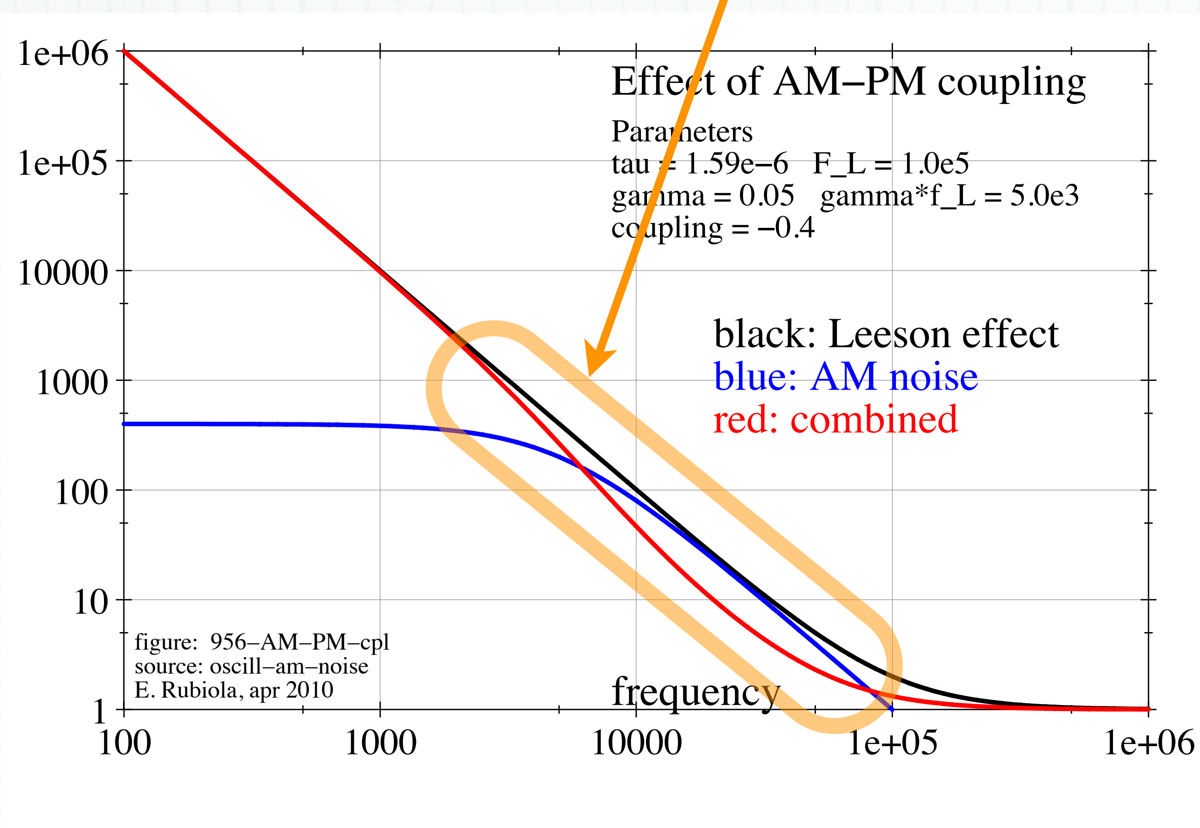
Effect of AM-PM noise coupling

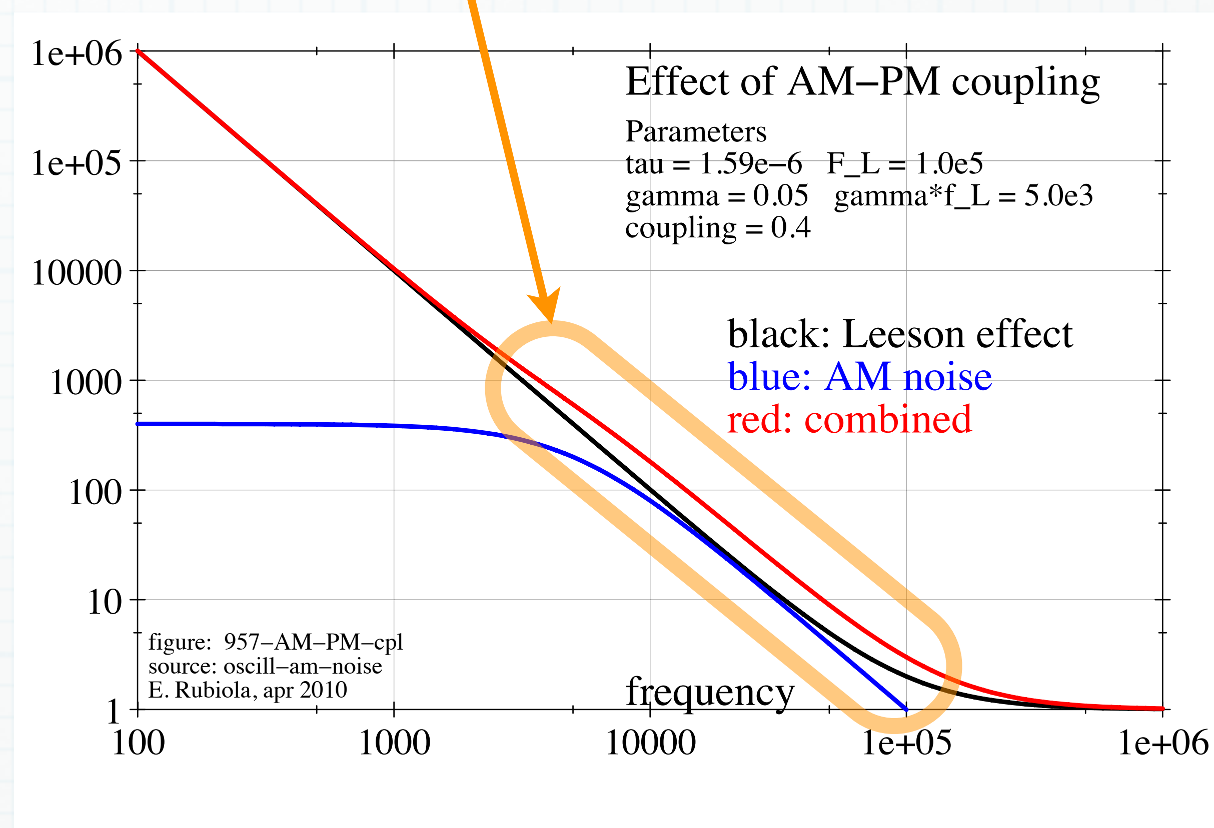


E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Noise transfer function, and spectra

AM-PM coupling shows up here

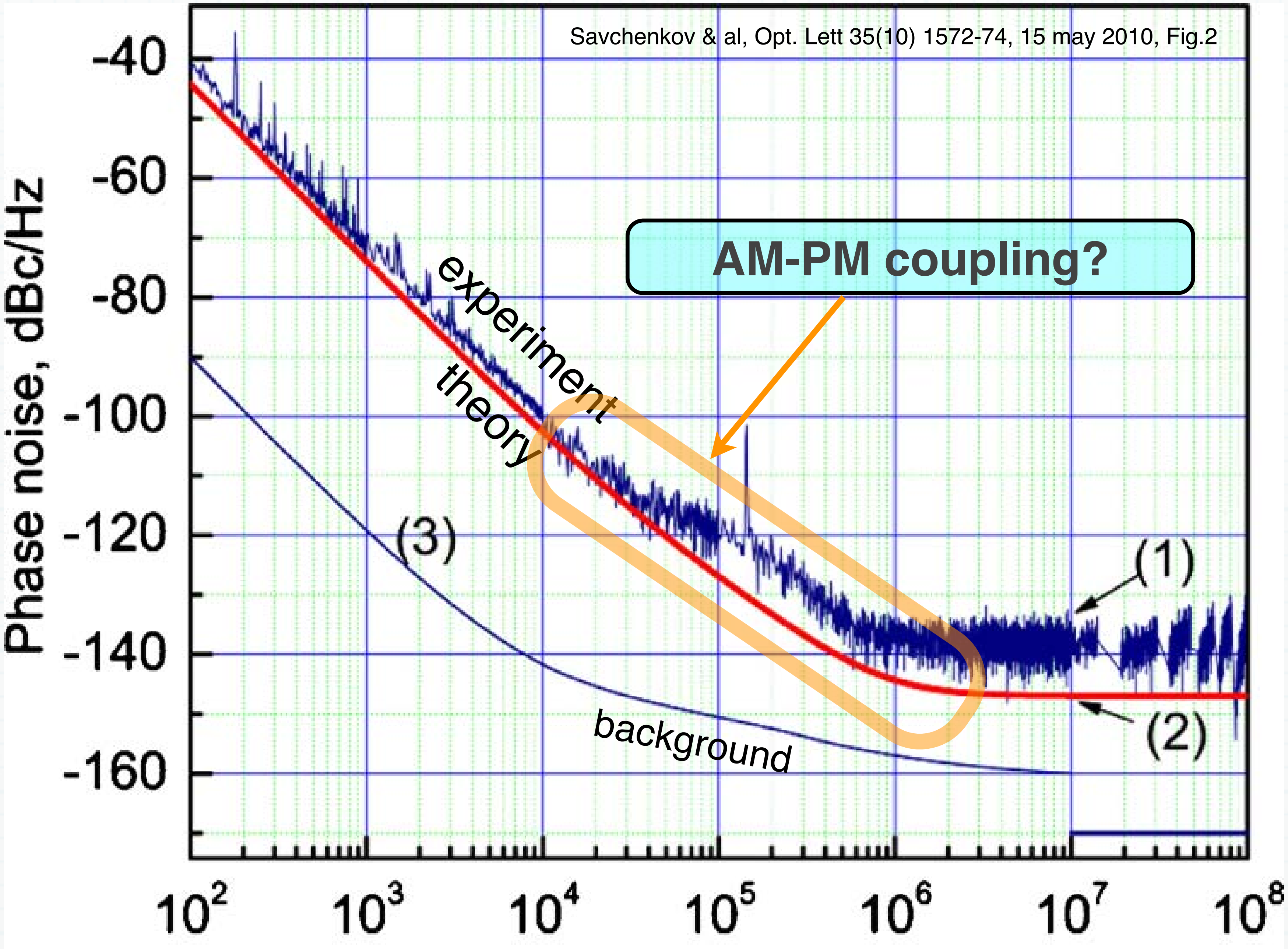




Notice that the AM-PM coupling can increase or decrease the PM noise

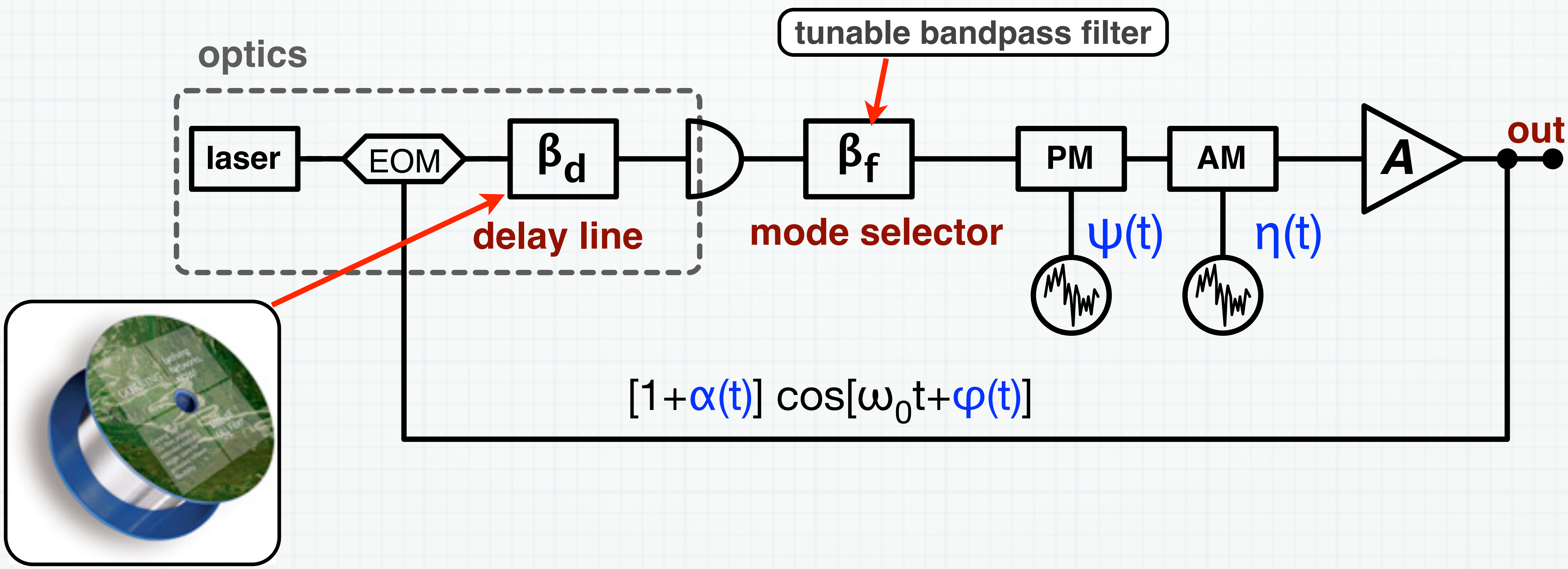
In a real oscillator, flicker noise shows up below some 10 kHz
In the flicker region, all plots are multiplied by 1/f

Noise spectra

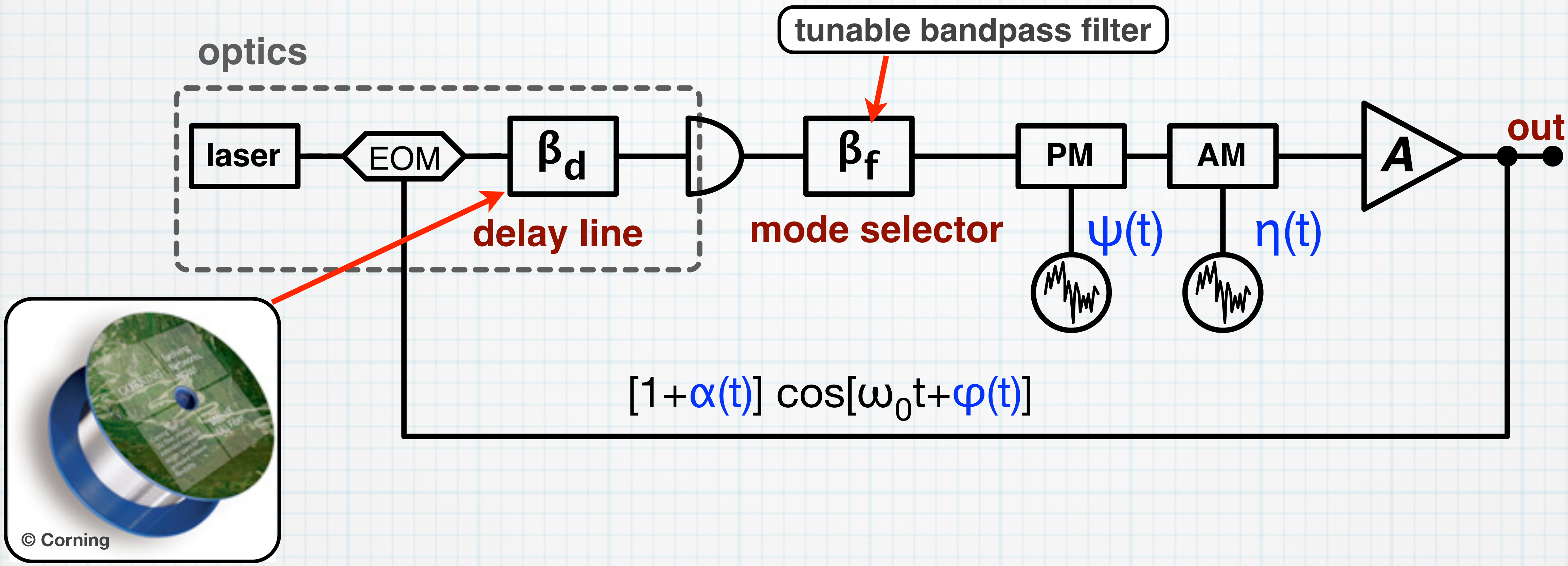


A. Savchenkov & al, Opt. Lett 35(10) 1572-74, 15 may 2010, Fig.2

Leeson effect in the delay-line oscillator



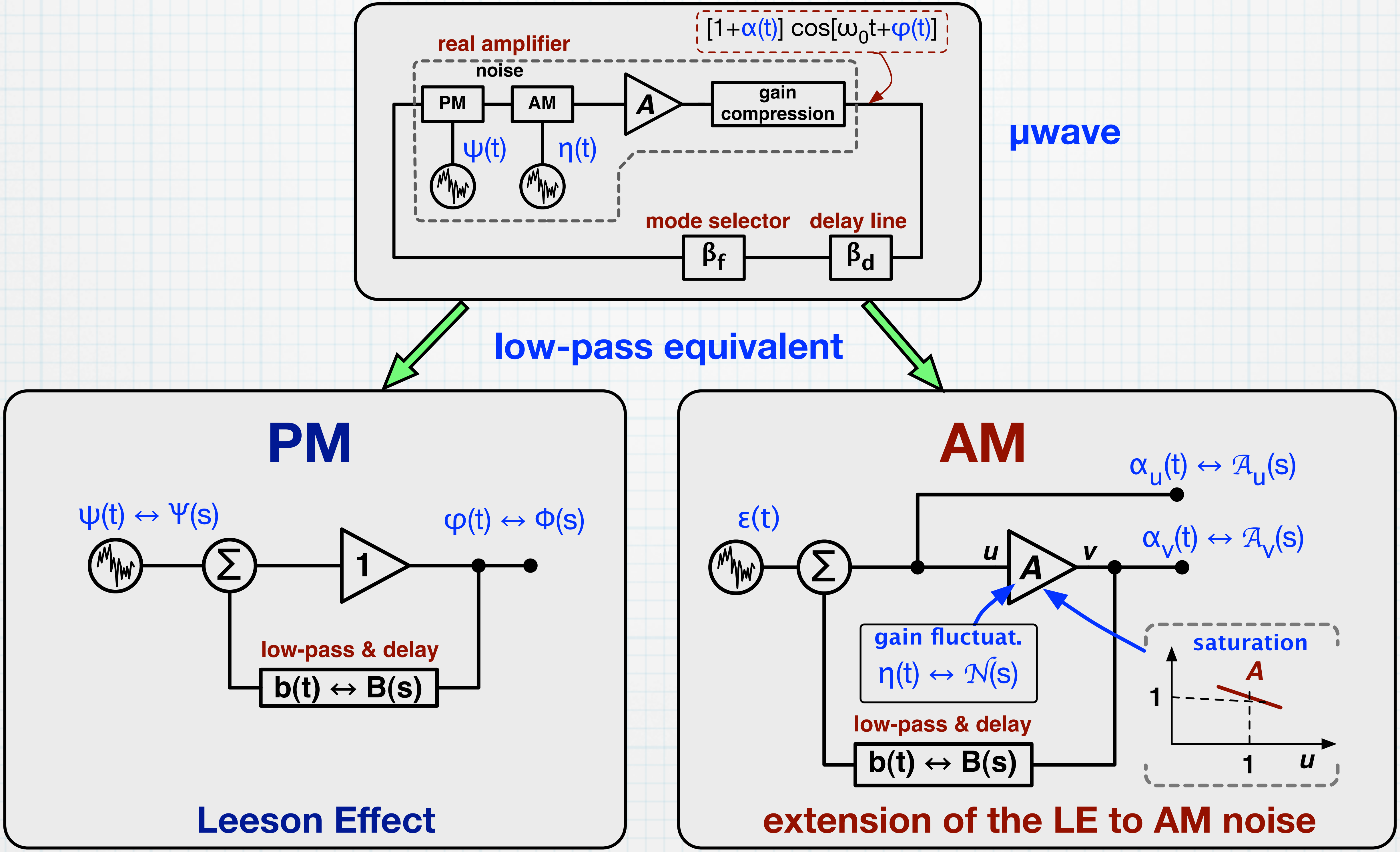
Motivations



- **Potential for very-low phase noise in the 100 Hz – 1 MHz range**
- **Invented at JPL, X. S. Yao & L. Maleki, JOSAB 13(8) 1725–1735, Aug 1996**
- **Early attempt of noise modeling, S. Römisch & al., IEEE T UFFC 47(5) 1159–1165, Sep 2000**
- **PM-noise analysis, E. Rubiola, *Phase noise and frequency stability in oscillators*, Cambridge 2008 [Chapter 5]**

- **Since, no progress in the analysis of noise at system level**
- **Nobody reported on the consequences of AM noise**

