Spring 2025

Scientific Instruments

— and —

Phase Noise and Frequency Stability in Oscillators

Lectures for PhD Students and Young Scientists

Updated
February 21, 2025

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Part 1: General

Part 2: Phase noise and oscillators

Part 3: The International System of Units SI

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Lecture 1

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Quantum noise
- Thermal noise
- Shot noise

ORCID 0000-0002-5364-1835

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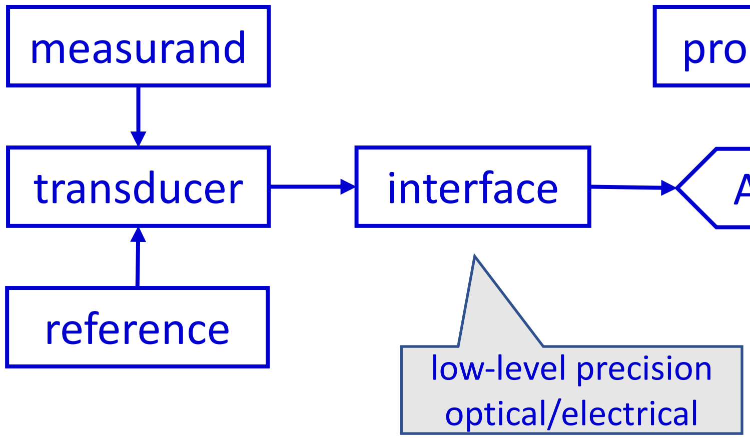
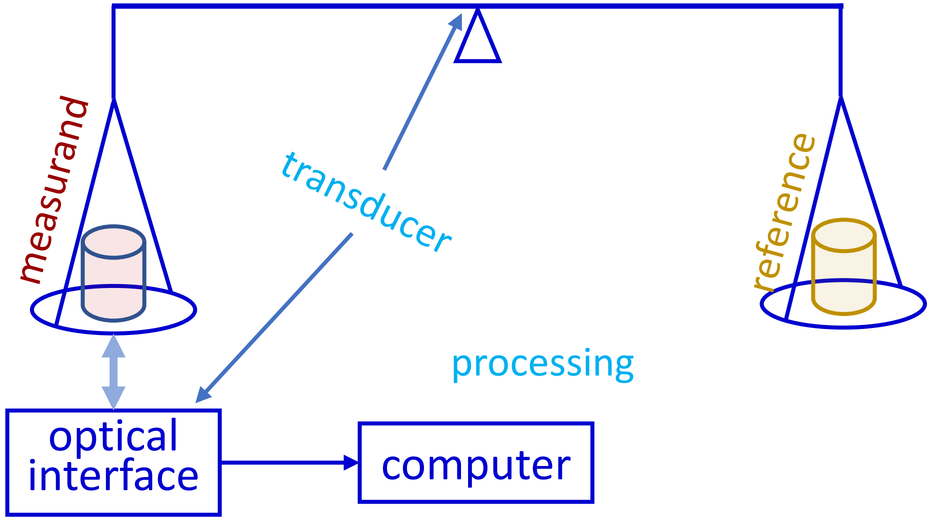
Misplaced

Instruments

basic scheme

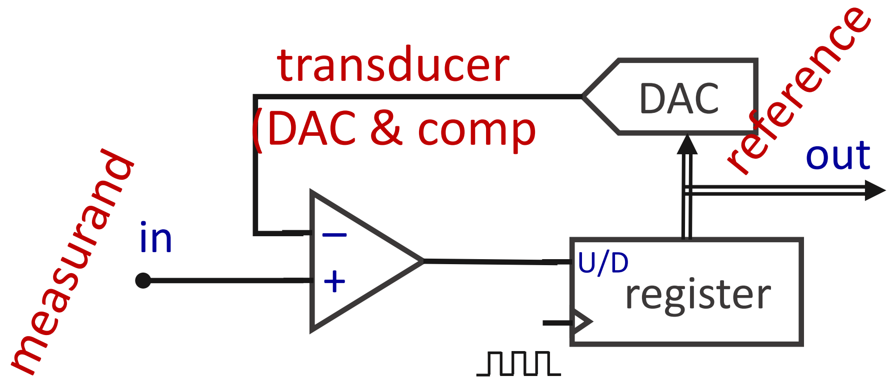


beam balance



caliper

Example, ADC



Up/Down counter
or Successive
Approximation
Register

Featured reading: P. Horowitz, W. Hill, *The Art of Electronics*, 3rd ed, Cambridge 2015

Thermal Noise

Planck constant $h = 6.02607015 \times 10^{-34}$ Js
Electron charge $e = 1.60207015 \times 10^{-19}$ C
Boltzmann constant $k = 1.380649 \times 10^{-23}$ J/K

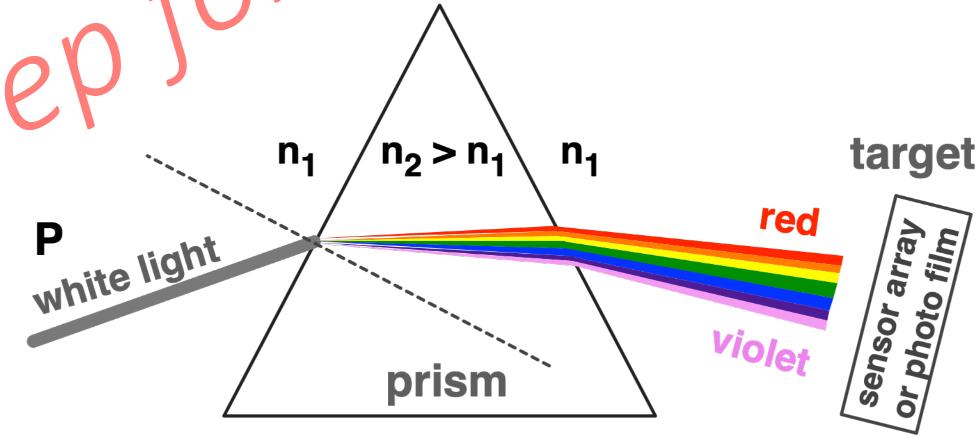
J. B. Johnson, Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928

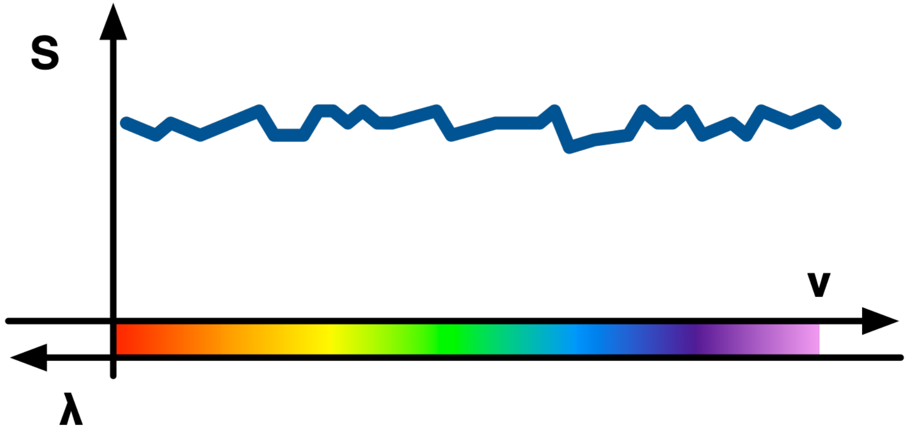
H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

The physical concept of spectrum

More precisely, Power Spectral Density

Keep for later





- The PSD is the distribution of power vs. frequency (power in 1-Hz bandwidth)
- The PS is the distribution of energy vs. frequency (energy in 1-Hz bandwidth)
- Power (energy) in physics is a square (integrated) quantity
- PSD –> W/Hz, or V^2/Hz, A^2/Hz, rad^2/Hz etc.

$$S_v(f) = \frac{\langle v_B^2(f) \rangle}{B}$$

Discrete: Δf is the resolution
Continuous: $\Delta f \to 0$

$$\frac{\text{average power in the bandwidth } B \text{ centered at } f}{\text{bandwidth } B}$$

The extended Planck law

Physical laws

Blackbody radiated energy

$$S(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1} \quad [\mathrm{W/Hz}]$$

At the receiver input

$$S(\nu) = h\nu + \frac{h\nu}{e^{h\nu/kT} - 1}$$

The additional $h\nu$ is the zero point energy

Nawrocki, Eq.1.13, Göbel-Siegner, Eq.2.10

Receiver

Thermal regime $h\nu \ll kT$

$$e^{h\nu/kT} \simeq 1 + h\nu/kT$$

$$S(\nu) = kT$$

Quantum regime $h\nu \gg kT$

$$e^{h\nu/kT} \gg 1$$

$$S(\nu) = h\nu$$

cutoff frequency $\nu_c = \frac{kT}{h}\ln(2)$

Featured reading:

Chapter 1, W. Nawrocki, *Introduction to quantum metrology* 2nd ed, Springer 2019

Chapter 2, E. O. Göbel, U. Siegner, *The new International System of units*, Wiley VCH 2019