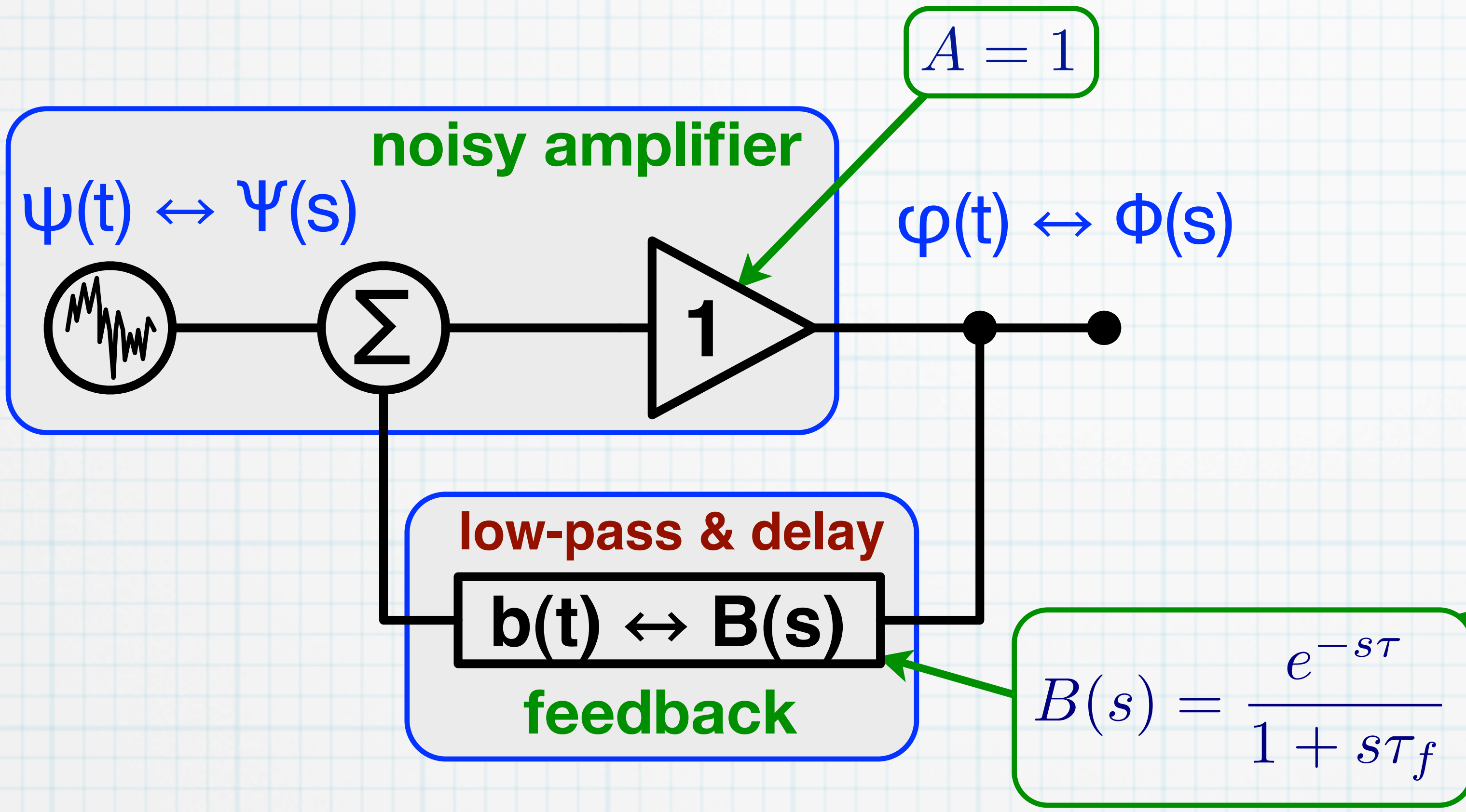
Low-pass representation of AM-PM noise



The amplifier
- "copies" the input phase to the out
- adds phase noise

The amplifier
- compresses the amplitude
- adds amplitude noise

Leeson effect



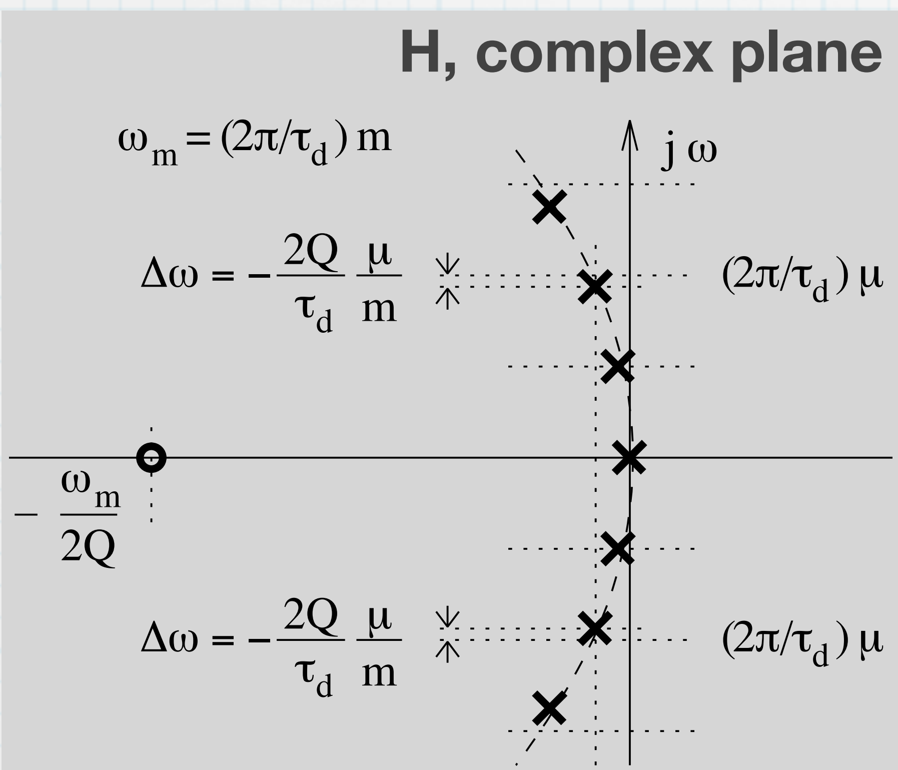
phase-noise transfer function

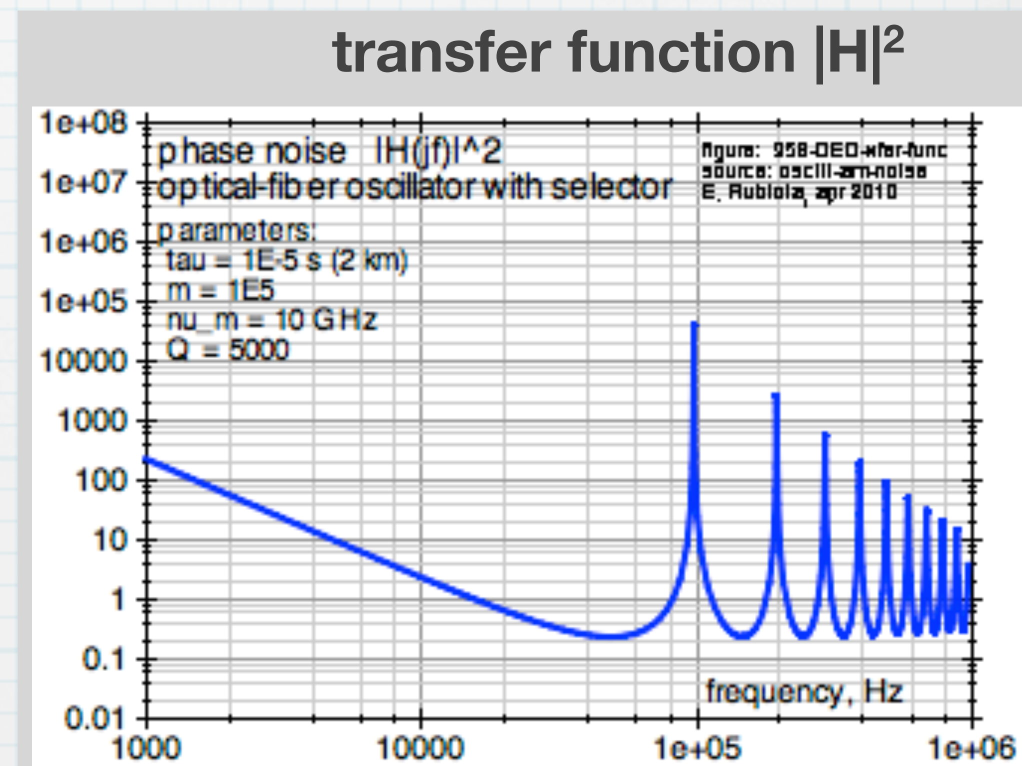
$$H(s) = \frac{\Phi(s)}{\Psi(s)}$$ **definition**

$$H(s) = \frac{1}{1+AB(s)}$$ **general feedback theory**

$$H(s) = \frac{1+s\tau_f}{1+s\tau_f - e^{-s\tau}}$$ **Leeson effect**

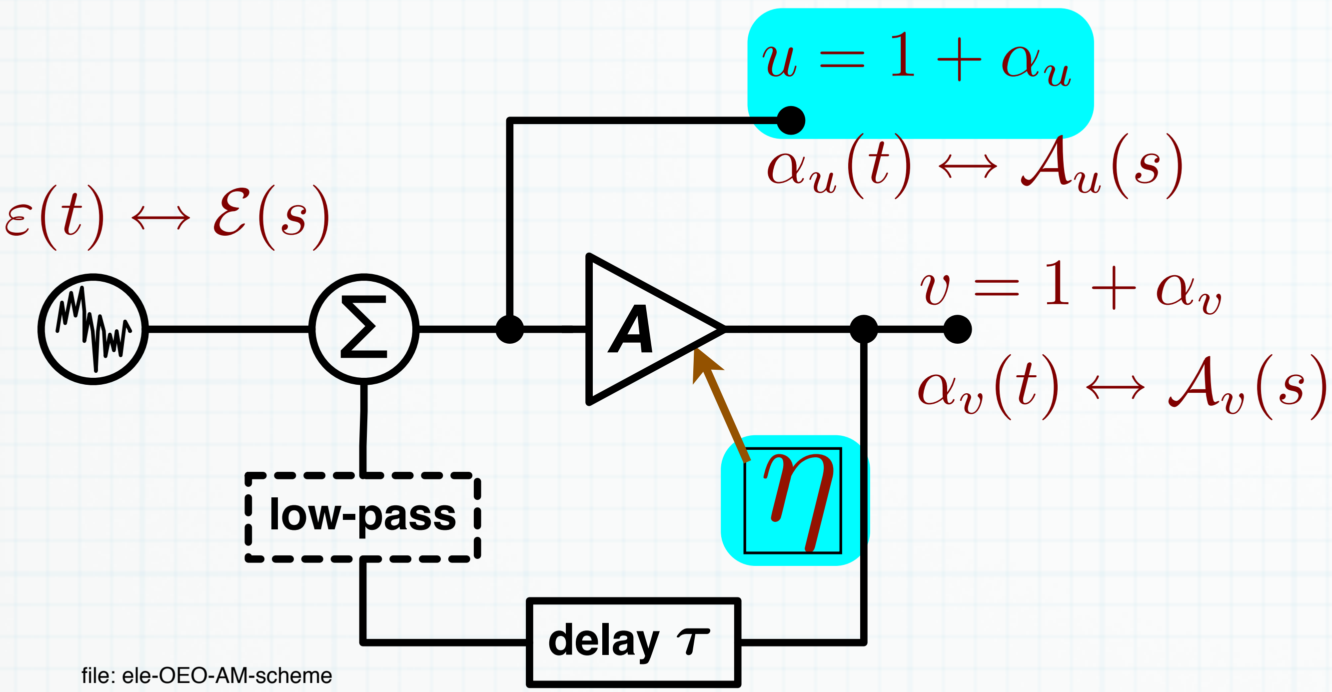
This figure is from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press





E. Rubiola, Phase noise and frequency stability in oscillators, Cambridge 2008

Gain fluctuations – output is *u*(*t*)



The low-pass has only 2nd order effect on AM

Linearize for low noise and use the Laplace transform

$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \eta(t) \leftrightarrow \mathcal{N}(s)$$

$$\mathrm{H}(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)}$$ definition

result

$$\mathrm{H}(s) = \frac{1}{1-(1-\gamma)e^{-s\tau}}$$

non-linear equation

$$u = A(t-\tau)\,u(t-\tau)$$

$$A = 1 - \gamma(u-1) + \eta$$

use u=α+1, expand and linearize for low noise

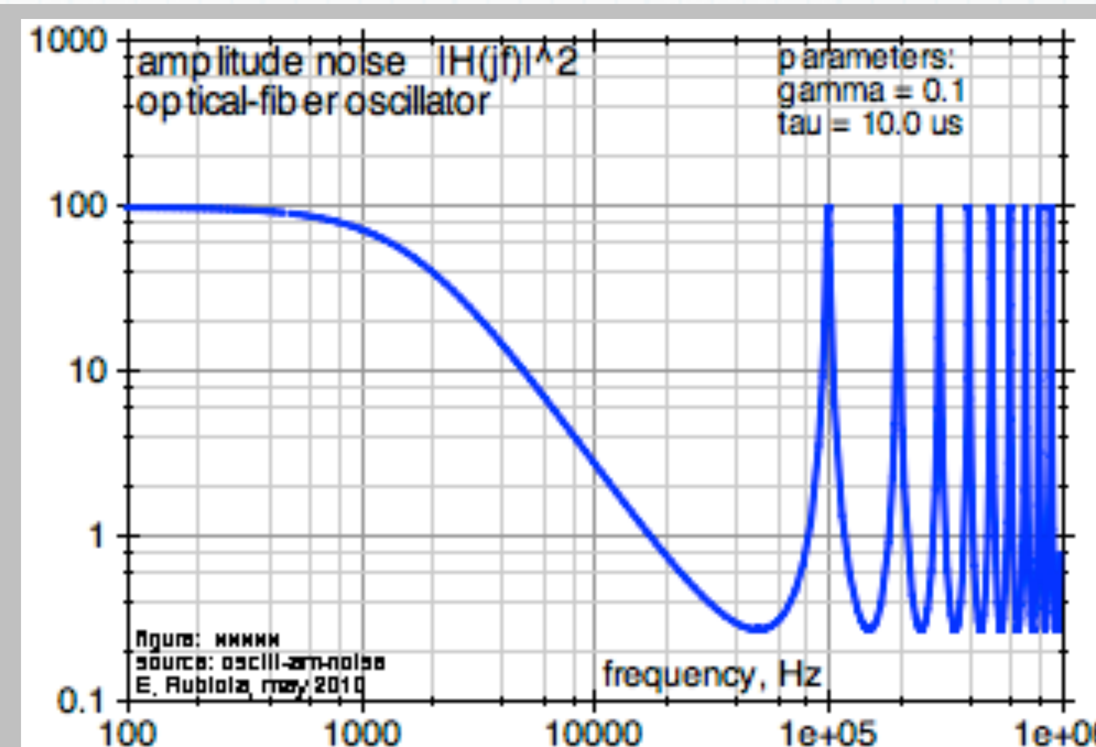
$$\alpha(t) = (1-\gamma)\alpha(t-\tau) - \gamma\alpha^2(t-\tau) \to 0$$
$$+\,\eta(t-\tau) + \eta(t-\tau)\alpha(t-\tau) \to 0$$

linearized equation

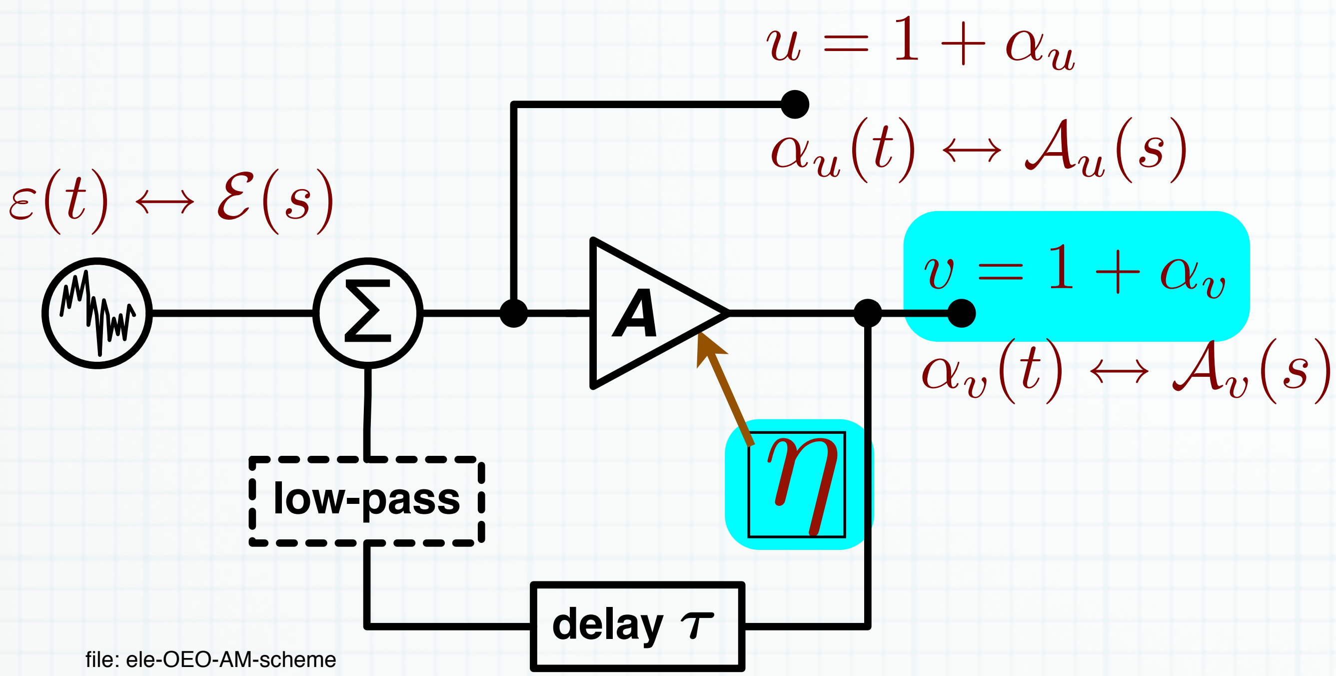
$$\alpha(t) = (1-\gamma)\alpha(t-\tau) + \eta(t-\tau)$$

Laplace transform

$$\mathcal{A}_u(s) = \left[1-(1-\gamma)e^{-s\tau}\right] = \mathcal{N}(s)$$



Gain fluctuations – output is *v*(*t*)



The low-pass has only 2nd order effect on AM

boring algebra relates α_v to α_u

$$v = Au$$

$$A = -\gamma(u-1) + 1 + \eta$$

$$v = [-\gamma(u-1) + 1 + \eta]\, u \qquad \text{use } u = \alpha + 1$$

$$v = [-\gamma\alpha_u + 1 + \eta]\,[1 + \alpha_u]$$

$$\cancel{1} + \alpha_v = \cancel{1} + \eta - \gamma\alpha_u + \alpha_u - \cancel{\alpha_u\eta} - \cancel{\gamma\alpha_u^2}$$

linearization for low noise

$$\alpha_v = (1-\gamma)\alpha_u + \eta$$

$$\alpha_u = \frac{\alpha_v - \eta}{1-\gamma}$$

$$\mathcal{A}_u(s)\left[1 - (1-\gamma)e^{-\imath\omega\tau}\right] = \mathcal{N}(s)$$

starting equation

$$\mathcal{A}_u(s) = \frac{\mathcal{A}_v(s) - \mathcal{N}(s)}{1-\gamma}$$

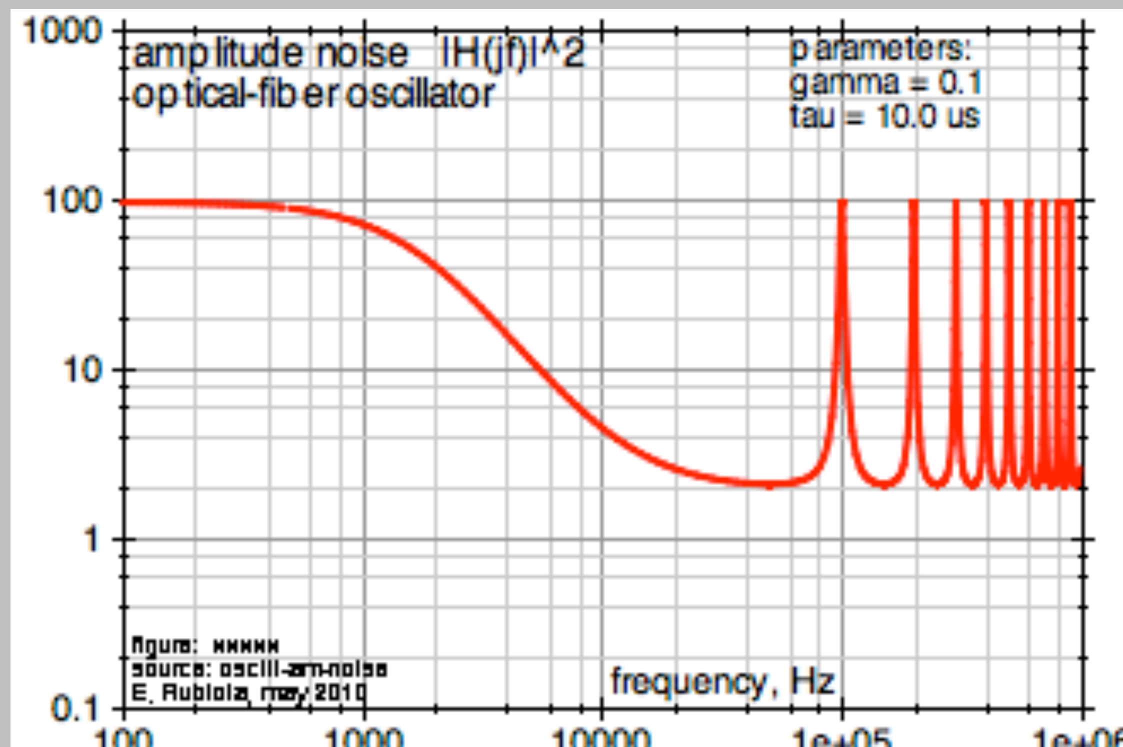
$$\left[1 + (1-\gamma)\left(1 - e^{-s\tau}\right)\right]\mathcal{A}_v(s) = \left[1 - (1-\gamma)e^{-s\tau}\right]\mathcal{N}(s)$$

$$\mathrm{H}(s) = \frac{\mathcal{A}_v(s)}{\mathcal{N}(s)}$$

definition

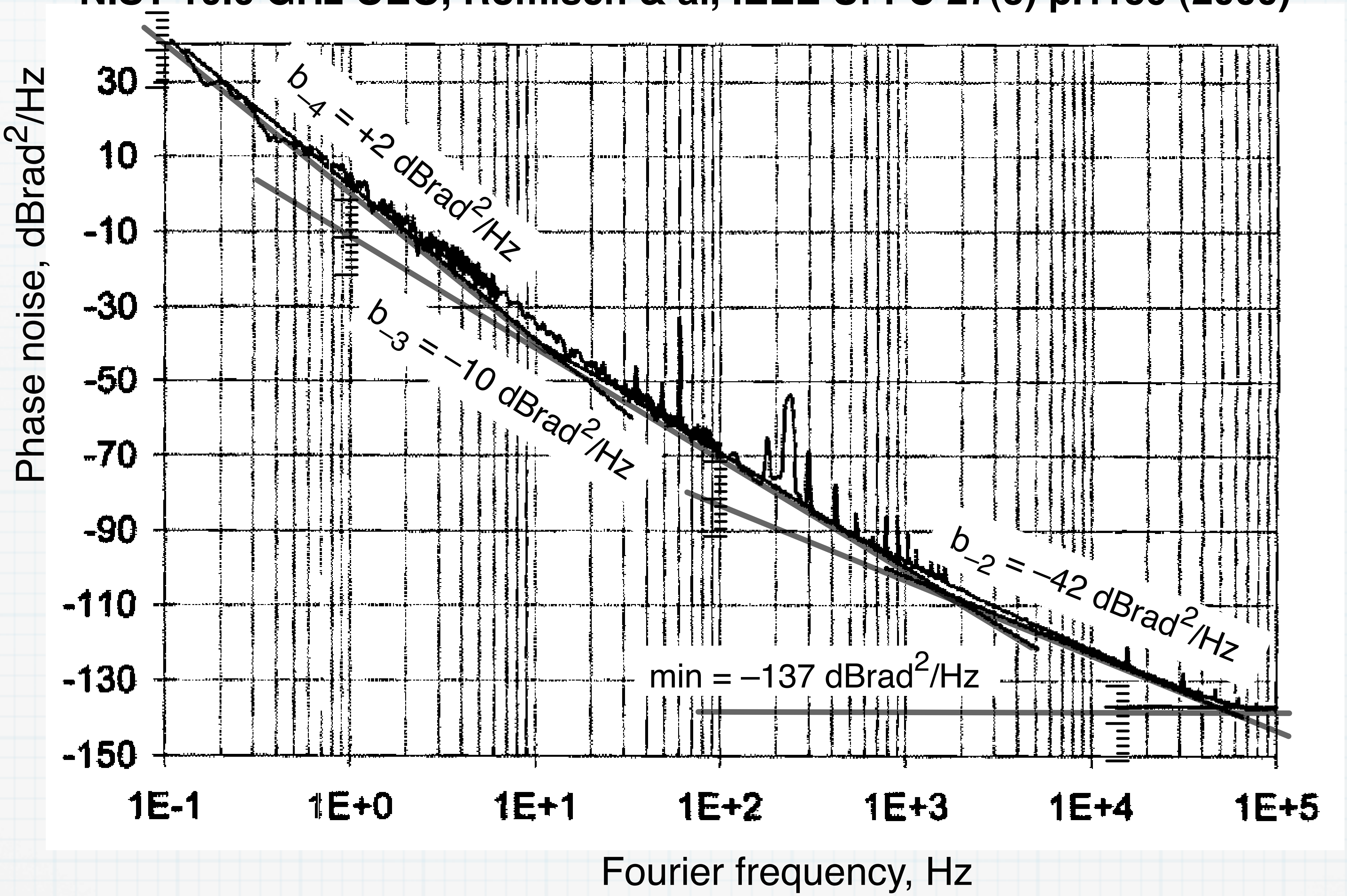
$$\mathrm{H}(s) = \frac{1 + (1-\gamma)\left(1 - e^{-s\tau}\right)}{1 - (1-\gamma)e^{-s\tau}}$$