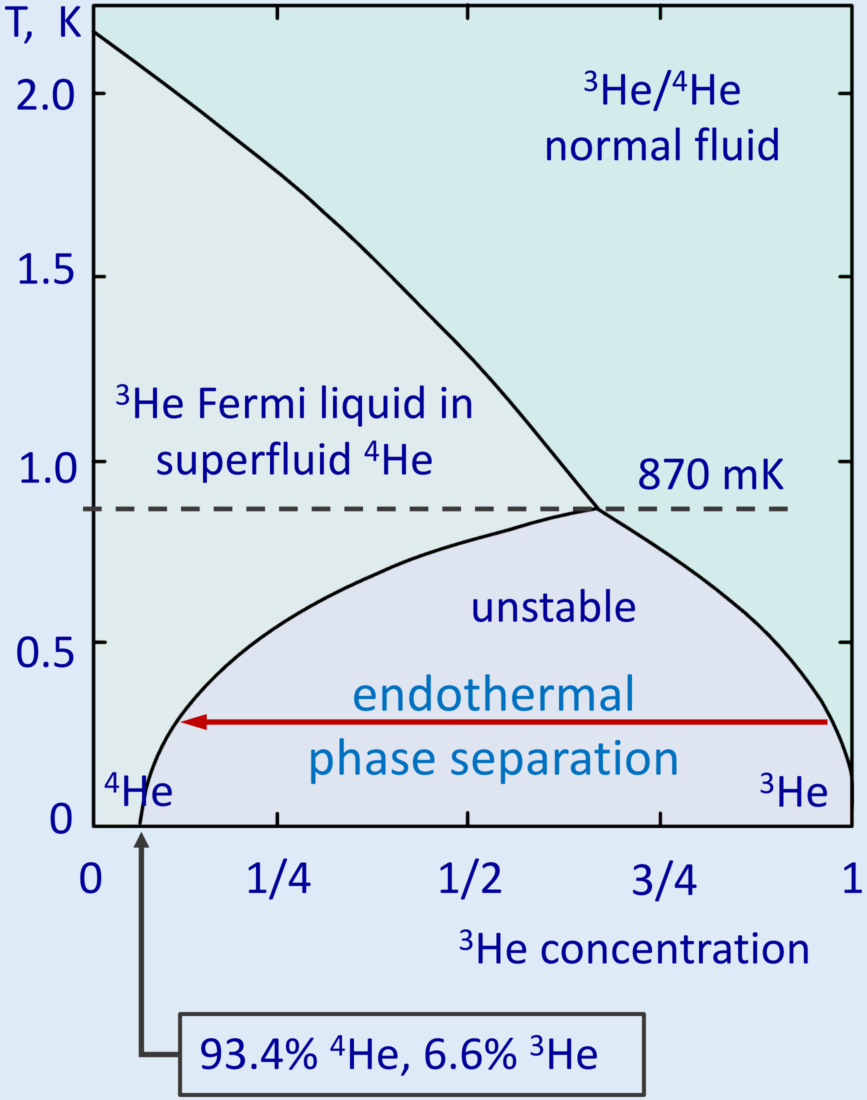
Cutoff frequency

$$\nu_c = \frac{kT}{h}\ln(2)$$

Reference	T, K	ν	λ
room	300	4.33 THz	69.2 µm
Liquid N_2	77	1.11 THz	270 µm
Liquid He	4.2	60.7 GHz	4.94 mm
^{3}He/^{4}He	0.01	144 MHz	2.08 m

POI – The dilution refrigerator

- ^{4}He is a boson
  - Superfluid at low temperature
- ^{3}He is a fermion
  - Pauli exclusion principle
  - Fermi liquid at low temperature
- Cooling process
  - Pre-cool the mixture to 1 K (cryocooler)
  - A capillary with large flow resistance cools to 0.5-0.7 K
  - The fluid is unstable
  - Phase separation is endothermal



Dilution refrigerator at the FEMTO-ST Institute

Theory: Heinz London, early 1950s
Implementation: 1964, Kamerlingh Onnes Lab, Leiden
H. K. Onnes (Nobel 1913) liquefied He (1908) and discovered the superconductivity of Hg (1911)

Featured reading:
Chapter 9, S. W. Van Sciver, Helium cryogenics 2nd ed., Springer 2012

The “Soul” of thermal noise

Thermal noise is blackbody radiation transmitted through an electrical line

It has two degrees of freedom, each has energy $kT/2$

electric and magnetic field

– or –

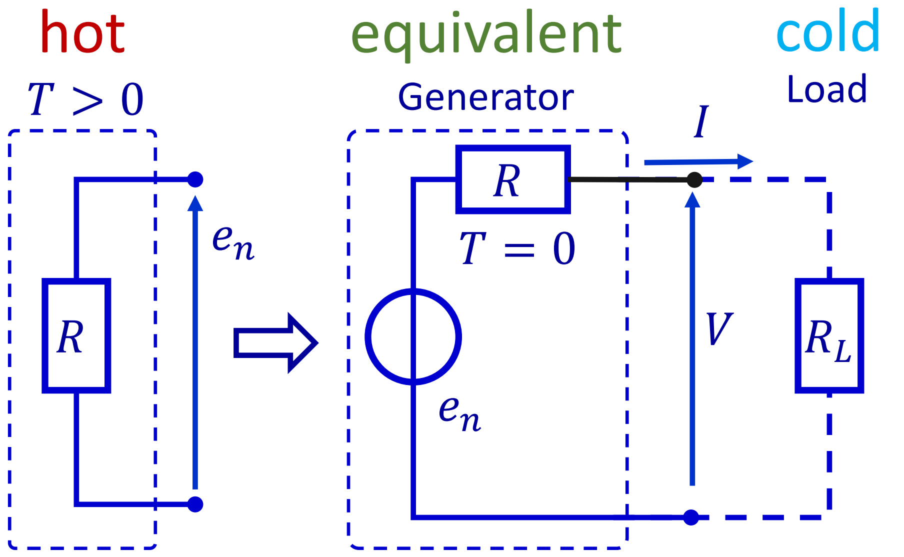
two polarization states

$$E_C = \frac{1}{2}CV^2 \rightarrow \frac{1}{2}kT$$

$$E_L = \frac{1}{2}LI^2 \rightarrow \frac{1}{2}kT$$

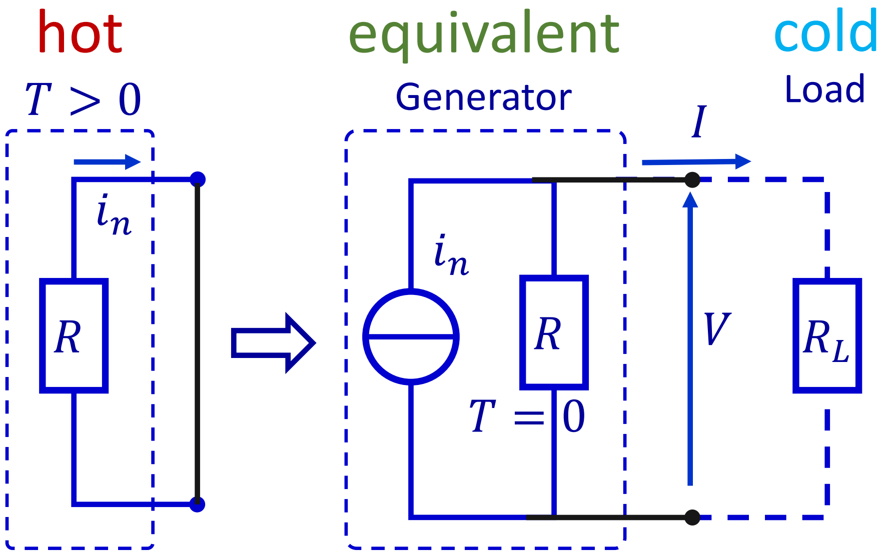
Thévenin and Norton models

Thévenin model



Thermal EMF $V_G \to e_n \equiv \sqrt{S_v}$ $[\mathrm{V}/\sqrt{\mathrm{Hz}}]$

Norton model



Thermal current $I_G \to i_n \equiv \sqrt{S_i}$ $[\mathrm{A}/\sqrt{\mathrm{Hz}}]$

Maximum power transfer $R_L = R_G$

$$V = \frac{1}{2} V_{\text{open}} \qquad I = \frac{1}{2} I_{\text{short}} \qquad P = \frac{1}{4} V_{\text{open}} I_{\text{short}}$$

Jargon: the *available* power/voltage/current is the $P/V/I$ delivered with $R_L = R$

Thermal noise

Terminated resistor (hot —> cold)

$$S = kT$$ W/Hz

$$S_V = kTR$$ V²/Hz

$$S_I = kT/R$$ A²/Hz

Two resistors at different temperature

$$S = k(T_2 - T_1)$$

$$S_V = 4kTR$$ Open circuit

$$S_I = 4kT/R$$ Short circuit

Bandwidth limited by cables / waveguide

Noise of a
50 Ω resistor

Reference	*T*, K	Available W/Hz	Open pV/√Hz	Short pA/√Hz
Room (approx.)	300	$4.14{\times}10^{-21}$	910	18.2
T_0 (RF electronics)	290	$4.00{\times}10^{-21}$	895	17.9
Dry ice (–78.5 °C)	194.7	$2.69{\times}10^{-21}$	733	14.7
Liquid N_2	77	$1.06{\times}10^{-21}$	461	9.22
Liquid He	4.2	$5.80{\times}10^{-23}$	108	2.15
^{3}He/^{4}He	0.01	$1.38{\times}10^{-25}$	5.25	0.105

J.B. Johnson JB,Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928
H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

The Harry Nyquist's article

Image user Quibik, Wikimedia

All schematics are from H. Nyquist 1928

Thermal equilibrium

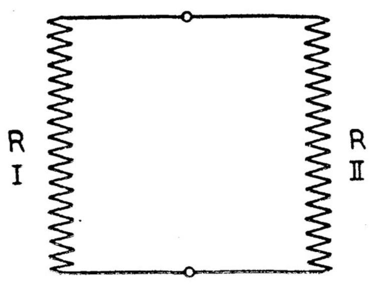


Fig. 1.