result

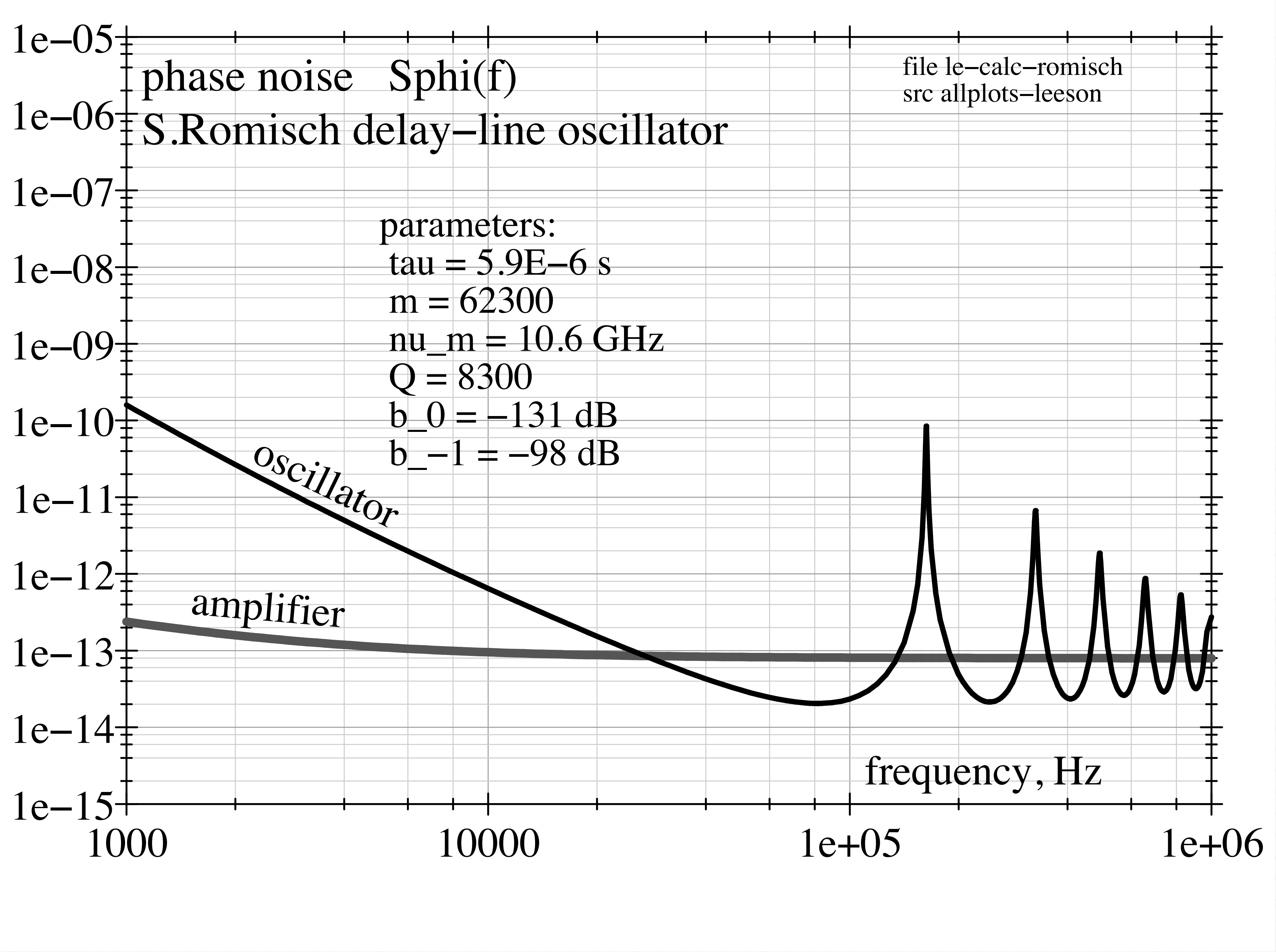


Opto-electronic oscillator

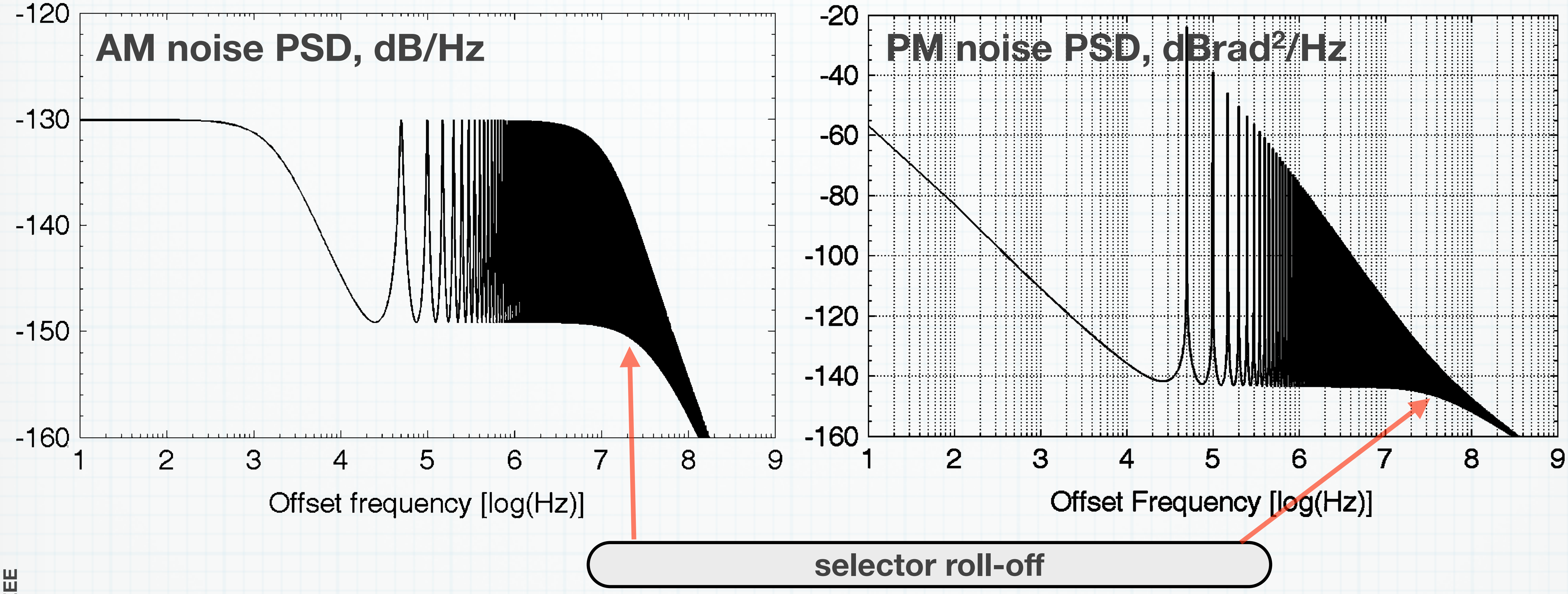
NIST 10.6 GHz OEO, Römisch & al, IEEE UFFC 27(5) p.1159 (2000)



Opto-electronic oscillator simulation



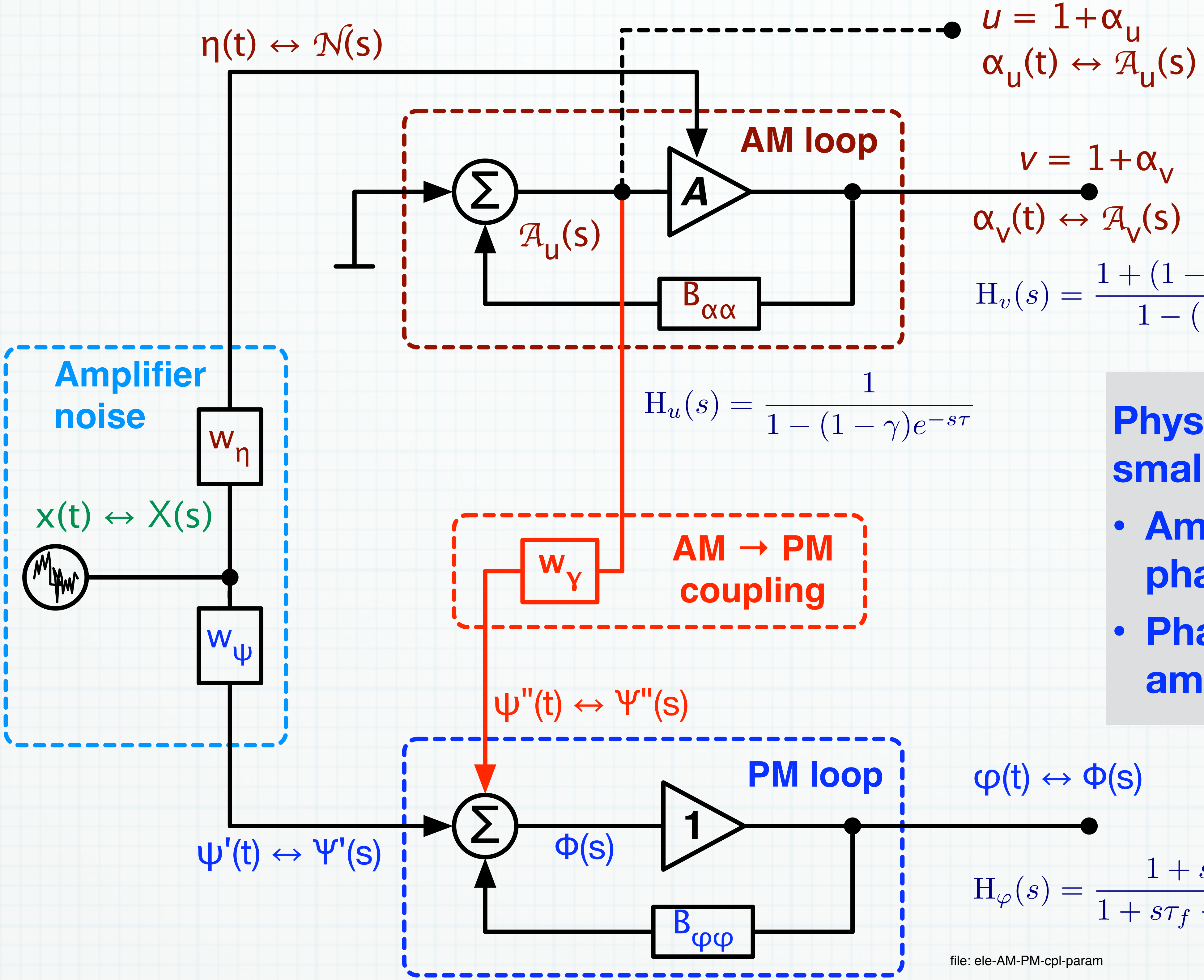
AM & PM spectra were anticipated



- Prediction is based on the stochastic diffusion (Langevin) theory
- However complex, the Langevin theory provides an independent check

Y.K. Chembo, K. Volyanskiy, L. Larger, E. Rubiola, P. Colet, & al., IEEE J. Quant. Electron. 45(2) p.178-186, Feb 2009

Parametric noise & AM-PM noise coupling



$$H_v(s) = \frac{1 + (1-\gamma)\,(1 - e^{-s\tau})}{1-(1-\gamma)e^{-s\tau}}$$

$$H_u(s) = \frac{1}{1-(1-\gamma)e^{-s\tau}}$$

$$H_\varphi(s) = \frac{1+s\tau_f}{1+s\tau_f - e^{-s\tau}}$$

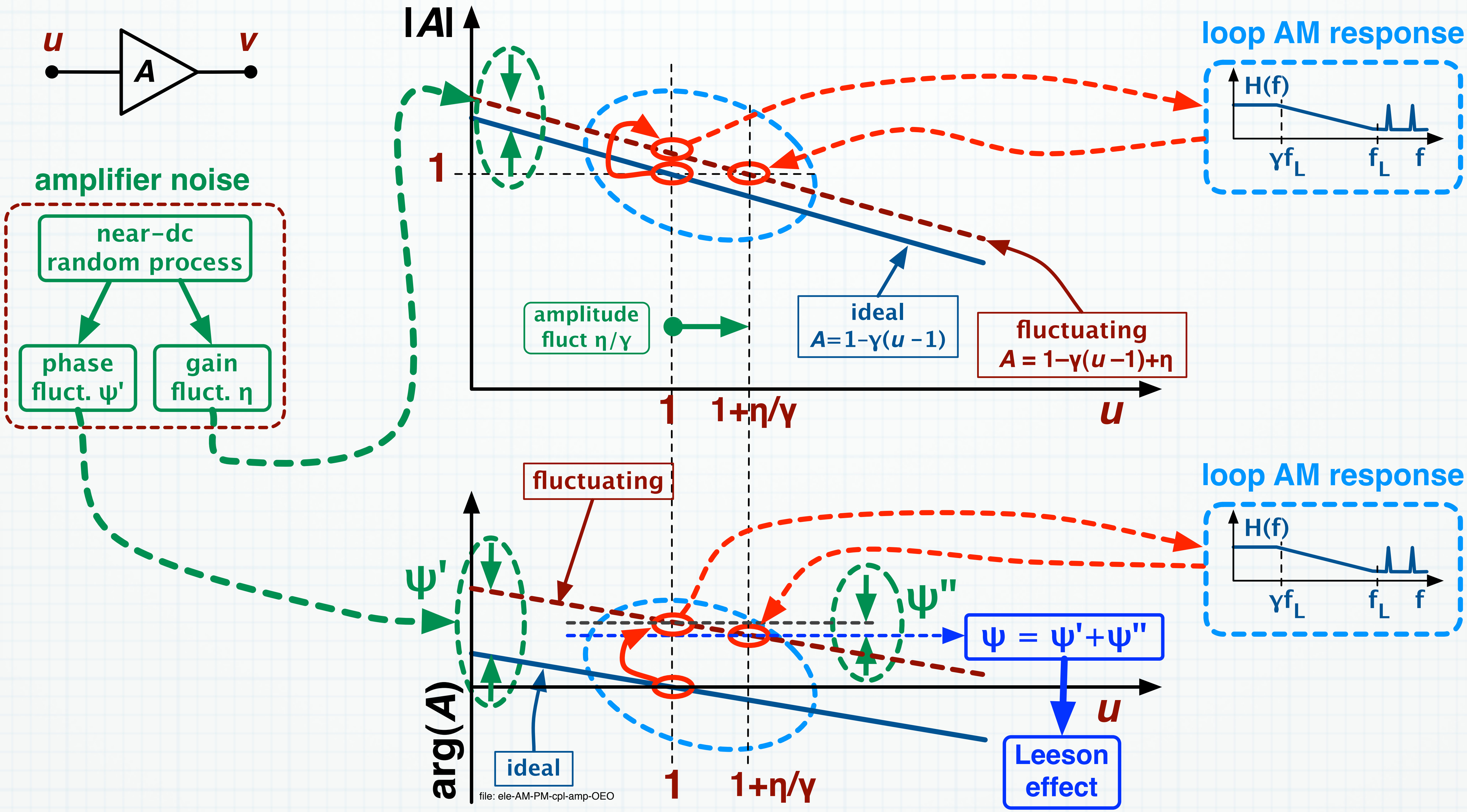
file: ele-AM-PM-cpl-param

Physics suggests that, for small fluctuations

- **Amplitude impacts on phase**
- **Phase has no effect on amplitude**

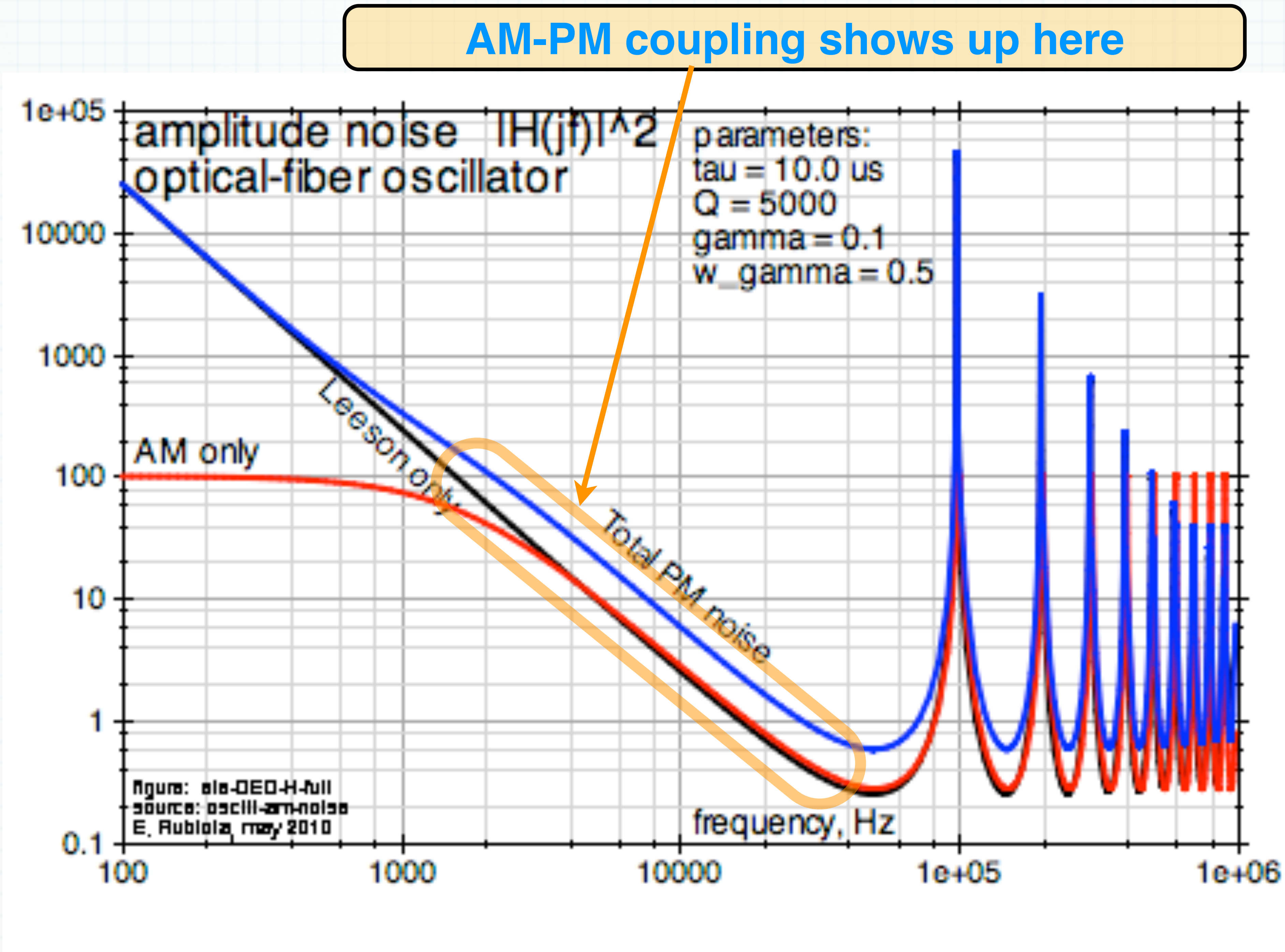
E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Effect of AM-PM noise coupling



E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

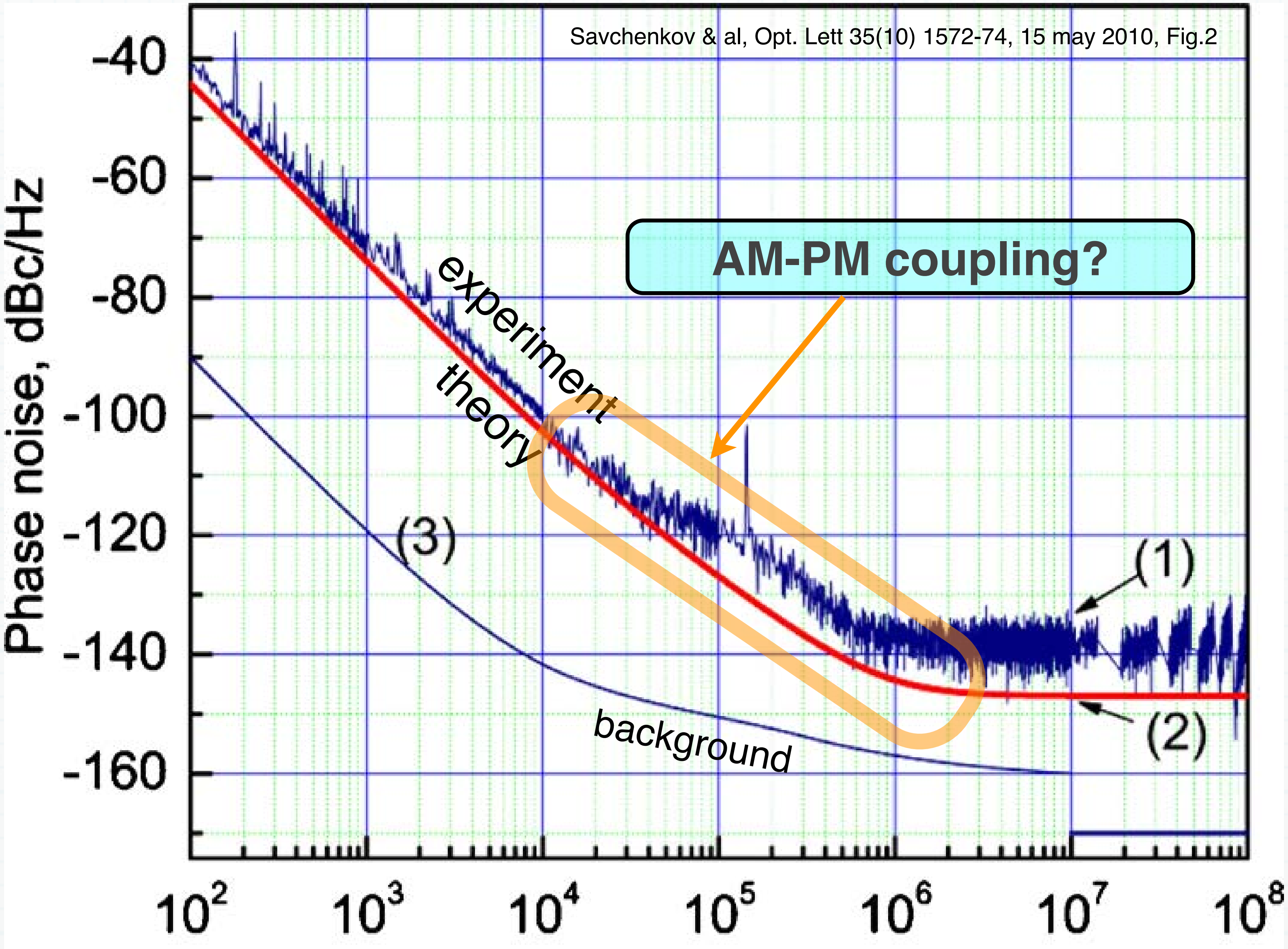
Noise transfer function and spectra



Notice that the AM-PM coupling can increase or decrease the PM noise

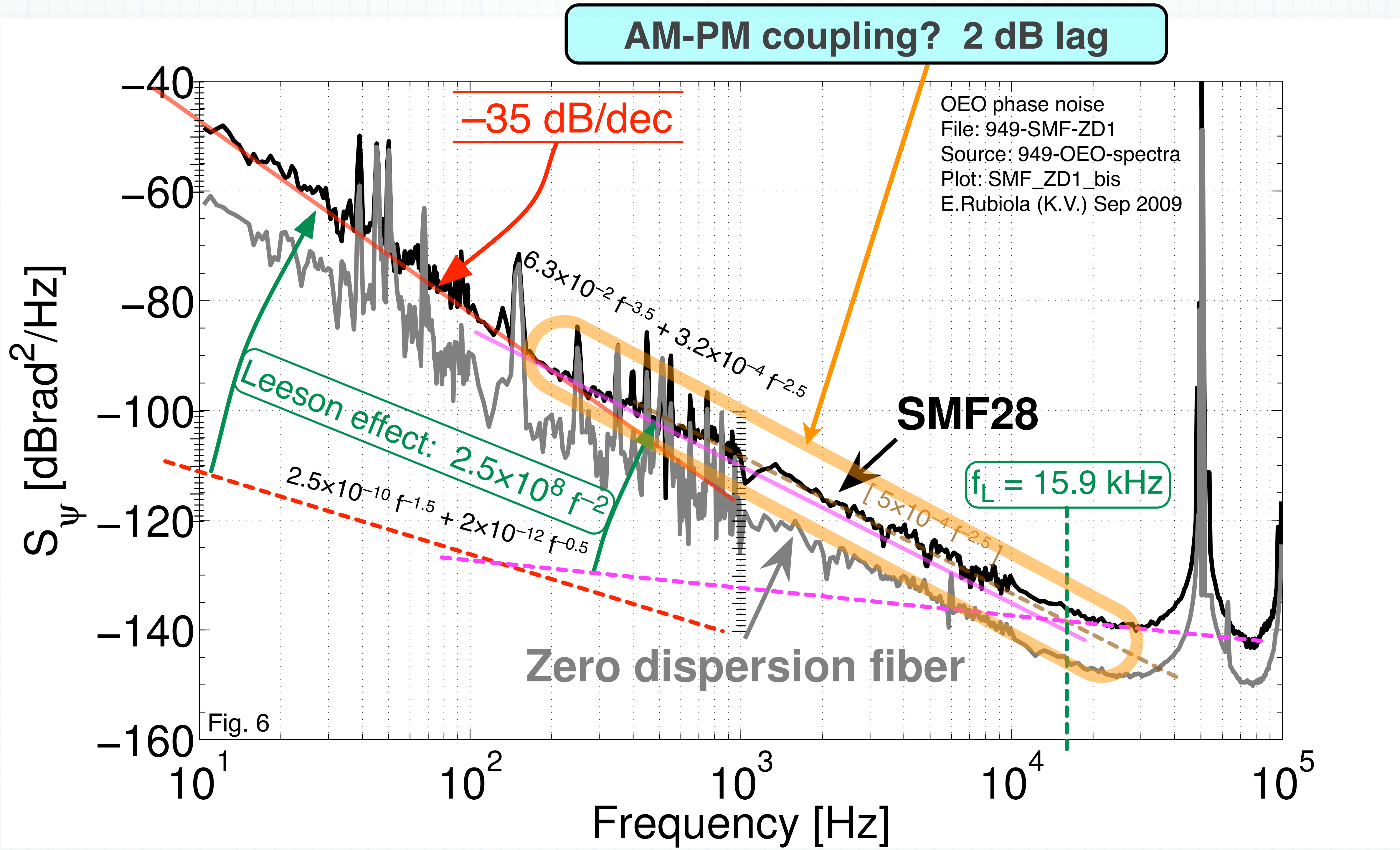
In a real oscillator, flicker noise shows up below some 10 kHz
In the flicker region, all plots are multiplied by 1/f