Thermal equilibrium also applies to any frequency (interval)

EMF E is a function of R, T and f only

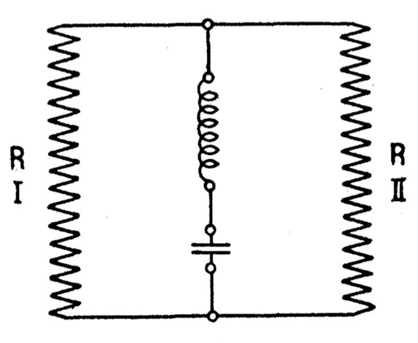


Fig. 2.

Loss-free, terminated electrical line

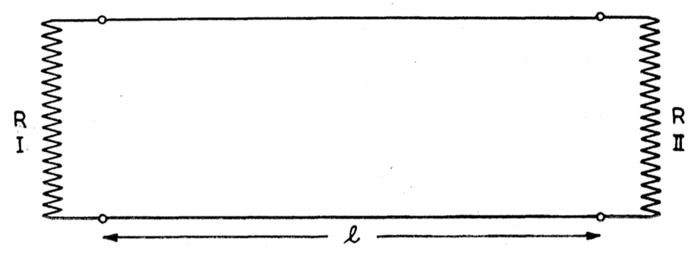


Fig. 3.

After thermal equilibrium, isolate the line (short at both ends).

Modes at $\nu = n\,c/\ell$

ν = frequency, c = velocity

Energy kT per mode

$$dE = 2\ell kT\,d\nu\,/\,c$$

Average power in frequency $d\nu$, and in time ℓ/c is $kT\,d\nu$

Extension to electrical circuits

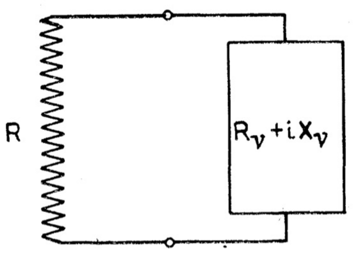


Fig. 4.

Energy per degree of freedom

$$h\nu/(e^{h\nu/kT}-1)$$

instead of kT

Conclusion

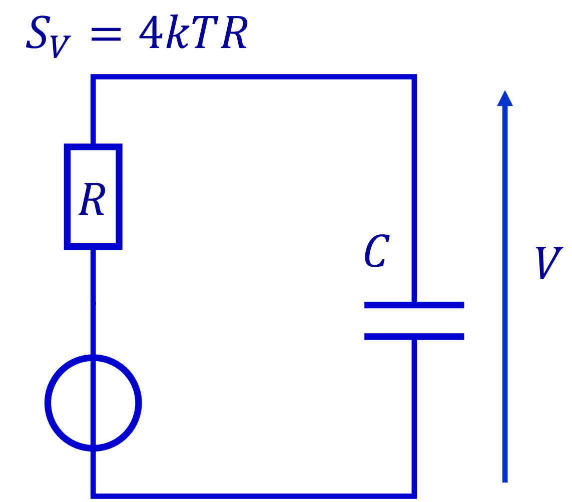
$$E_\nu^2\,d\nu = 4R_\nu\,h\,d\nu\,/\,(e^{h\nu/kT}-1)$$

J. B. . Johnson, Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928

H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

Thermal noise across a capacitor

Beware of CMOS gates and Track/Hold circuits



$$\langle V^2 \rangle = kT/C, \qquad \forall R$$

0.1 pF	200 µV	20 aC	125 e
1 pF	64 µV	64 aC	400 e
10 pF	20 µV	200 aC	1250 e
100 pF	6.4 µV	640 aC	4000 e
1 nF	2 µV	2 fC	12500 e

Proof (stat physics)

Capacitor $E = \frac{1}{2} C V^2$

The energy fluctuation per degree of freedom is

$$E = kT/2$$

at thermal equilibrium

Mean square fluctuation

$$C\Delta(V^2/2) = kT/2$$

Conclusion

$$\langle V^2 \rangle = kT/C$$

Sarpeshkar R, Delbruck T, Mead CA - White Noise in MOS Transistors and Resistors - Circuits and Devices, November 1993

Proof (circuit theory)

Voltage

$$S_V = 4kTR$$

Transfer function

$$|H(f)|^2 = \frac{1}{1 + (2\pi f RC)^2}$$

Mean square fluctuation

$$\langle V^2 \rangle = \int_0^\infty 4kTR\, |H(f)|^2 \, df$$

Conclusion, R cancels, and

$$\langle V^2 \rangle = kT/C$$

Trivial exercise

Shot Noise

Electron charge $e = 1.60207015 \times 10^{-19}$ C

W. Schottky, „Über spontane Stromschwankungen in verschiedenen Elektrizitatsleitern“, Annalen der Physik 362(23) p541-567, 1918 (in German). Get free pdf from Zenodo