Noise spectra



A. Savchenkov & al, Opt. Lett 35(10) 1572-74, 15 may 2010, Fig.2

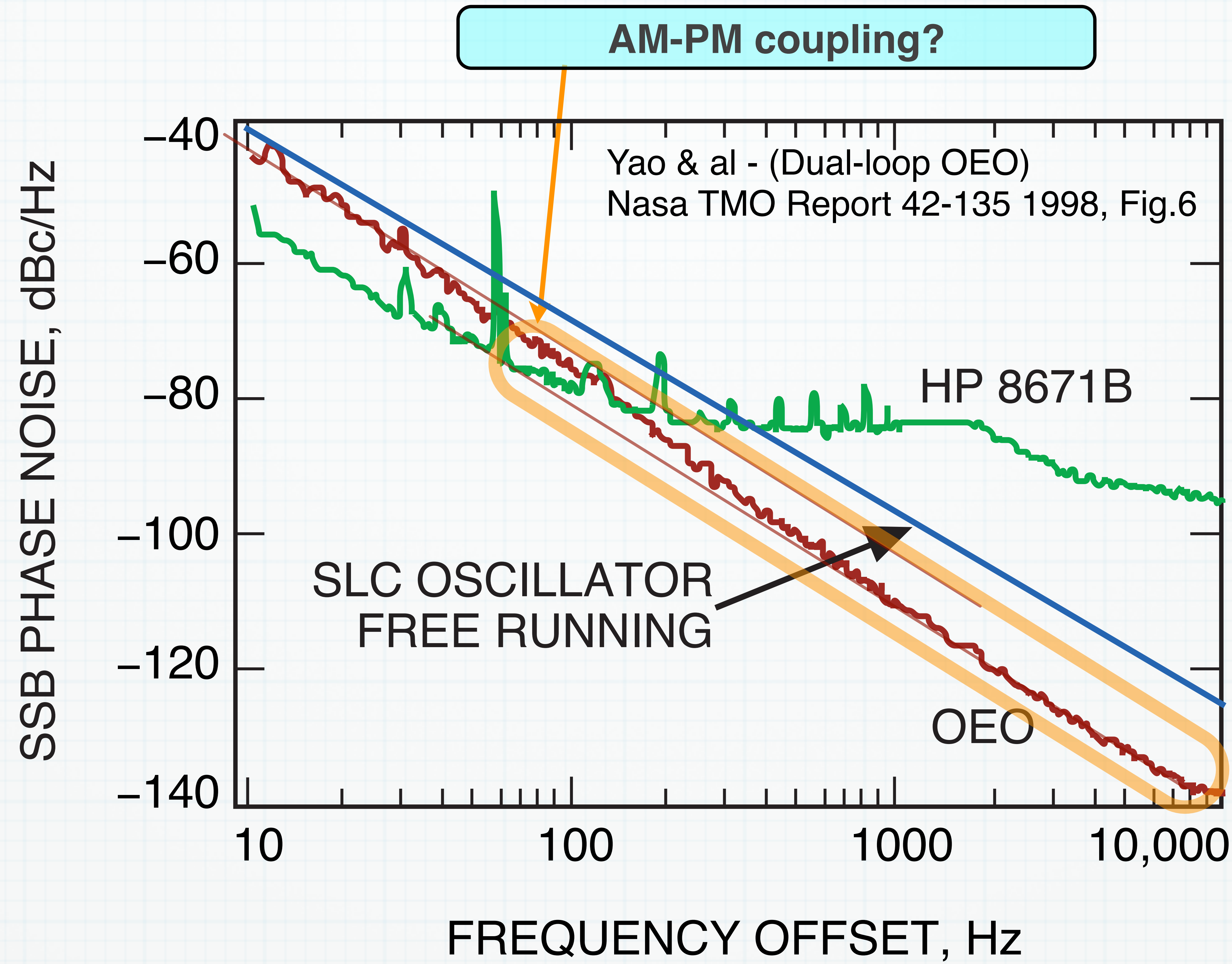
Noise spectra



Unfortunately, the awareness of this model come after the end of the experiments

Spectrum from K. Volyanskiy & al., IEEE JLT (Submitted, Apr. 2010)

Noise spectra



X.S.Yao & al., NASA TMO Report 42-135 (1998), Fig. 6

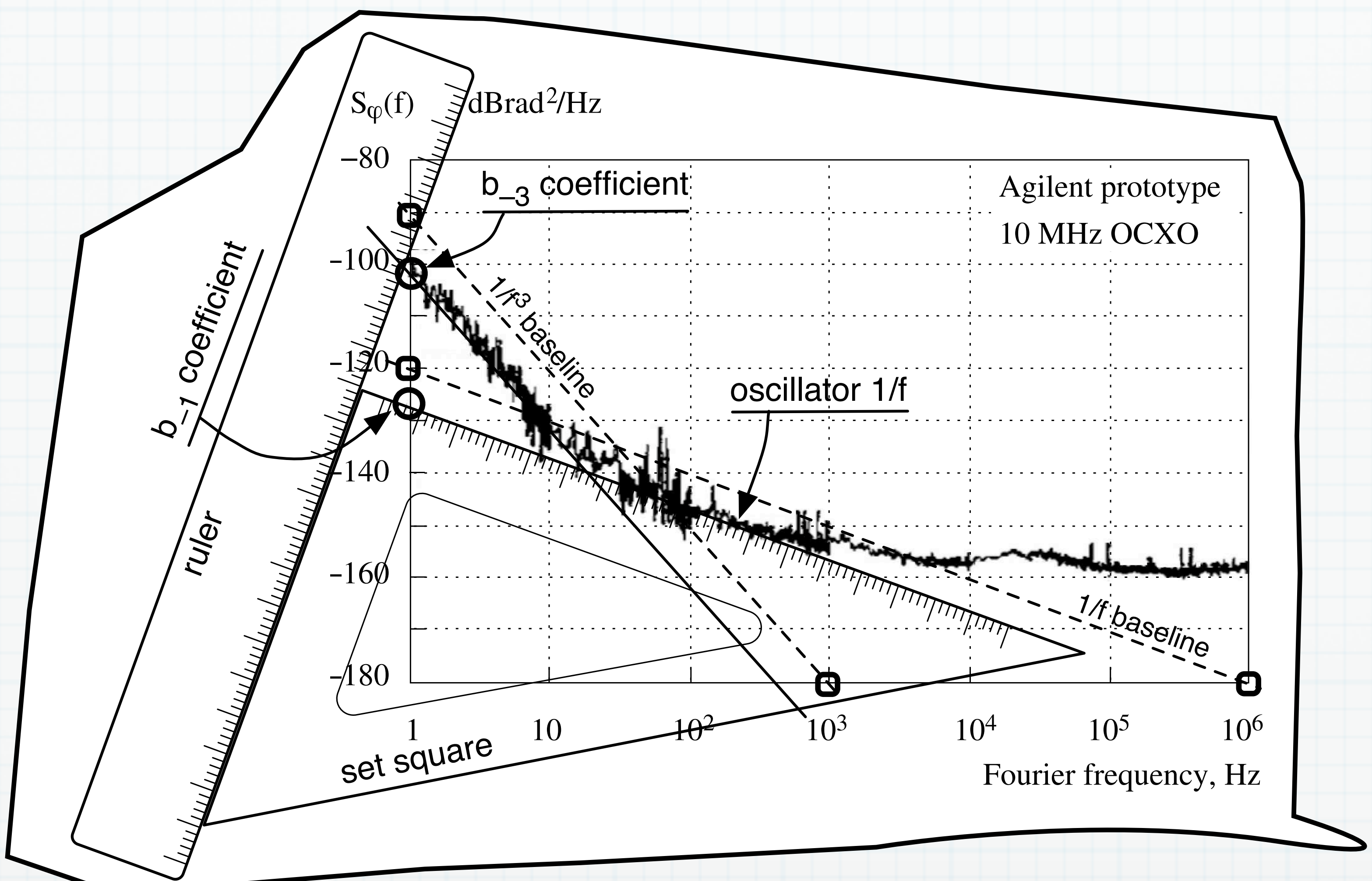
Oscillator Hacking

Analysis of commercial oscillators

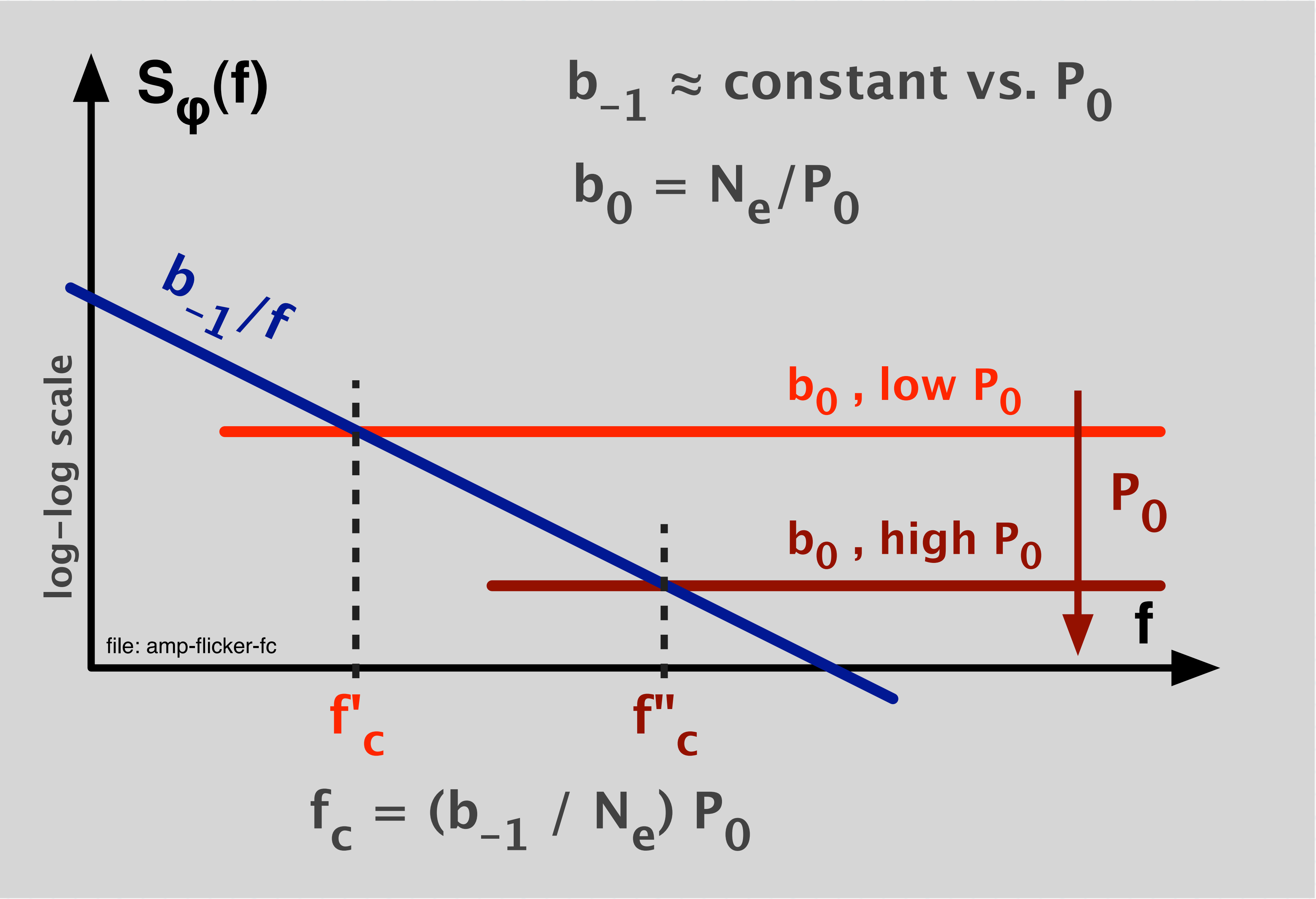
The purpose of this section is to help to understand the oscillator inside from the phase noise spectra, plus some technical information. I have chosen some commercial oscillators as an example.

The conclusions about each oscillator represent only my understanding based on experience and on the data sheets published on the manufacturer web site.

You should be aware that this process of interpretation is not free from errors. My conclusions were not submitted to manufacturers before writing, for their comments could not be included.

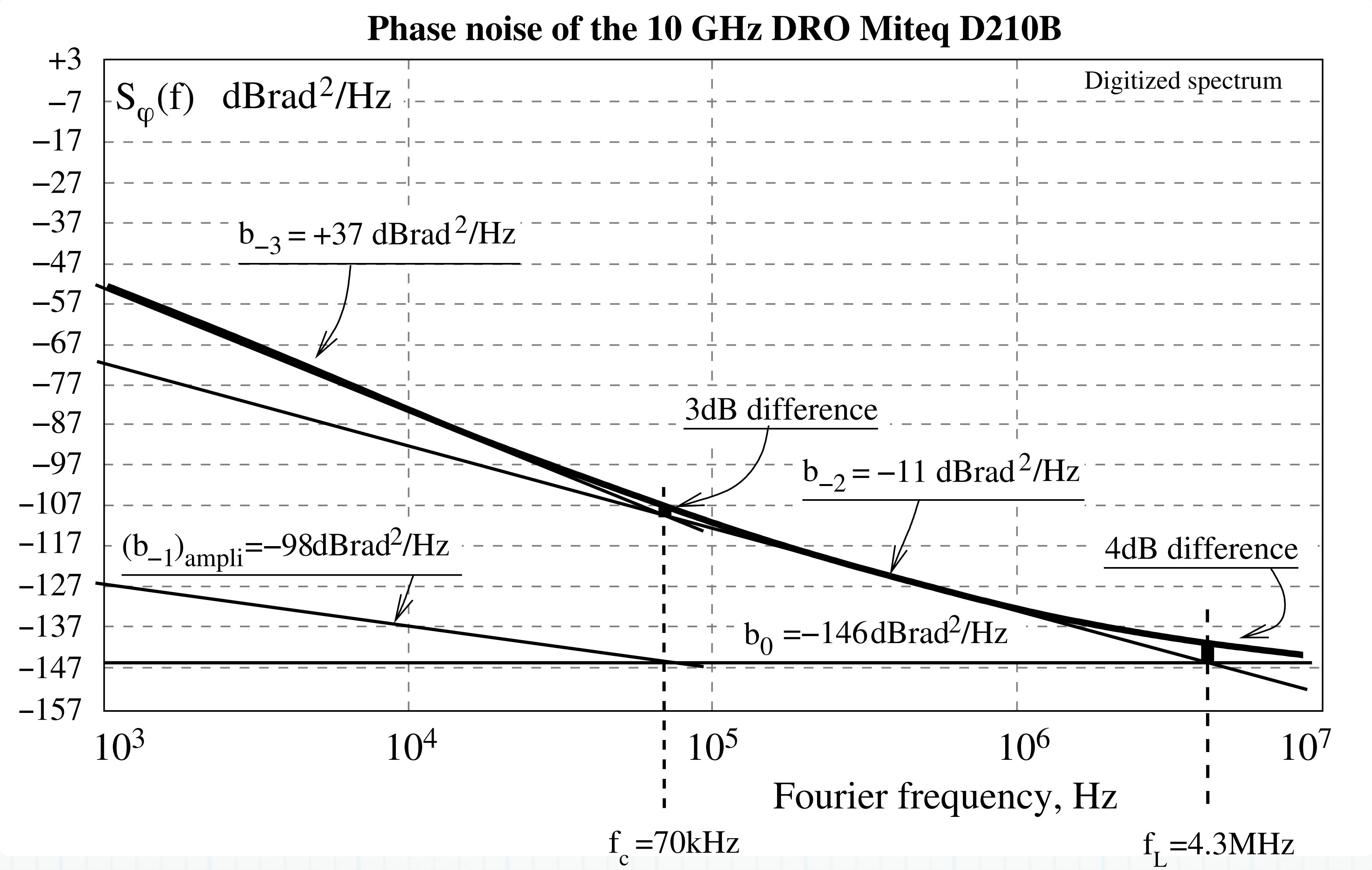


Amplifier white and flicker noise



The corner frequency f_c, sometimes specified in data sheets
is a misleading parameter because it depends on P_0

Miteq D210B, 10 GHz DRO



From the table

$\sigma^2_y = h_0/2\tau + 2\ln(2)h_{-1}$

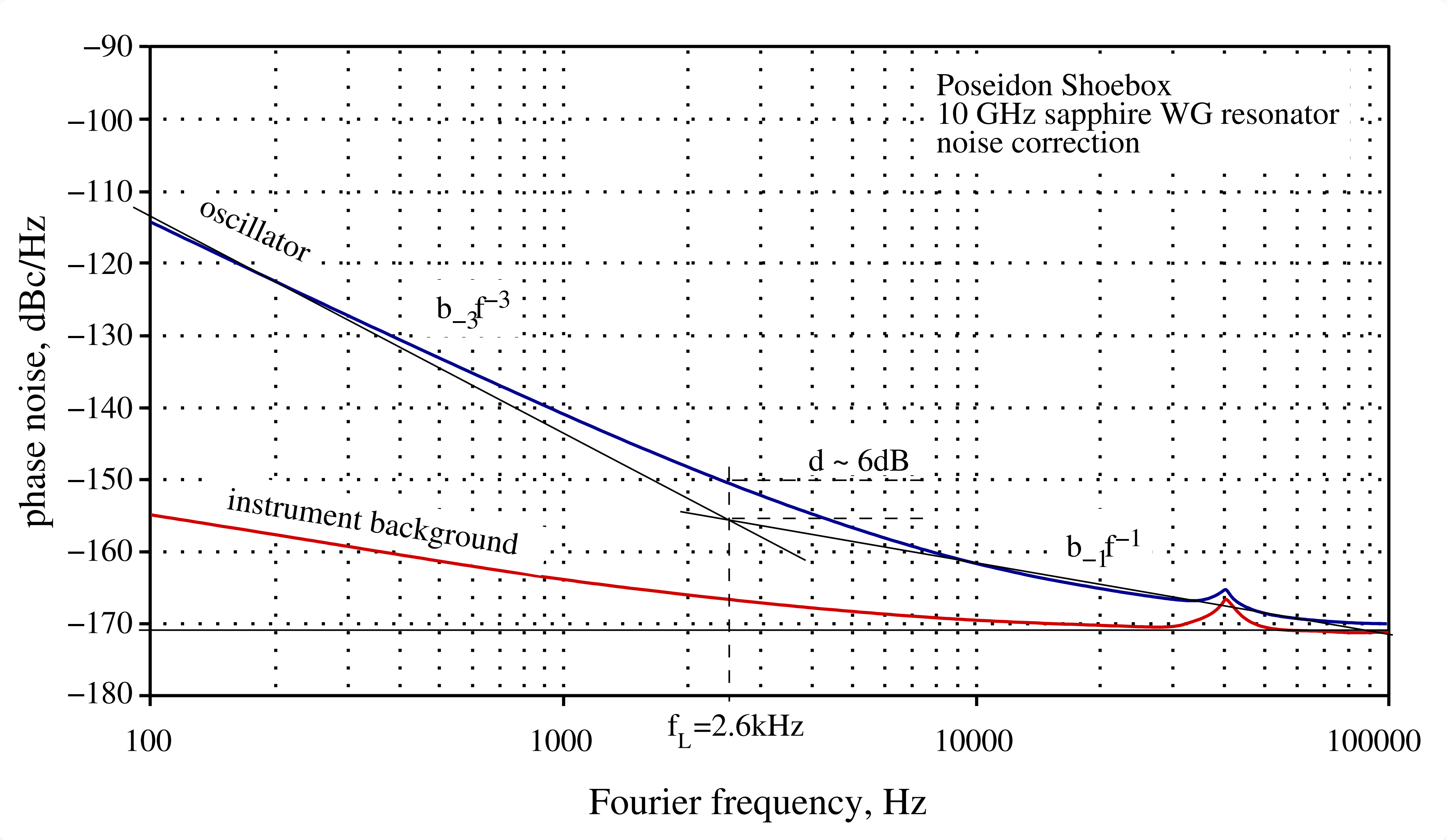
$h_0 = b_{-2}/\nu^2_0$

$h_{-1} = b_{-3}/\nu^2_0$

- **$kT_0 = 4\times10^{-21}$ W/Hz (–174 dBm/Hz)**
- **floor –146 dBrad²/Hz, guess F = 1.25 (1 dB) => $P_0 = 2\ \mu W$ (–27 dBm)**
- **$f_L = 4.3$ MHz, $f_L = \nu_0/2Q$ => Q = 1160**
- **$f_c = 70$ kHz, $b_{-1}/f = b_0$ => $b_{-1} = 1.8\times10^{-10}$ (–98 dBrad²/Hz) [sust.ampli]**
- **$h_0 = 7.9\times10^{-22}$ and $h_{-1} = 5\times10^{-17}$ => $\sigma_y = 2\times10^{-11}/\sqrt{\tau} + 8.3\times10^{-9}$**

Poseidon* Scientific Instruments – Shoebox 10 GHz sapphire whispering-gallery (1)

The spectrum is © Poseidon. The figure is from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press



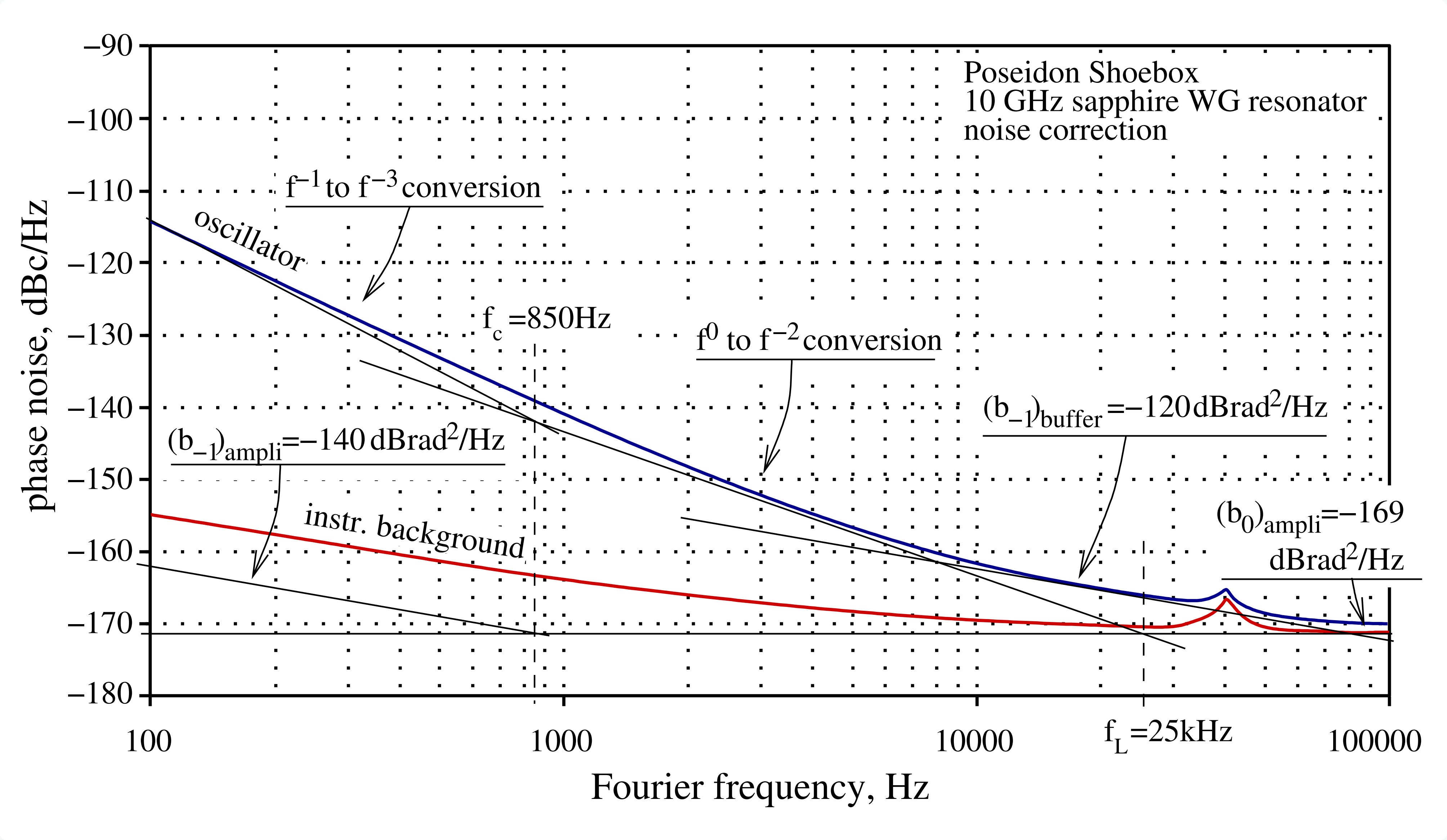
$f_L = \nu_0/2Q = 2.6$ kHz => $Q = 1.8\times10^6$

This incompatible with the resonator technology.
Typical Q of a sapphire whispering gallery resonator:
2×10^5 @ 295K (room temp), 3×10^7 @ 77K (liquid N), 4×10^9 @ 4K (liquid He).
In addition, d ~ 6 dB does not fit the power-law.

The interpretation shown is wrong, and the Leeson frequency is somewhere else

* PSI no longer exists as an independent Company. Now with Raytheon?

Poseidon Scientific Instruments – Shoebox 10 GHz sapphire whispering-gallery (2)



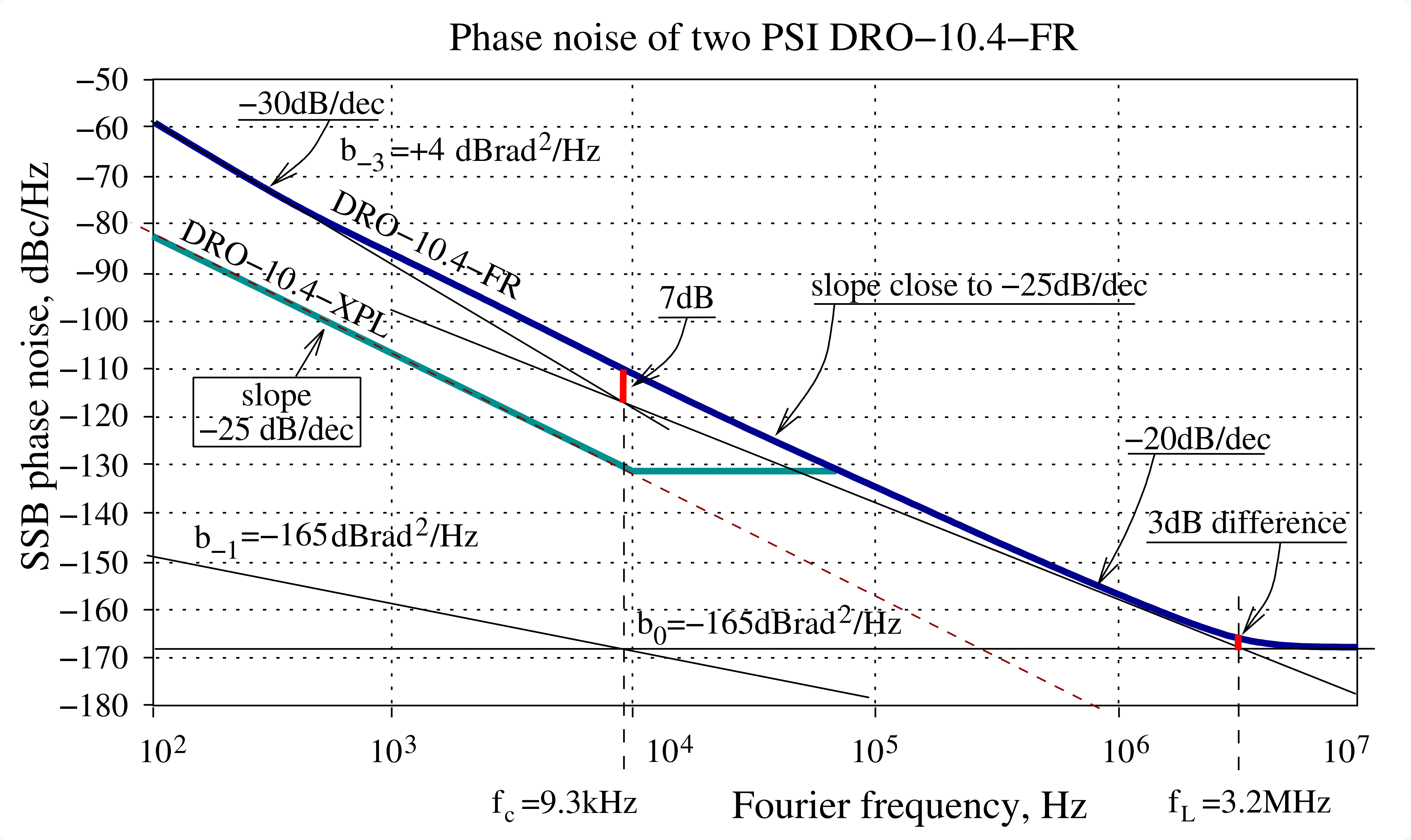
The 1/f noise of the output buffer is higher than that of the sustaining amplifier (a complex amplifier with interferometric noise reduction / or a Pound control)

In this case both 1/f and 1/f² are present

white noise –169 dBrad²/Hz, guess F = 5 dB (interferometer) => P_0 = 0 dBm
buffer flicker –120 dBrad²/Hz @ 1 Hz => good microwave amplifier

$f_L = \nu_0/2Q$ = 25 kHz => Q = 2×10⁵ (quite reasonable)
f_c = 850 Hz => flicker of the interferometric amplifier –139 dBrad²/Hz @ 1 Hz

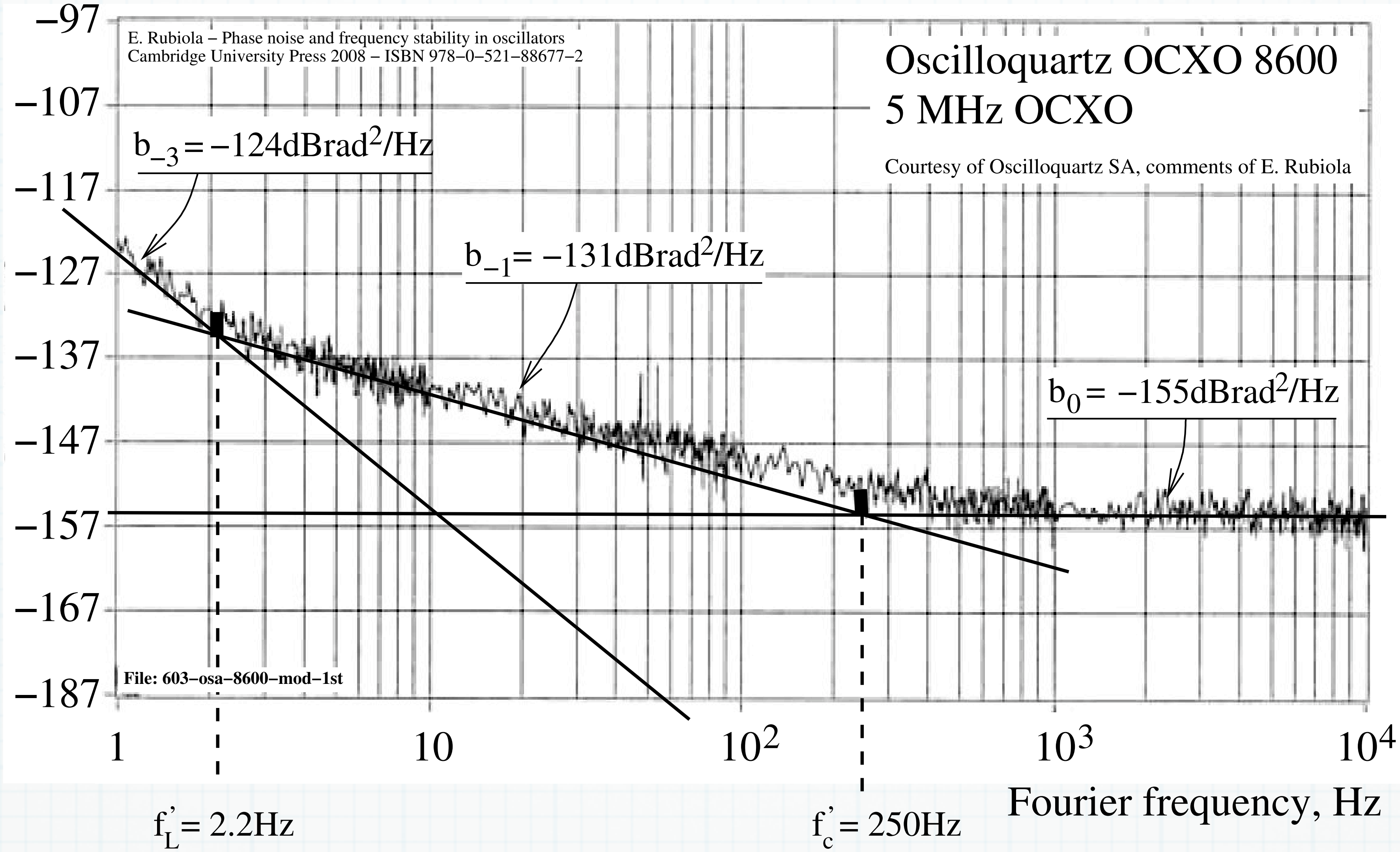
Poseidon Scientific Instruments
10 GHz dielectric resonator oscillator (DRO)



- floor –165 dBrad²/Hz, guess F = 1.25 (1 dB) => P_0 = 160 μW (–8 dBm)
- f_L = 3.2 MHz, $f_L = \nu_0/2Q$ => Q = 625
- f_c = 9.3 kHz, $b_{-1}/f = b_0$ => $b_{-1} = 2.9\times10^{-13}$ (–125 dBrad²/Hz) [sust.ampli, too low]

Slopes are not in agreement with the theory

Example – Oscilloquartz 8600 (wrong)



ANALYSIS

1 – floor $S_{\varphi 0}$ = –155 dBrad²/Hz, guess F = 1 dB → P_0 = –18 dBm

2 – ampli flicker S_φ = –132 dBrad²/Hz @ 1 Hz → good RF amplifier

3 – merit factor $Q = \nu_0/2f_L = 5{\cdot}10^6/5 = 10^6$ (seems too low)

4 – take away some flicker for the output buffer:

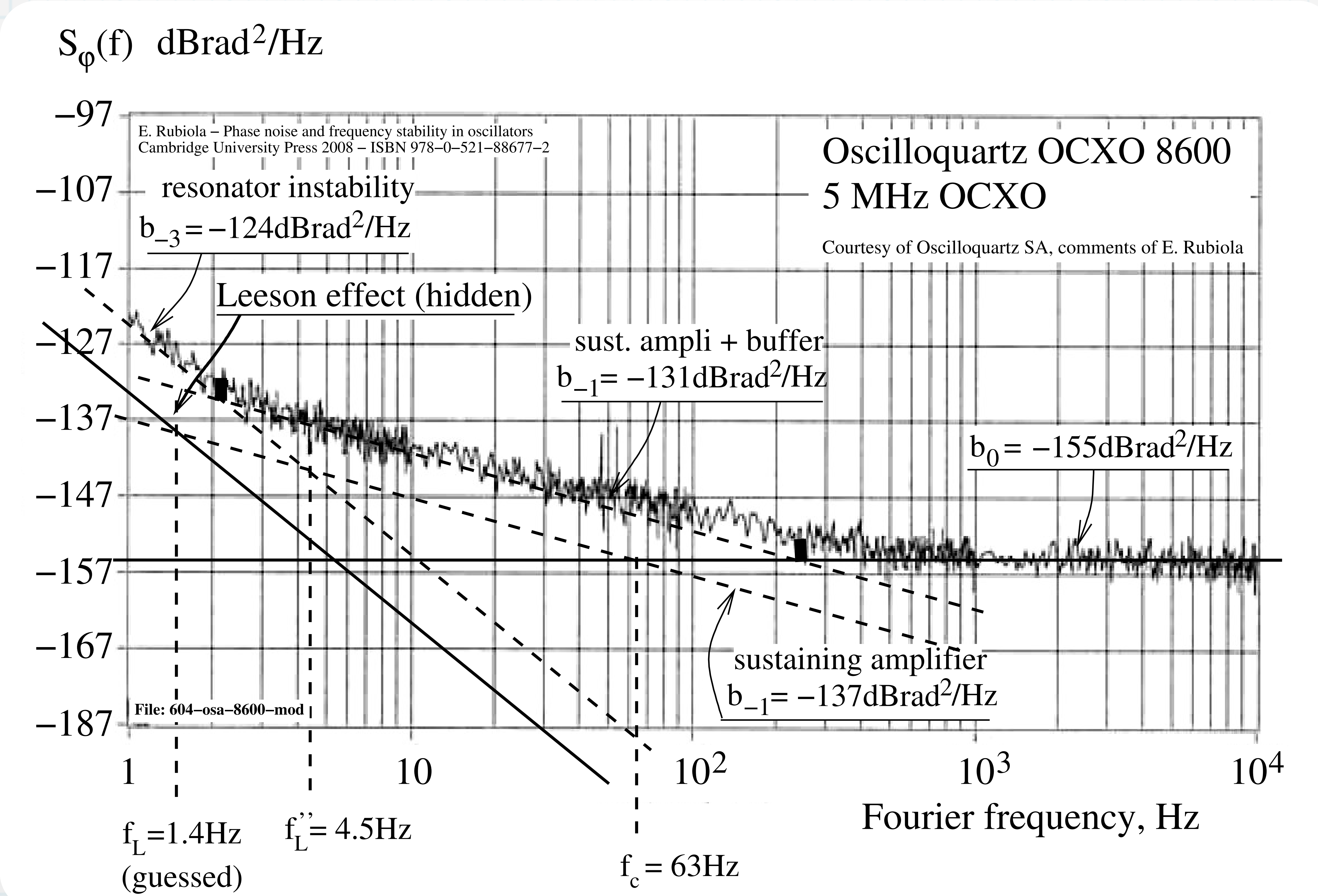
- flicker in the oscillator core is lower than –132 dBrad²/Hz @ 1 Hz
- f_L is higher than 2.5 Hz
- the resonator Q is lower than 10^6

This is inconsistent with the resonator technology (expect $Q > 10^6$).

The true Leeson frequency is lower than the frequency labeled as f_L

The $1/f^3$ noise is attributed to the fluctuation of the quartz resonant frequency

Example – Oscilloquartz 8600 (trusted)

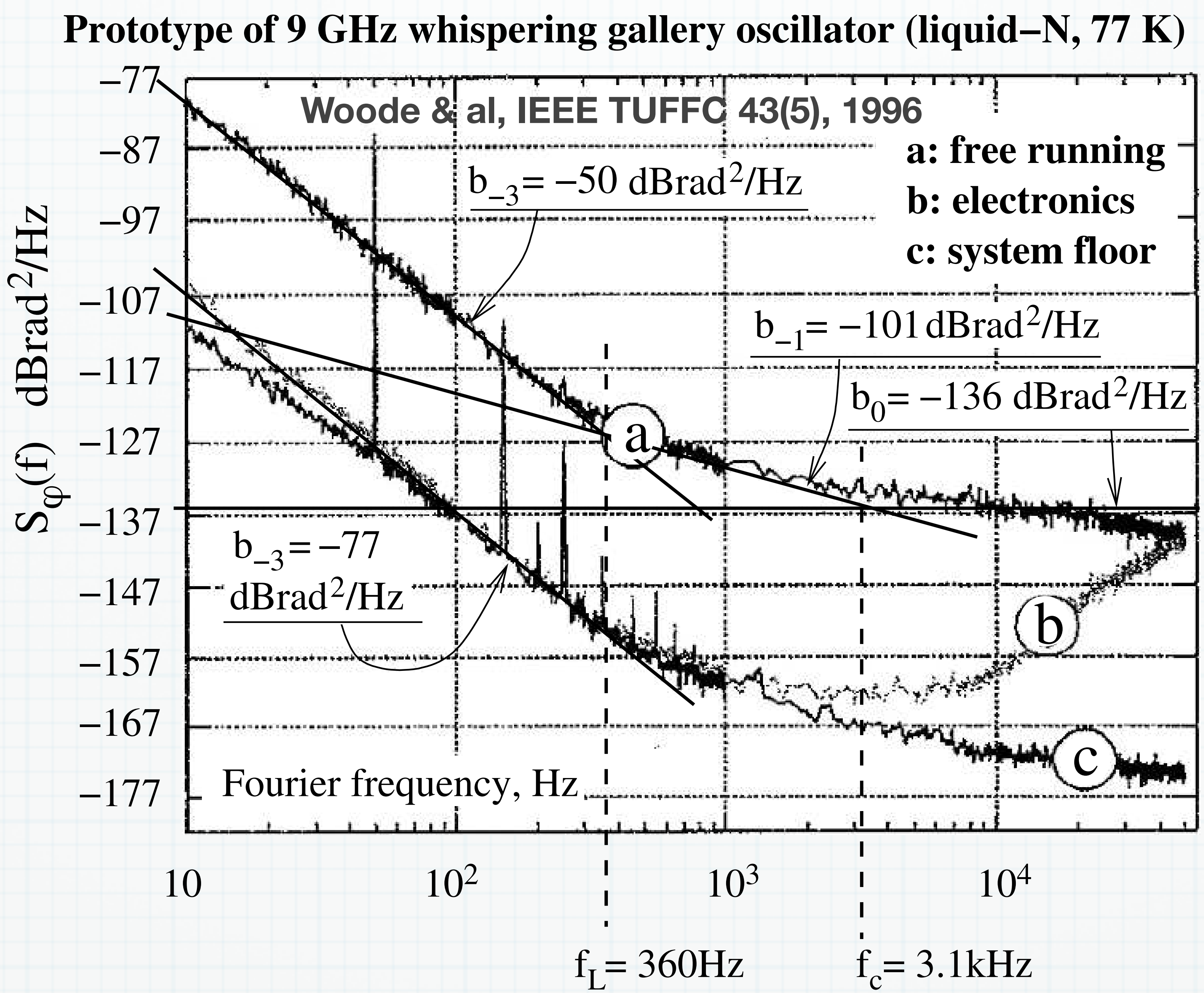


F=1dB b_0 => P_0=–18 dBm

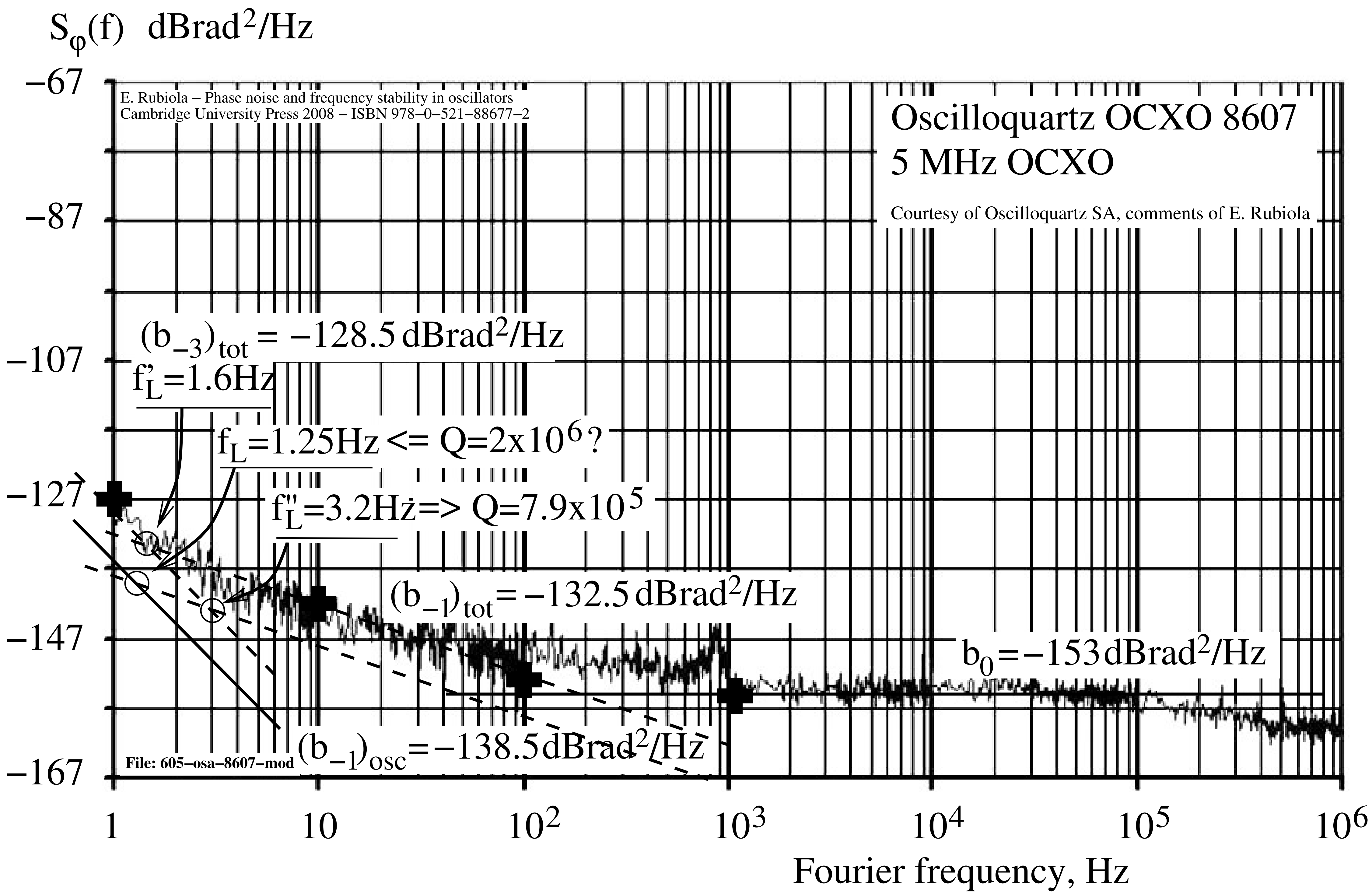
$(b_{-3})_{osc}$ => $\sigma_y = 1.5\times10^{-13}$, $Q = 5.6\times10^{5}$ (too low)

$Q \stackrel{?}{=} 1.8\times10^{6}$ => $\sigma_y = 4.6\times10^{-14}$ Leeson (too low)