Open access English translation "On spontaneous current fluctuations in various electrical conductors" by Martin Burkhardt, with additional editing by Anthony Yen

The exponential distribution

A cell emitting particles at random, at the average rate of ϕ events/s

In the literature we often find λ instead of ϕ, and x instead of t

Probability Density Function

PDF $p(t;\phi) = \phi e^{-\phi t}, t \geq 0$

Mean $\mu = 1/\phi$, Variance $\sigma^2 = 1/\phi^2$

$$\mu = \int t\, p(t;\phi)\, dt = 1/\phi$$

$$\sigma^2 = \int (t-\mu)^2\, p(t;\phi)\, dt = 1/\phi^2$$

Properties

Memoryless $\mathbb{P}\{T > s + t | T > s\} = \mathbb{P}\{T > t\}$

T is the waiting time

- Statistically, T is the same starting at 0 or at s, if the particle did not show up
- Maximum differential entropy —> maximum entropy for a given μ

This describes “emissions” in physics

- Electrons and holes in a junction
- Photons
- Radioactive decay (assuming that the nuclei are not lost)

Featured reading:
W. Feller, *Introduction to probability theory and its applications*, 2nd ed, Wiley. Vol. I, 1957, Vol. II, 1970

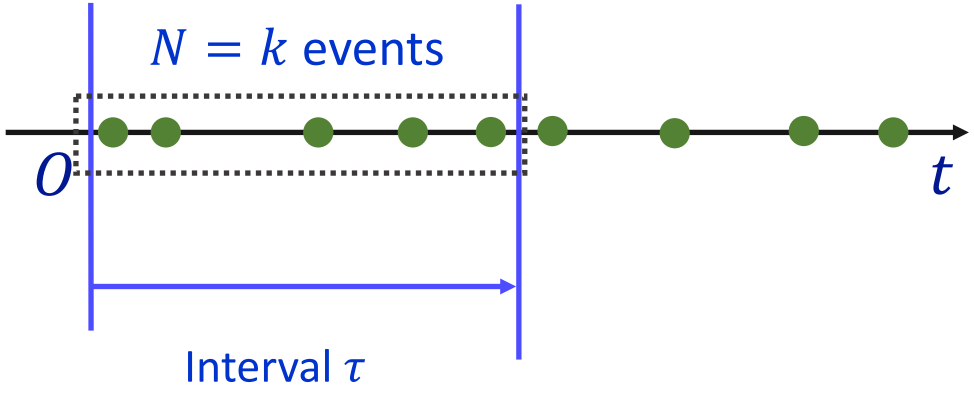
Vol. 1, Sec. XVII-6 provides a proof that in a memory-less process, the tail of the distribution has to be of the form $u = \exp -\lambda t$ (or zero), and nothing else. See also vol.II, Sec. I-3

Homogeneous Poisson process

An ensemble of memoryless and statistically independent cells emitting at random at the average rate (flux) of ϕ events/s

$$\mathbb{P}\{N(\tau)=k\}=\frac{(\phi\tau)^k}{k!}e^{-\phi\tau}$$

$\mathbb{P}$ is the probability that the number N of particles emitted from time 0 to τ equals k



My notebook vol. XXIII p. 49

Properties

average

$\mathbb{E}\{N(\tau)\}=\phi t$ written as $\boxed{\mu=\phi\tau}$

variance

$\mathbb{E}\{[N(\tau)-\mu]^2\}=\phi t$ $\boxed{\sigma^2=\phi\tau}$

signal-to-noise ratio

$SNR=\sigma/\mu$ $\boxed{SNR=\sqrt{N}}$

physical meaning of ϕ

$$\lim_{t\to\infty}\frac{N(t)}{t}=\phi$$

average no of events / time,
flux in the case of particle emission

W. Feller, Introduction to Probability Theory and Its Applications, vol.II, 2nd ed., Wiley 1970

Shot noise

Electrical charge

e	$\mathbb{E}(Q) = \phi\tau e$	[C]
e^2	$\mathbb{V}(Q) = \phi\tau e^2$	[C²]
$e^2\tau$	$S_Q(f) = 2\phi\tau^2 e^2$	[C²/Hz]

Electrical current

e/τ	$\mathbb{E}(I) = \phi e$	[A]
e^2/τ^2	$\mathbb{V}(I) = \phi\tau(e/\tau)^2$	[C²]
e^2/τ	$S_I(f) = 2\phi\tau^2(e^2/\tau^2)$	
	$= 2\phi e^2 = 2eI$	[A²/Hz]