Whispering gallery oscillator, liquid-N_2 temp



Example – Oscilloquartz 8607

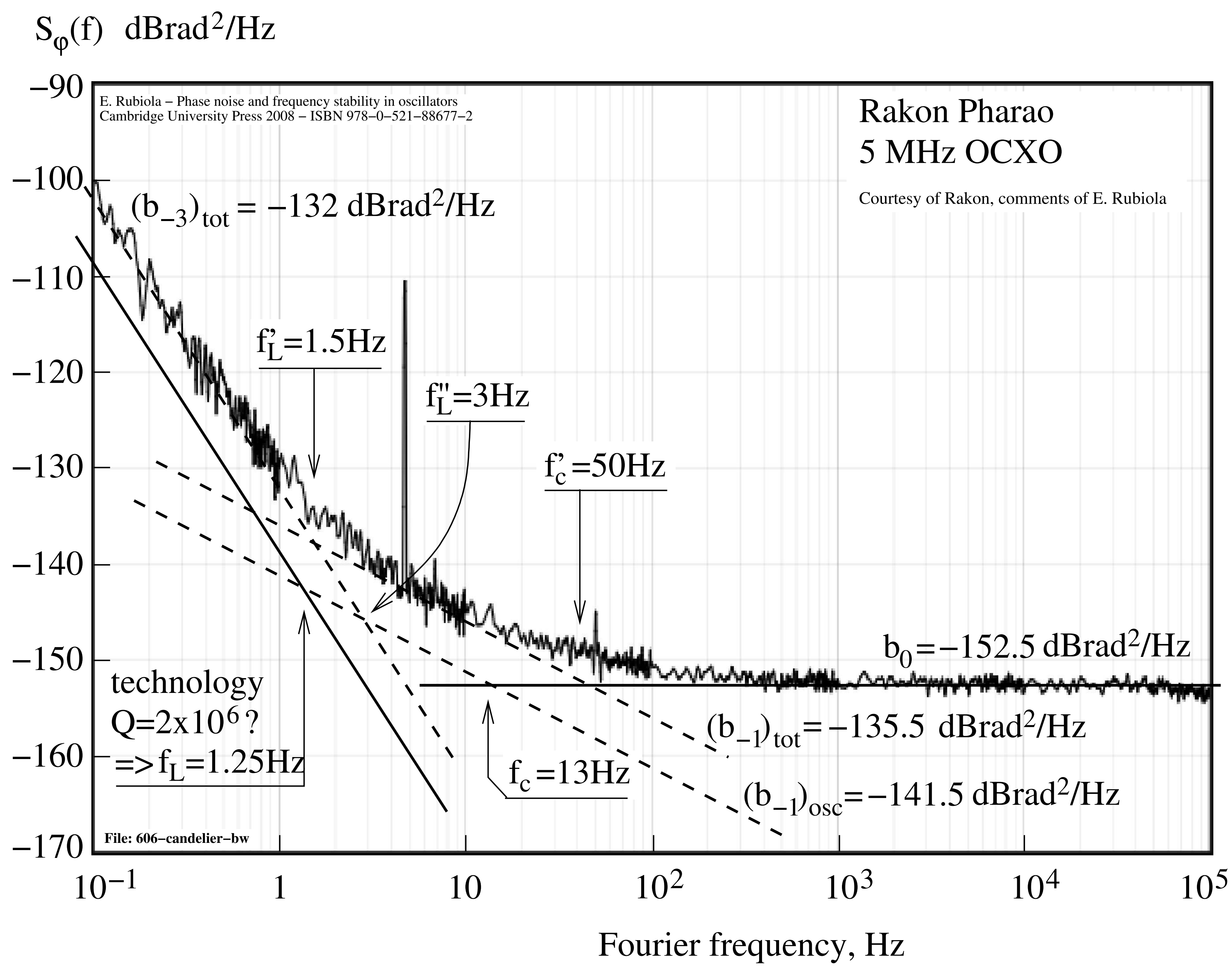


F=1dB b_0 => P_0=–20 dBm

$(b_{-3})_{osc}$ => $\sigma_y=8.8\times10^{-14}$, $Q=7.8\times10^5$ (too low)

$Q \stackrel{?}{=} 2\times10^6$ => $\sigma_y=3.5\times10^{-14}$ Leeson (too low)

Example – CMAC Pharao

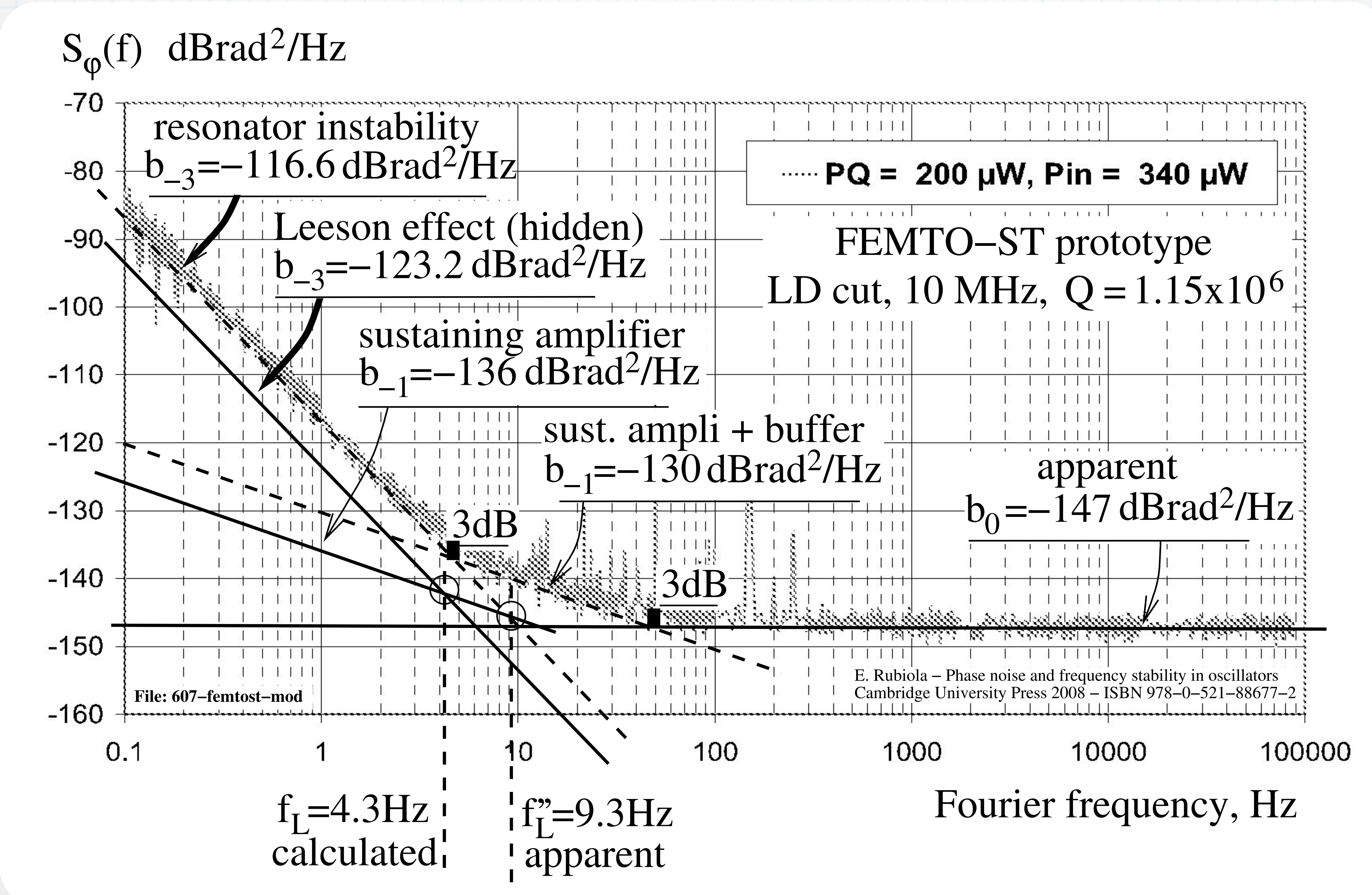


F=1dB b_0 => P_0=–20.5 dBm

$(b_{-3})_{osc}$ => $\sigma_y = 5.9\times10^{-14}$, $Q = 8.4\times10^5$ (too low)

$Q \stackrel{?}{=} 2\times10^6$ => $\sigma_y = 2.5\times10^{-14}$ Leeson (too low)

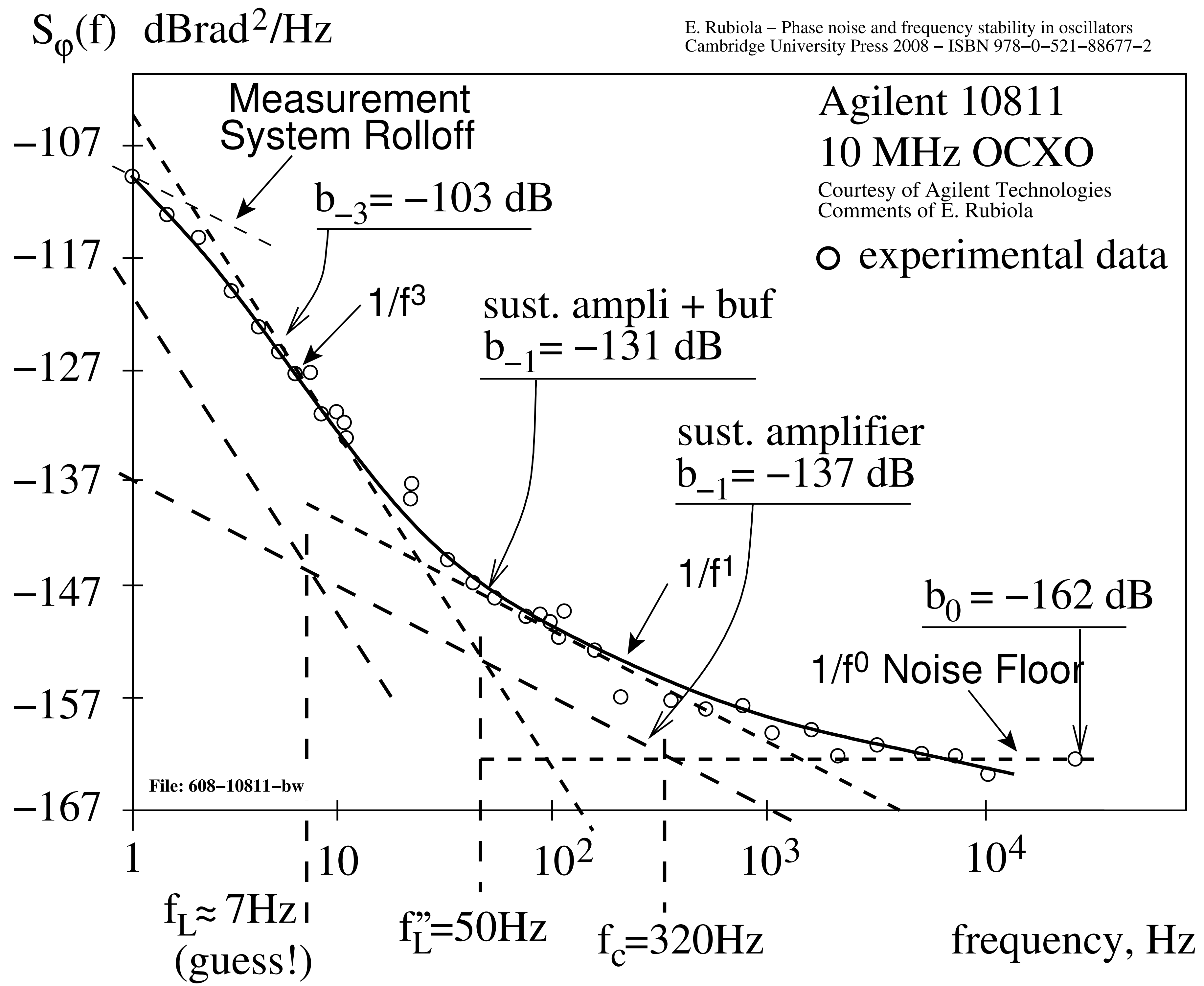
Example – FEMTO-ST prototype



F=1dB b_0 => P_0=–26 dBm $(b_{-3})_{osc}$ => $\sigma_y=1.7\times10^{-13}$, Q=$5.4\times10^5$ (too low)

(there is a problem) Q=1.15×10^6 => $\sigma_y=8.1\times10^{-14}$ Leeson (too low)

Example – Agilent 10811

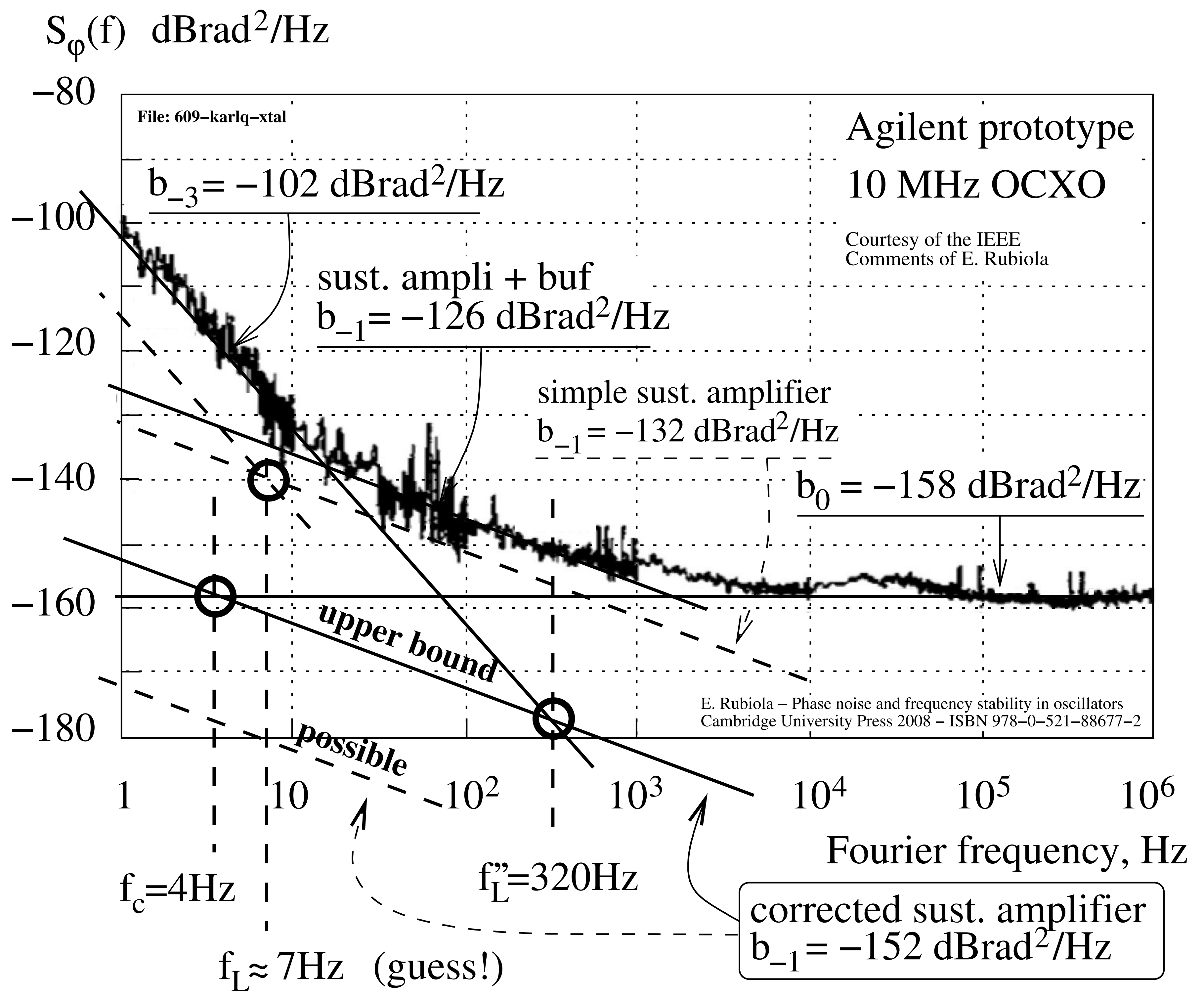


F=1dB b_0 => P_0=–11 dBm

$(b_{-3})_{osc}$ => $\sigma_y=8.3\times10^{-13}$, $Q=1\times10^5$ (too low)

$Q \stackrel{?}{=} 7\times10^5$ => $\sigma_y=1.2\times10^{-13}$ Leeson (too low)

Example – Agilent prototype

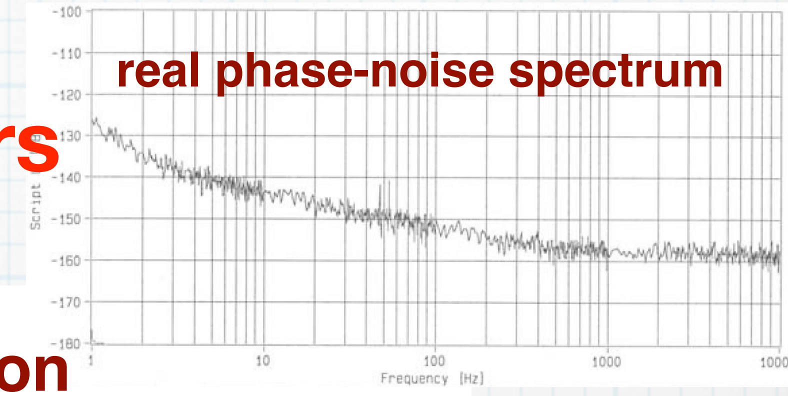


F=1dB b_0 => P_0=–12 dBm

$(b_{-3})_{osc}$ => $\sigma_y = 9.3\times10^{-13}$ Q=1.6x10⁵

$Q \stackrel{?}{=} 7\times10^5$ => $\sigma_y = 2.1\times10^{-13}$ (Leeson)

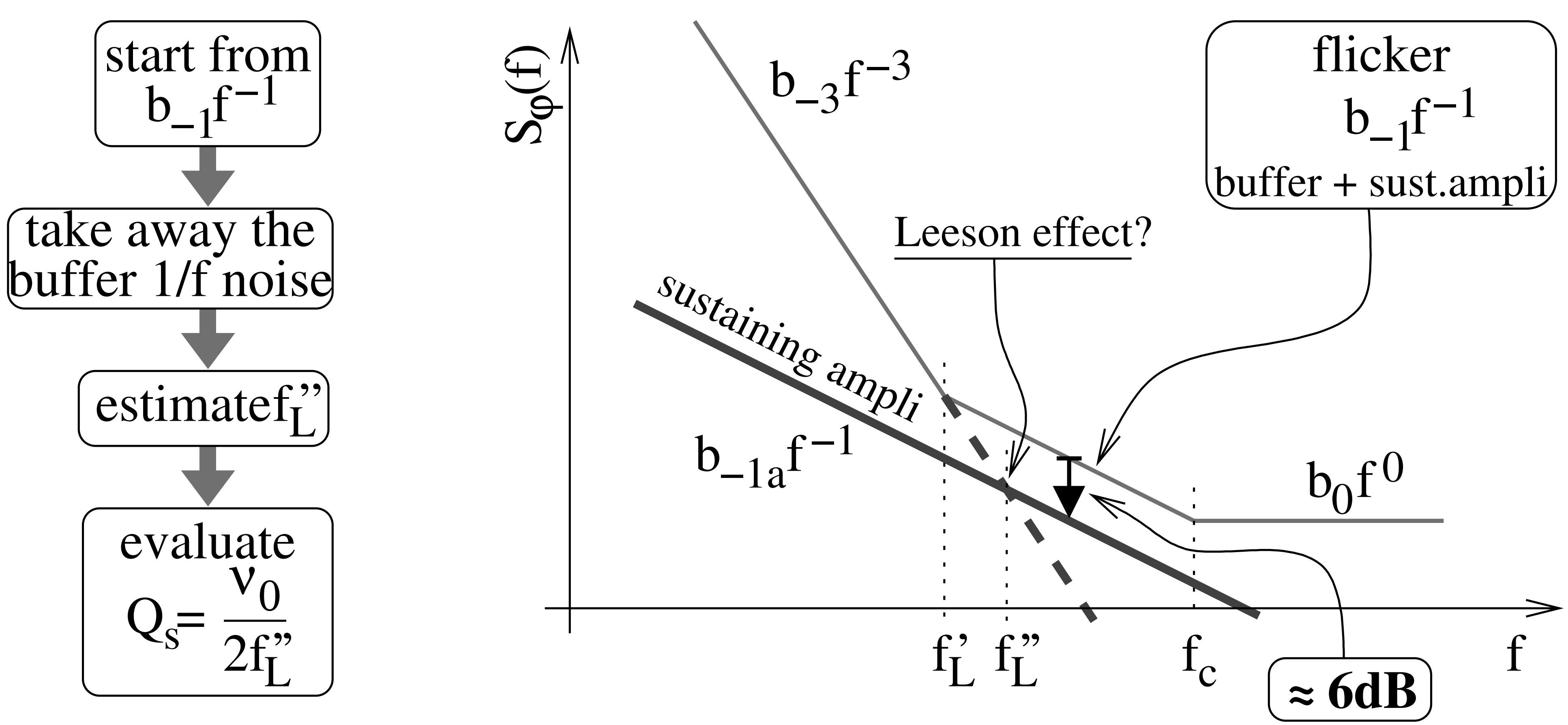
Interpretation of $S_\varphi(f)$ [1]



Only quartz-crystal oscillators

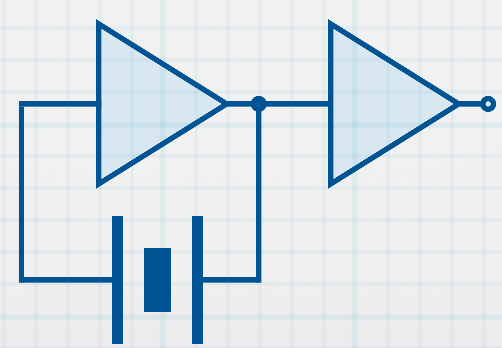
E. Rubiola – Phase noise and frequency stability in oscillators
Cambridge University Press 2008 – ISBN 978–0–521–88677–2

after parametric estimation



File: 602a–xtal–interpretation

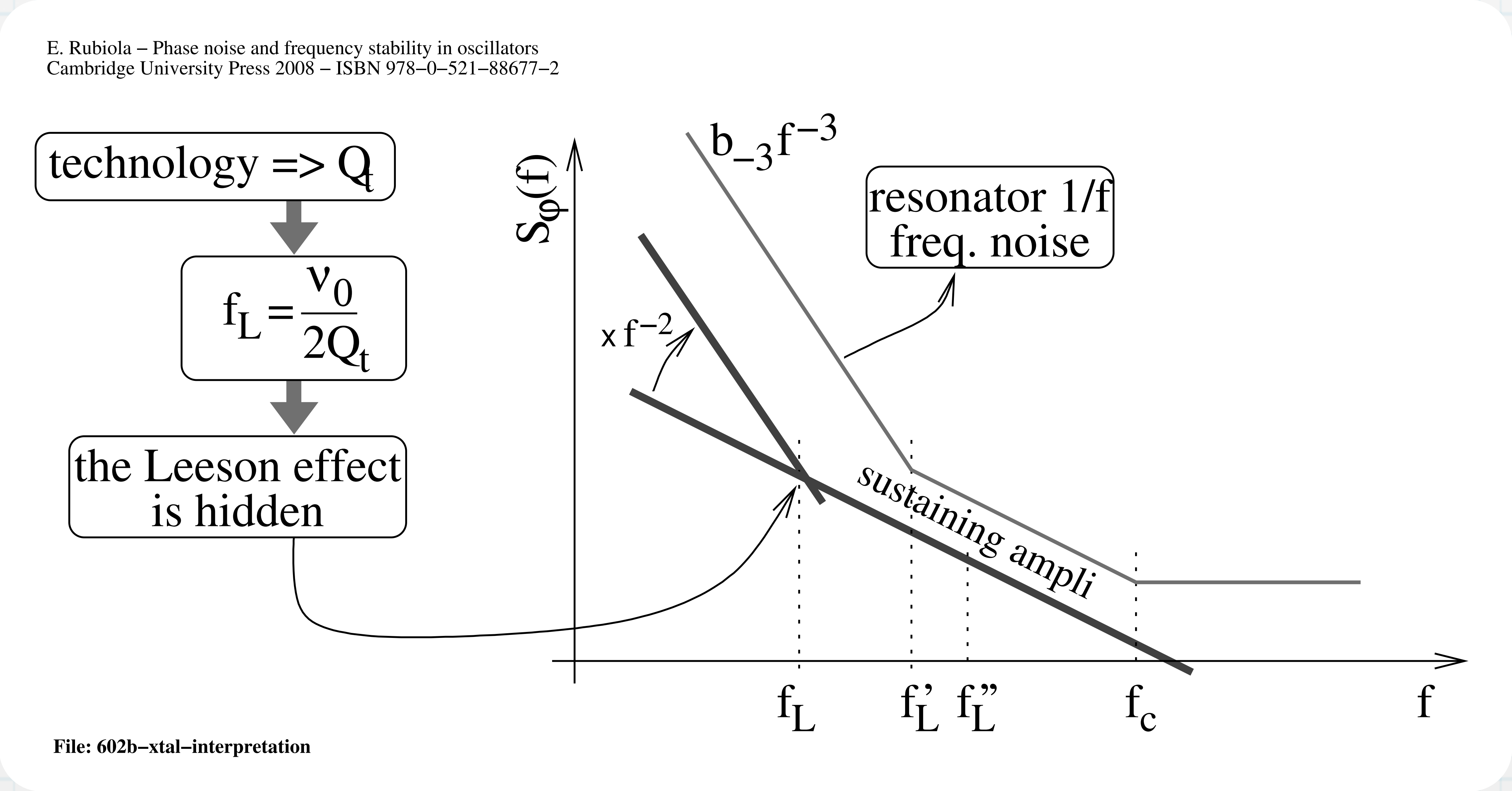
Sanity check:
– power P_0 at amplifier input
– Allan deviation σ_y (floor)



2–3 buffer stages => the sustaining amplifier contributes ≲ 25% of the total 1/f noise

Interpretation of $S_\varphi(f)$ [2]

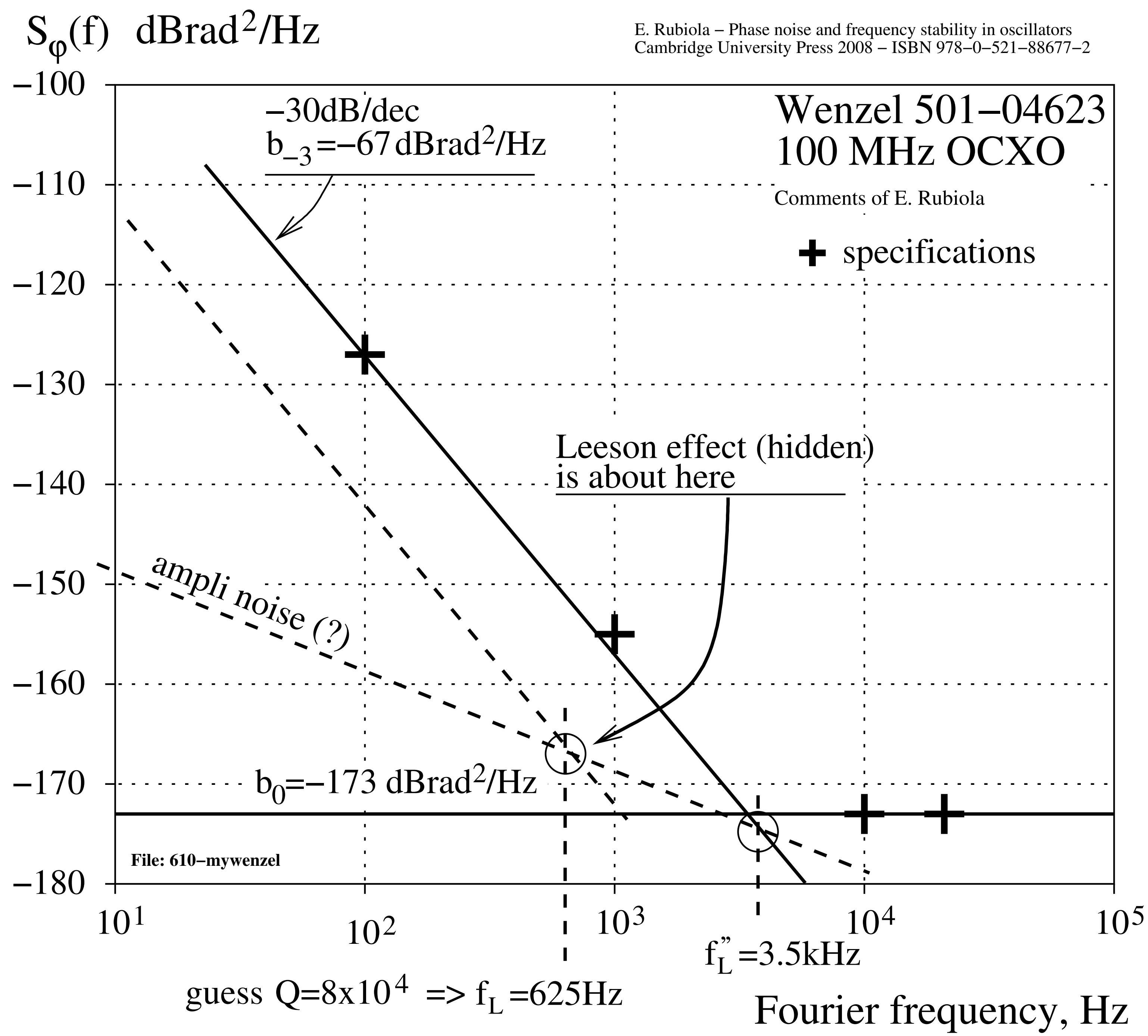
Only quartz-crystal oscillators



Figures are from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press

Technology suggests a quality factor Q_t. In all xtal oscillators we find $Q_t \gg Q_s$

Example – Wenzel 501-04623



Data are from the manufacturer web site. Interpretation and mistakes are of the authors.

Estimating $(b_{-1})_{ampli}$ is difficult because there is no visible 1/f region

F=1dB b_0 => P_0=0 dBm

$(b_{-3})_{osc}$ => $\sigma_y=5.3\times10^{-12}$ $Q=1.4\times10^4$

$Q \overset{?}{=} 8\times10^4$ => $\sigma_y=9.3\times10^{-13}$ (Leeson)

Quartz-oscillator summary

Oscillator	ν_0	$(b_{-3})_{\text{tot}}$	$(b_{-1})_{\text{tot}}$	$(b_{-1})_{\text{amp}}$	f'_L	f''_L	Q_s	Q_t	f_L	$(b_{-3})_{\text{L}}$	R	Note
Oscilloquartz 8600	5	-124.0	-131.0	-137.0	2.24	4.5	5.6×10^5	1.8×10^6	1.4	-134.1	10.1	(1)
Oscilloquartz 8607	5	-128.5	-132.5	-138.5	1.6	3.2	7.9×10^5	2×10^6	1.25	-136.5	8.1	(1)
Rakon Pharao	5	-132.0	-135.5	-141.1	1.5	3	8.4×10^5	2×10^6	1.25	-139.6	7.6	(2)
FEMTO-ST LD prot.	10	-116.6	-130.0	-136.0	4.7	9.3	5.4×10^5	1.15×10^6	4.3	-123.2	6.6	(3)
Agilent 10811	10	-103.0	-131.0	-137.0	25	50	1×10^5	7×10^5	7.1	-119.9	16.9	(4)
Agilent prototype	10	-102.0	-126.0	-132.0	16	32	1.6×10^5	7×10^5	7.1	-114.9	12.9	(5)
Wenzel 501-04623	100	-67.0	-132 ?	-138 ?	1800	3500	1.4×10^4	8×10^4	625	-79.1	15.1	(6)
unit	MHz	dB rad^2/Hz	dB rad^2/Hz	dB rad^2/Hz	Hz	Hz	(none)	(none)	Hz	dB rad^2/Hz	dB	

Notes

(1) Data are from specifications, full options about low noise and high stability.

(2) Measured by Rakon on a sample. Rakon confirmed that $2\times10^6 < Q < 2.2\times10^6$ in actual conditions.

(3) LD cut, built and measured in our laboratory, yet by a different team. Q_t is known.

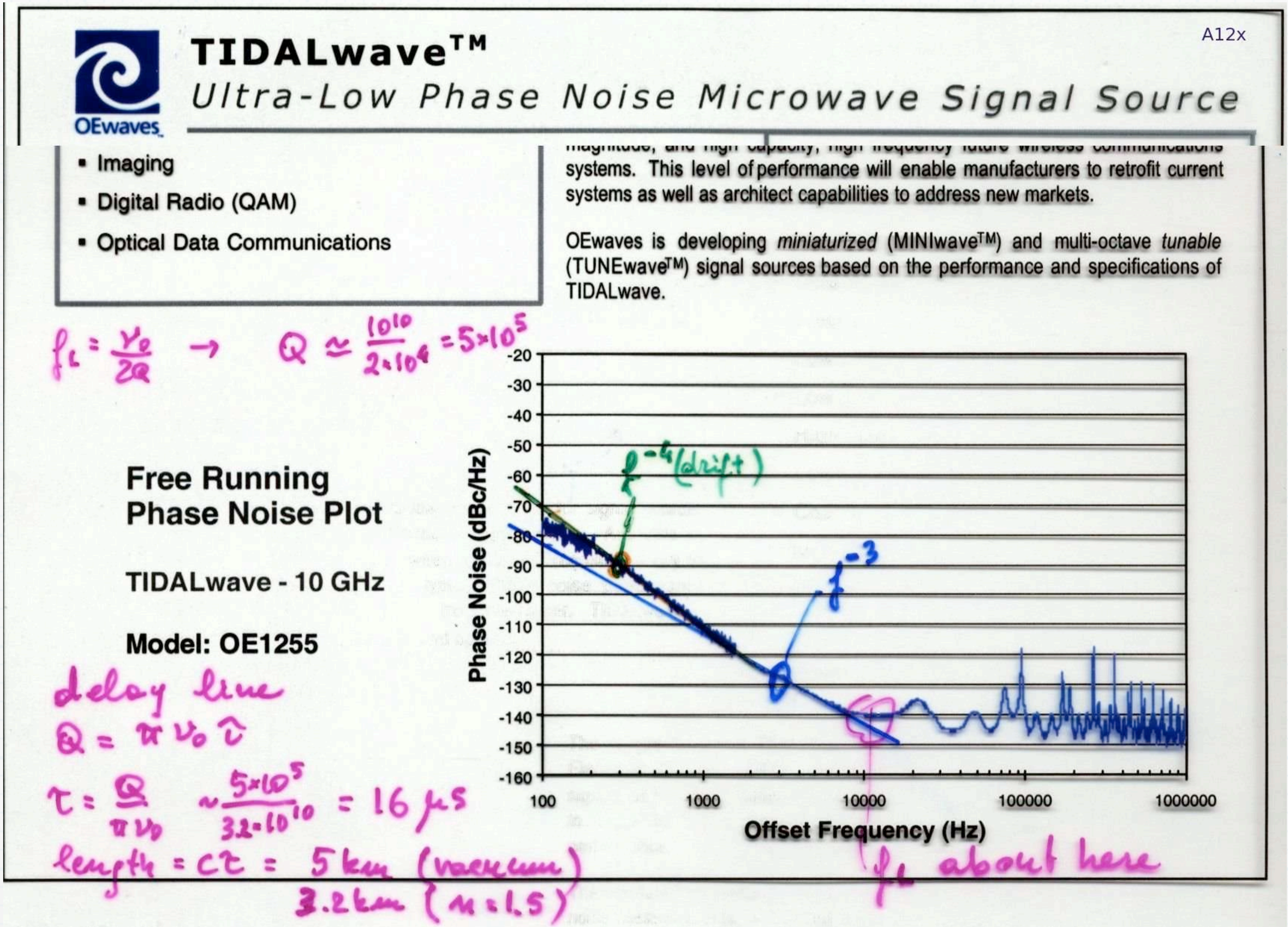
(4) Measured by Hewlett Packard (now Agilent) on a sample.

(5) Implements a bridge scheme for the degeneration of the amplifier noise. Same resonator of the Agilent 10811.

(6) Data are from specifications.

$$R = \left.\frac{(\sigma_y)_{\text{oscill}}}{(\sigma_y)_{\text{Leeson}}}\right|_{\text{floor}} = \sqrt{\frac{(b_{-3})_{\text{tot}}}{(b_{-3})_L}} = \frac{Q_t}{Q_s} = \frac{f''_L}{f_L}$$

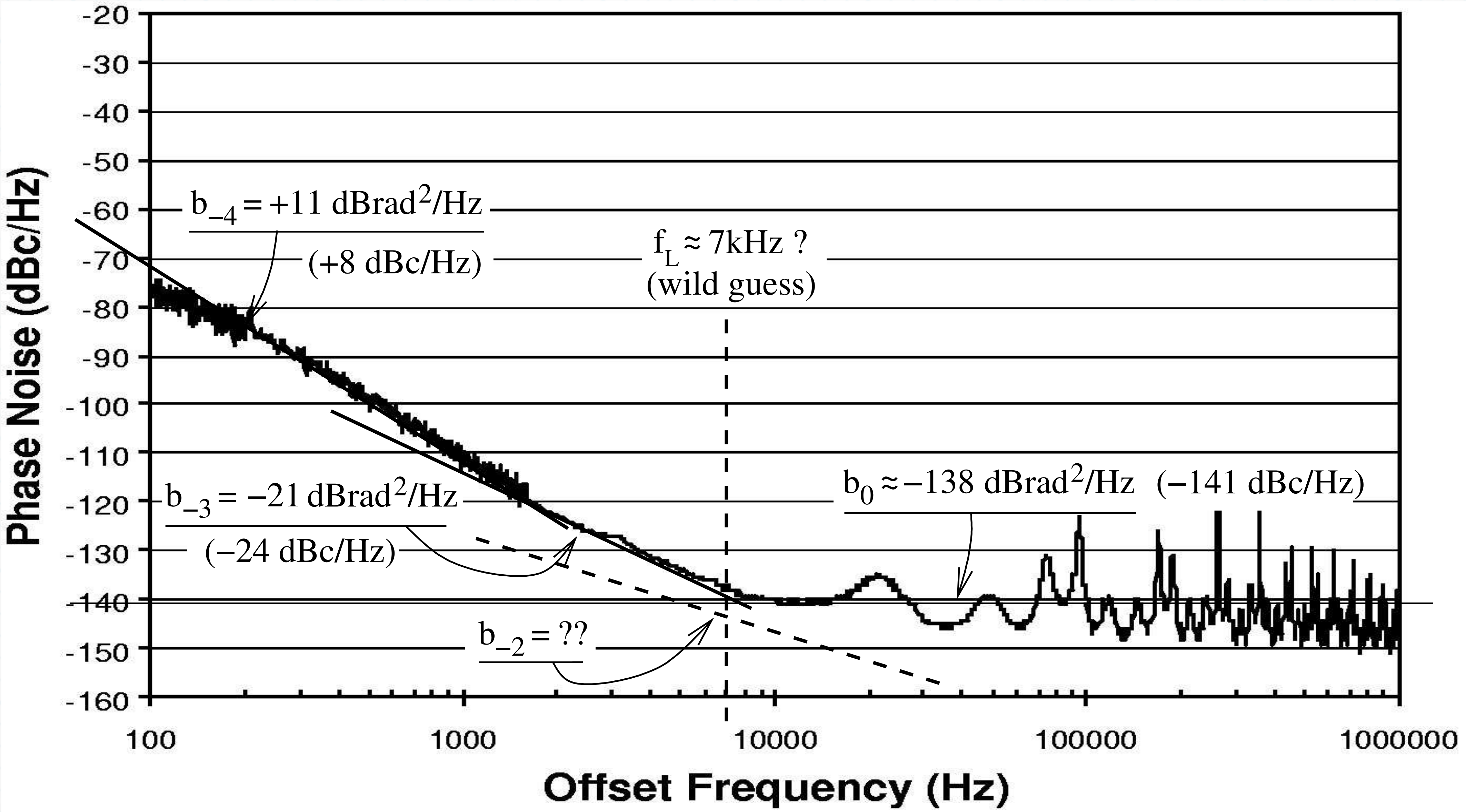
Opto-electronic oscillator



Courtesy of OEwaves (handwritten notes are mine). Cut from the oscillator specifications available at the URL http://www.oewaves.com/products/pdf/TDALwave_Datasheet_012104.pdf

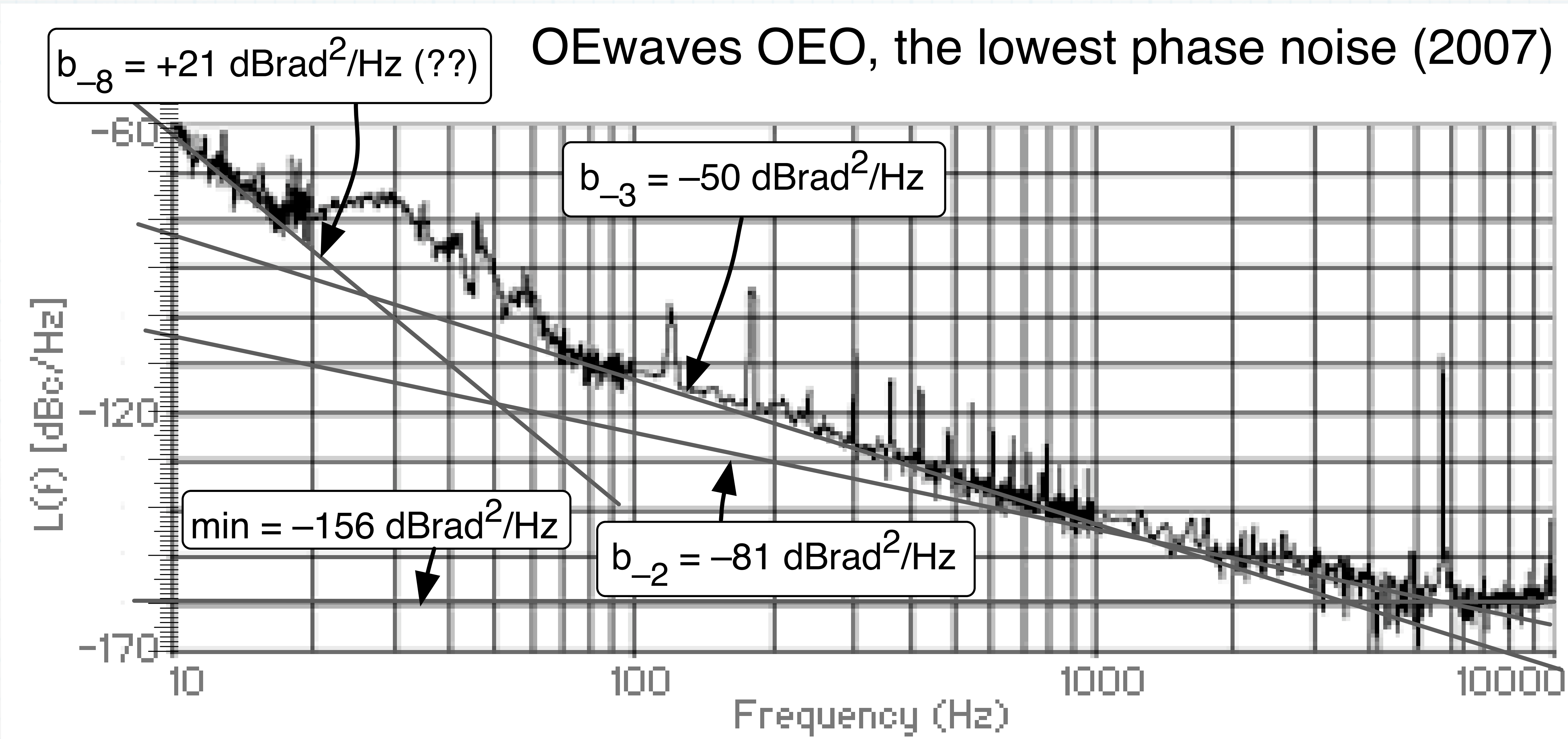
Opto-electronic oscillator

OEwaves Tidalwave photonic oscillator (10 GHz)



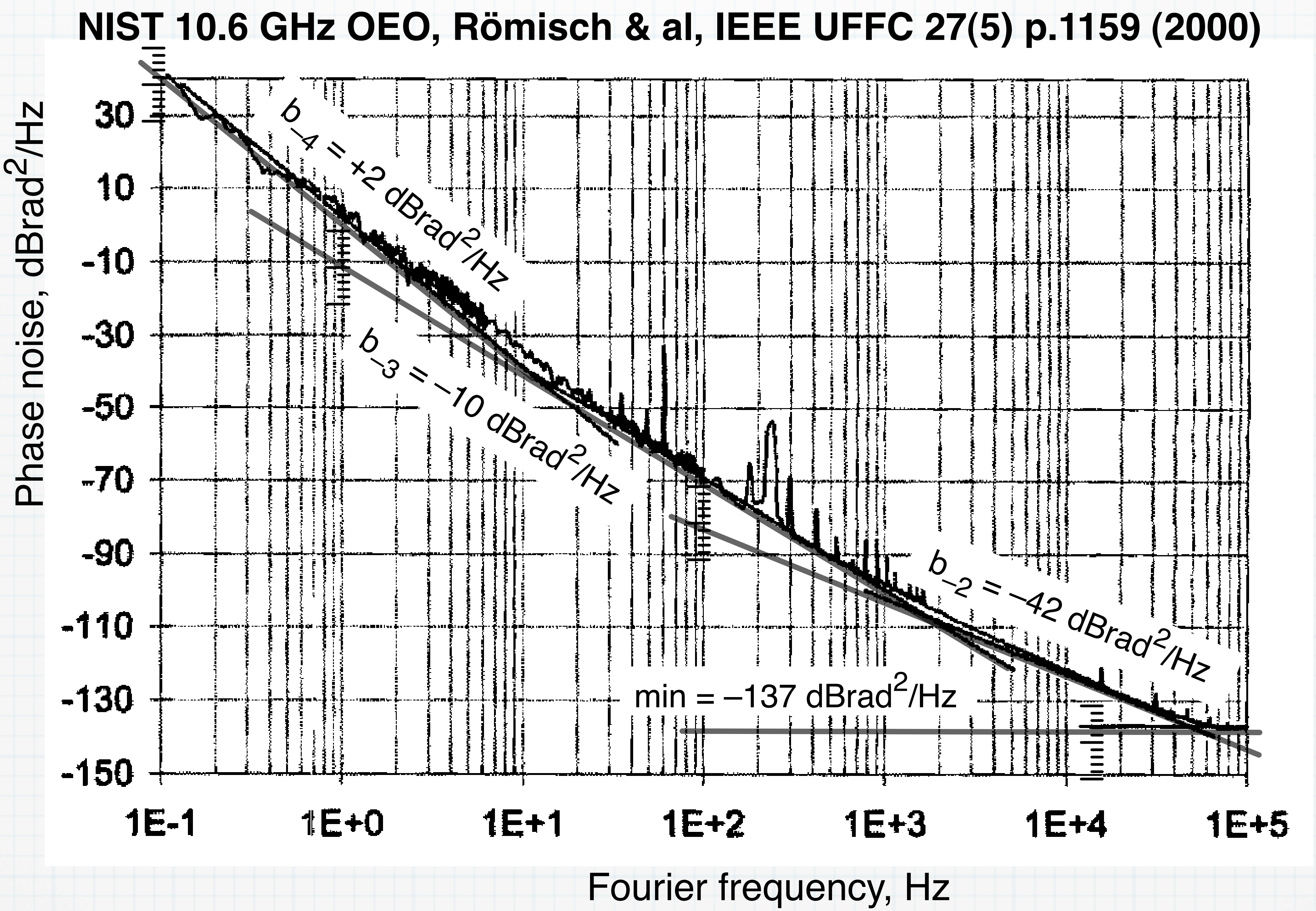
Courtesy of OEwaves (handwritten notes are mine). Cut from the oscillator specifications available at the URL http://www.oewaves.com/products/pdf/TDALwave_Datasheet_012104.pdf