Photon energy

$h\nu$	$\mathbb{E}(Q) = \phi\tau h\nu$	[J]
$(h\nu)^2$	$\mathbb{V}(Q) = \phi\tau(h\nu)^2$	[J²]
$(h\nu)^2\tau$	$S_Q(f) = 2\phi\tau^2(h\nu)^2$	[J²/Hz]

Photon power

$h\nu/\tau$	$\mathbb{E}(I) = \phi h\nu$	[W]
$(h\nu)^2/\tau^2$	$\mathbb{V}(I) = \phi\tau(h\nu/\tau)^2$	[W²]
$(h\nu)^2/\tau$	$S_I(f) = 2\phi\tau^2[(h\nu)^2/\tau^2)$	
	$= 2\phi(h\nu)^2$	[W²/Hz]

More on the shot noise in electrical current

Metallic conductors

- Long-range correlation between electrons
- The electrical current propagates as a field, not as separate electrons

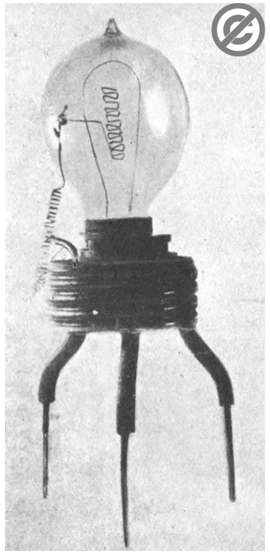
There is no shot noise

Shot noise in resistors is quite small too

Semiconductor junctions, vacuum tubes, electron guns…

- Carriers propagate as separate electrical charges
- There is no correlation between carriers

Shot noise shows up



Fleming diode

https://upload.wikimedia.org/wikipedia/commons/4/4d/Fleming_valves.jpg (excerpt)

Trick – Shot noise *may cancel* at the output of emitter-follower amplifiers

Horowitz P, Hill W, , *The Art of Electronics* 3rd ed (2015), §8.3.5

Quantum Limit

Planck constant $h = 6.02607015{\times}10^{-34}$ Js
Electron charge $e = 1.60207015{\times}10^{-19}$ C
Boltzmann constant $k = 1.380649{\times}10^{-23}$ J/K

This section is based upon

E. O. Göbel, U. Siegner, The New International System of Units (SI), Wiley VCH 2019

See also

M. Gläser, M. Kochsiek (Ed.), Handbook of Metrology vol.1-2, Wiley VCH 2010 (Amazon DE)

V. B. Braginsky, F. Ya. Khalili, Quantum Measurement, Cambridge 1992

Fundamental quantum limit

Photon energy

$$E = h\nu$$

Heisenberg Principle:
The minimum action H is

$$H \gtrsim h$$

If p and x are momentum and position,

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

Planck constant
$h = 6.02607015 \times 10^{-34}$ Js
(exact)

Application to the measurement

Energy extracted from the system in the time τ

$$E \gtrsim h/\tau \quad \text{or} \quad E \gtrsim hB$$

$B = 1/\tau$

Measurement bandwidth

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019
Also: V. B. Braginsky, F. Ya. Khalili, Quantum Measurement, Cambridge 1992

Quantum limit in the capacitor

$$E \gtrsim h/\tau$$

Voltage

Energy

$$\boxed{\frac{1}{2}CV^2} \gtrsim \frac{h}{\tau}$$

$$V \gtrsim \sqrt{\frac{2h}{\tau C}}$$

Charge

Energy

$$\boxed{\frac{1}{2}\frac{Q^2}{C}} \gtrsim \frac{h}{\tau}$$

$$Q \gtrsim \sqrt{\frac{2hC}{\tau}}$$

Use large C and τ

$C = 1.5$ nF, $\tau = 10$ ms

$V = 9.4$ pV

Use small C and large τ

$C = 2$ pF, $\tau = 10$ ms

$Q = 5.15\times10^{-22}$ C

$Q \ll e = 1.6\times10^{-19}$ C

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Quantum limit in the inductor

$$E\tau \gtrsim h$$

Current

$$\overset{\text{Energy}}{\boxed{\frac{1}{2}LI^2}} \gtrsim \frac{h}{\tau}$$

$$I \gtrsim \sqrt{\frac{2h}{\tau L}}$$

Magnetic flux

$$\overset{\text{Energy}}{\boxed{\frac{1}{2}\frac{\Phi^2}{L}}} \gtrsim \frac{h}{\tau}$$

$$\Phi \gtrsim \sqrt{\frac{2hL}{\tau}}$$

Use large τ and L

$L = 200$ mH, $\tau = 100$ ms

$I = 25.7$ aA

Use small L and large τ

$L = 2.5$ nH, $\tau = 100$ ms

$\Phi = 5.8\times10^{-21}$ Wb

$\mu_0 \simeq 1.257$ μH/m
$L = \mu_0\ell \rightarrow \ell = 2$ mm
$\Phi_0 = \frac{h}{2e} = 2.0678\times10^{-15}$ Wb