Opto-electronic oscillator



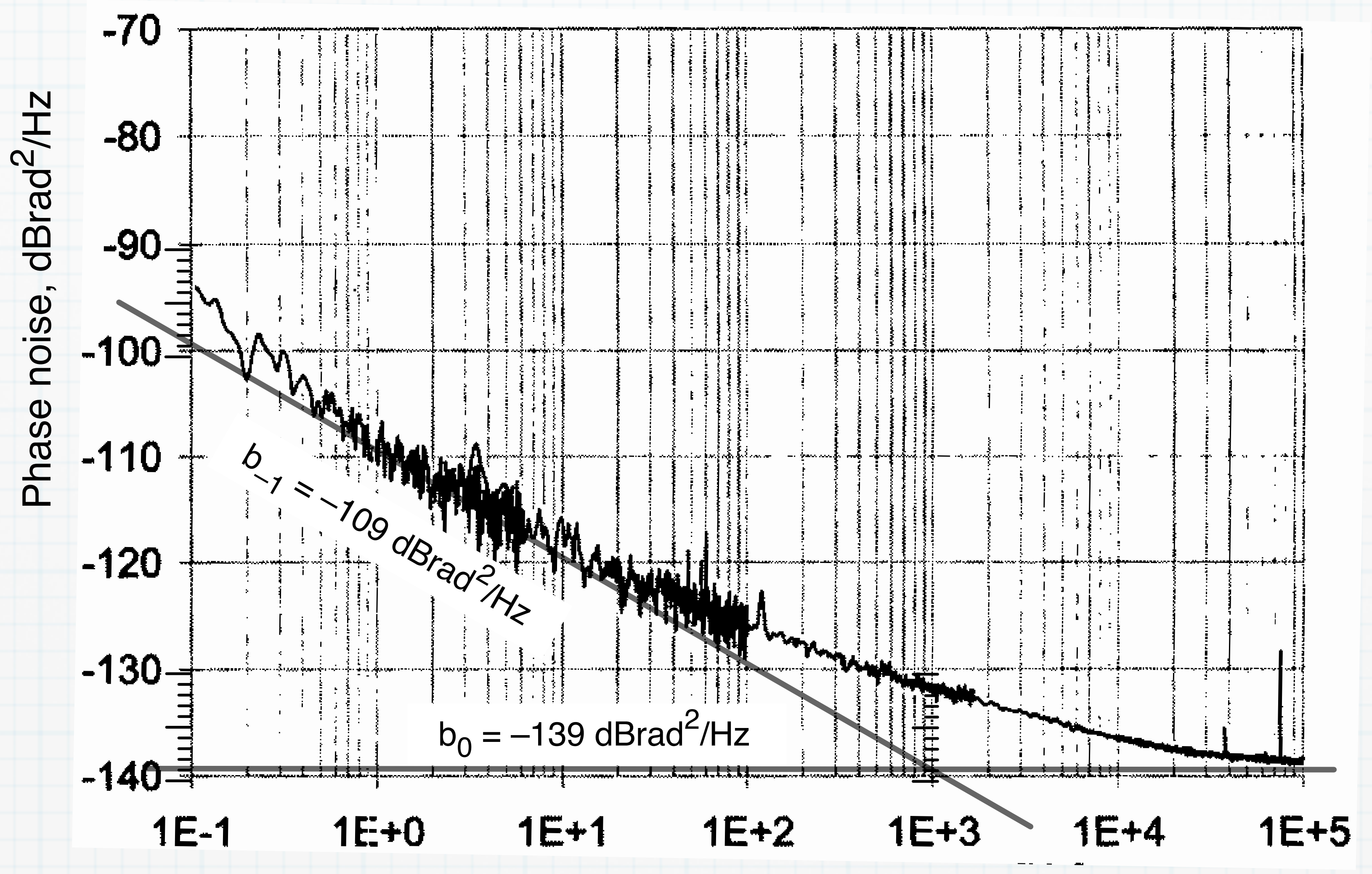
Courtesy of OEwaves, notes are mine

Opto-electronic oscillator

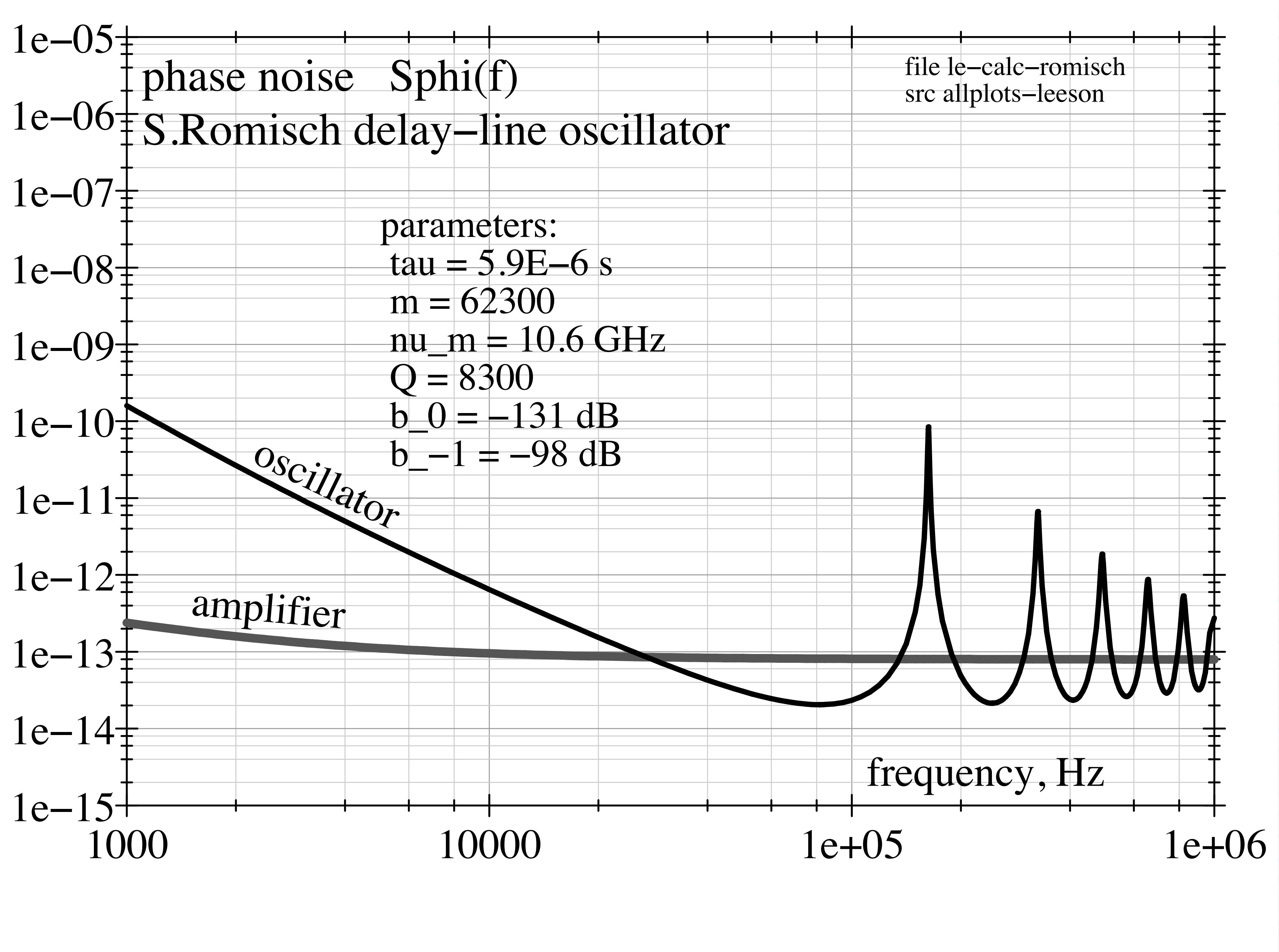


Opto-electronic oscillator (amplifier)

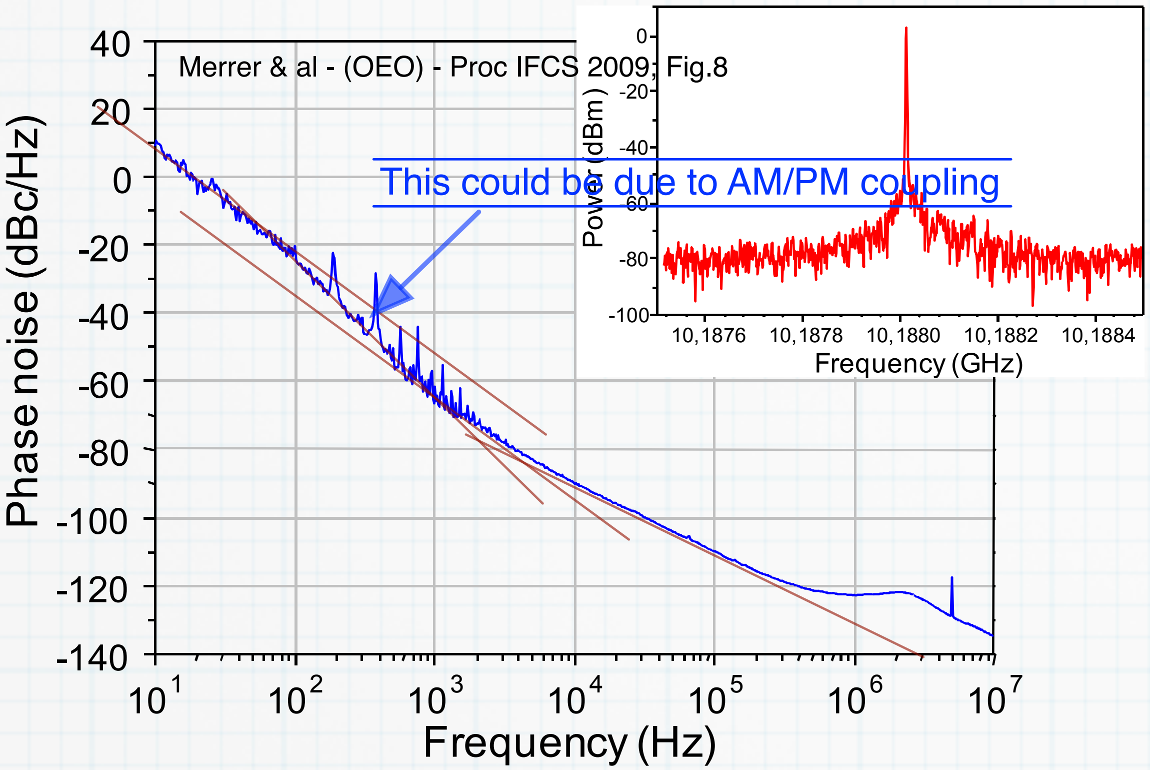
OEO amplifier, Römisch & al, IEEE UFFC 27(5) p.1159 (2000)

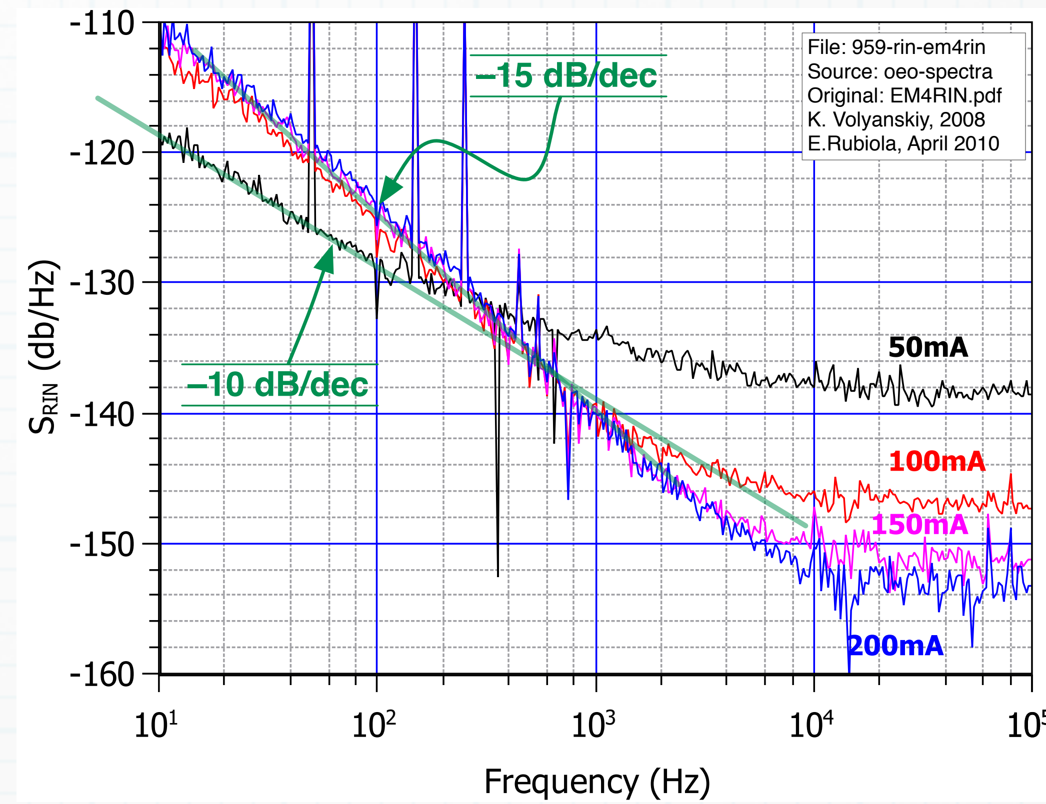


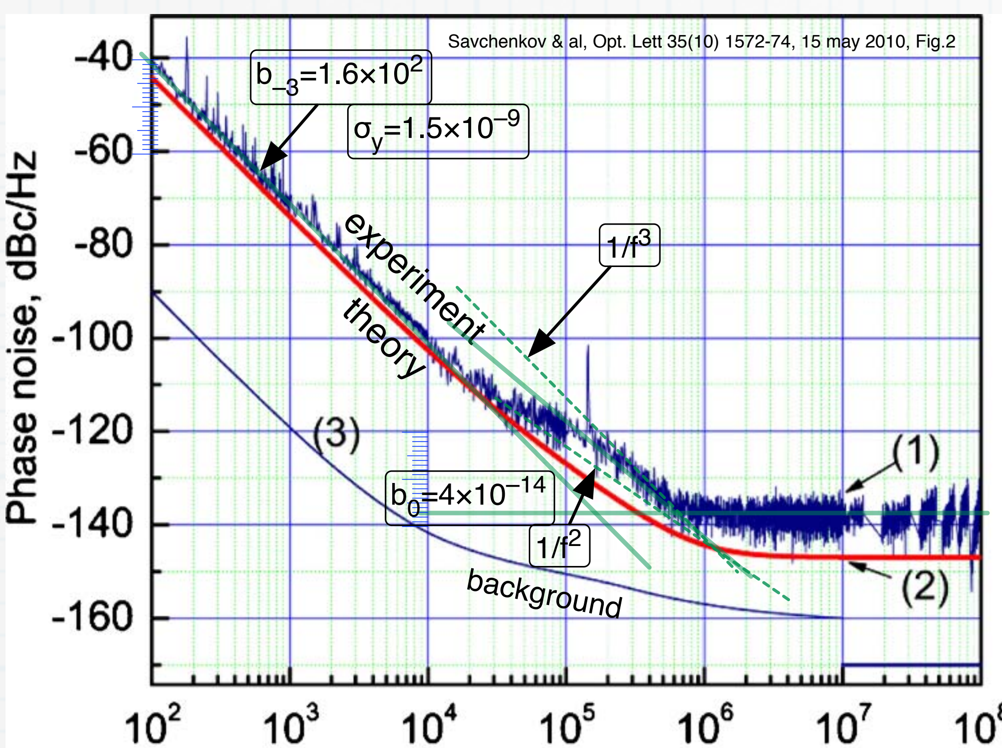
Opto-electronic oscillator simulation

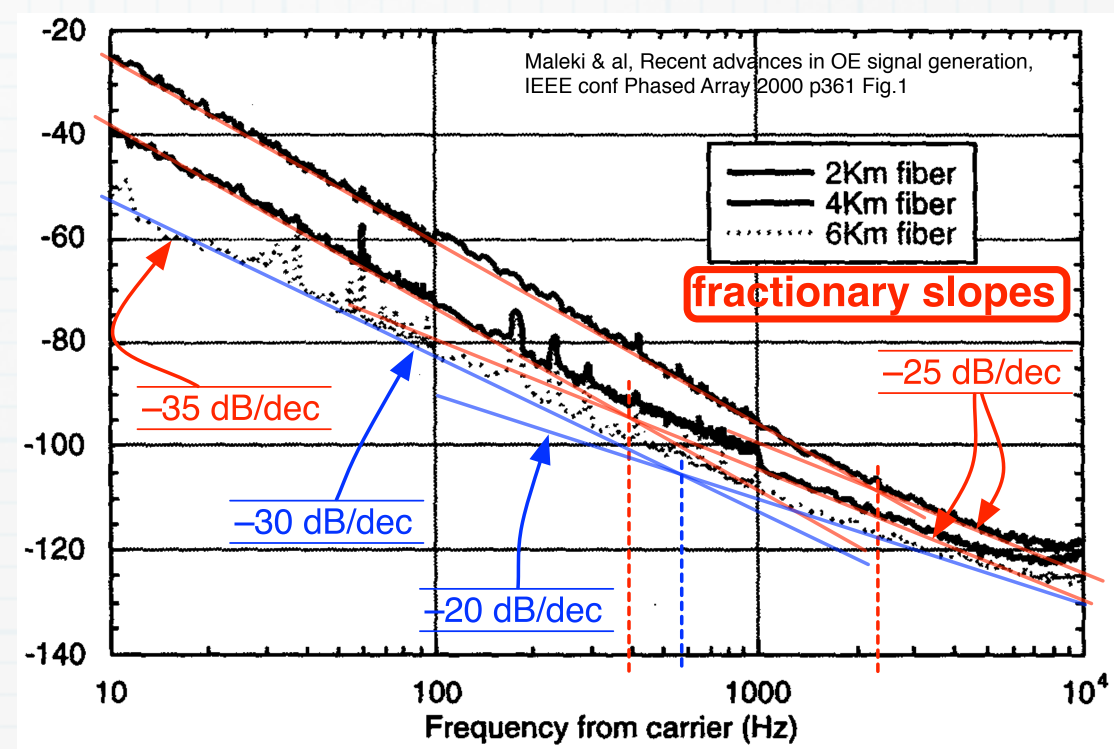


Things may not be that simple



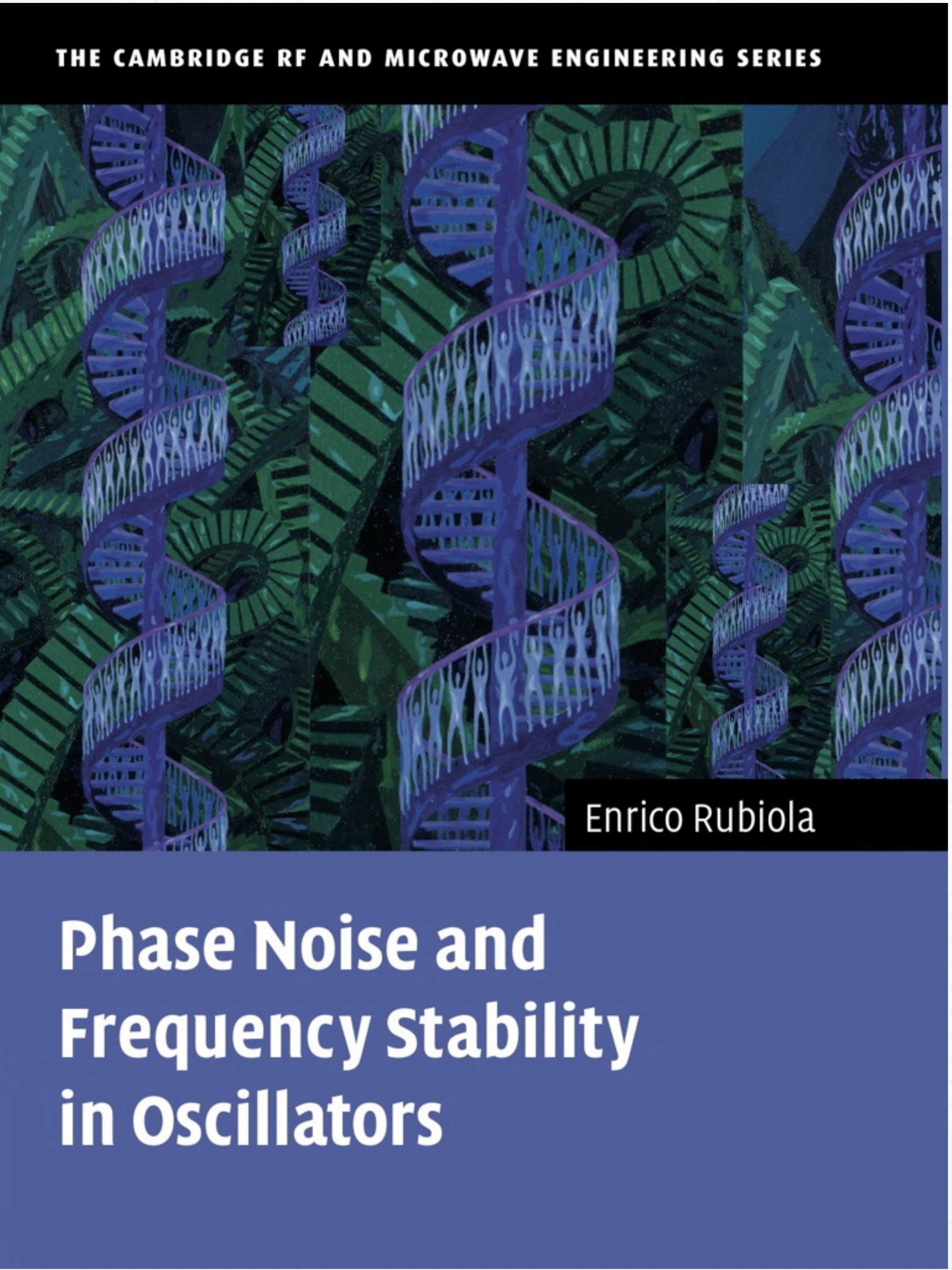






Conclusions

Phase noise and frequency stability in oscillators



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Contents

- Forewords (L. Maleki, D. B. Leeson)
- Phase noise and frequency stability
- Phase noise in semiconductors & amplifiers
- Heuristic approach to the Leson effect
- Phase noise and feedback theory
- Noise in delay-line oscillators and lasers
- Oscillator hacking
- Appendix

E. Rubiola
Experimental methods in AM-PM noise metrology
— book project —

Front cover: The Wind Machines
Artist view of the AM and PM noise
Courtesy of Roberto Bergonzo, http://robertobergonzo.com

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I am grateful to Lute Maleki and to John Dick for numerous discussions during my visits at the NASA JPL, which are the first seed of my approach to the oscillator noise

I am indebted to a number of colleagues and friends for invitations and stimulating discussions
- dr. Tim Berenc, ANL, Argonne
- dr. Holger Schlarb, DESY, Hambourg
- prof. Theodor W. Hänsch and dr. Thomas Udem, MPQ, München

This material would never have existed without continuous discussions, help and support of Vincent Giordano, FEMTO-ST, over about 15 years

This presentation is based on

E.Rubiola, *Phase noise and frequency stability in oscillators*, Cambridge 2008,

and on the complementary material

E. Rubiola, R. Brendel, A generalization of the Leeson effect, arXiv:1004.5539 [physics.ins-det]

Please visit my home page http://rubiola.org

Dave and Enrico at the end of a tutorial

IEEE Frequency Control Symposium, S. Francisco, Ca, 1–5 May 2011

Photo by Barbara Leeson, Dave's wife

Summary of relevant points

- The Leeson effect consists in a phase-to-frequency conversion
  - fully explained as a phase (noise) integration
  - takes place below $f_L = \nu_0/2Q$
- The step response provides analytical solutions and physical inside. (Same formalism introduced by Oliver. Heaviside in network theory)
- Buffer noise and resonator instability add to the Leeson effect
- Amplifier phase noise
  - white noise: S_φ scales down as the carrier power P_0
  - flicker noise: S_φ is independent of P_0
- Numerous oscillator spectra can be interpreted successfully
- The amplitude-noise response is similar to phase noise, but gain compression provides stabilization at low frequencies
- The theory indicates that amplitude-phase coupling results in a deviation from the polynomial law
- Unified AM/PM noise that applies to resonator-oscillators and to delay-line oscillators, including optical oscillators

A bunch of free material is available on my home page

http://rubiola.org