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Quantum limit in the resistor

$$E\tau \gtrsim h$$

Voltage

$$\overset{\text{Energy}}{\boxed{\frac{V^2}{R}\tau}} \gtrsim \frac{h}{\tau}$$

$$V \gtrsim \sqrt{hR}\,\frac{1}{\tau}$$

Current

$$\overset{\text{Energy}}{\boxed{RI^2\tau}} \gtrsim \frac{h}{\tau}$$

$$I \gtrsim \sqrt{\frac{h}{R}\frac{1}{\tau}}$$

Use small R and large τ

$R = 50\ \Omega,\ \tau = 100$ ms

$V = 1.82$ fV

Use large R and τ

$R = 1\ \mathrm{M}\Omega,\ \tau = 100$ ms

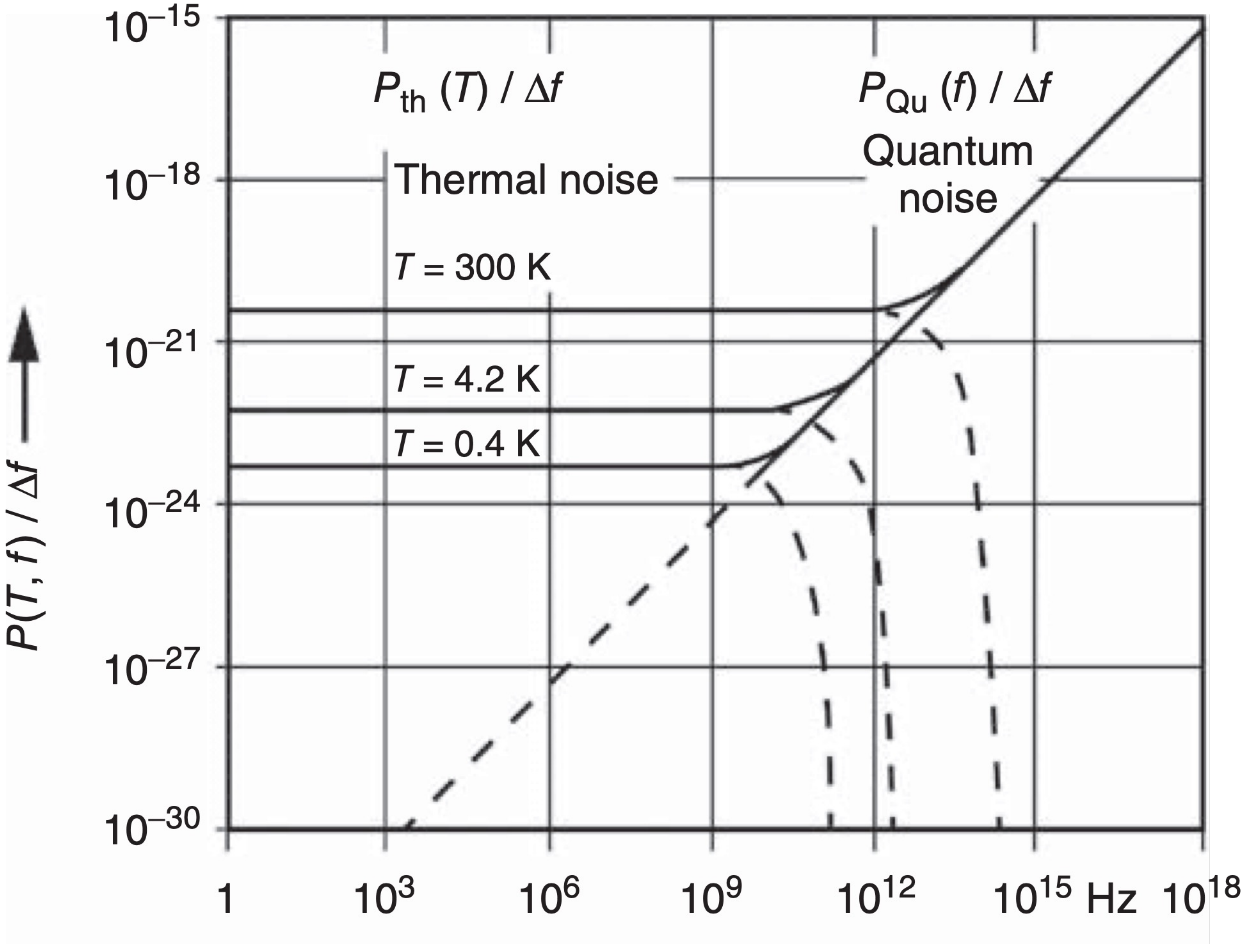
$I = 2.57\times10^{-19}$ A

$e = 1.6\times10^{-19}$ C

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Thermal vs quantum noise

Odd notation, $P/\Delta f$ is a PSD



This figure is from

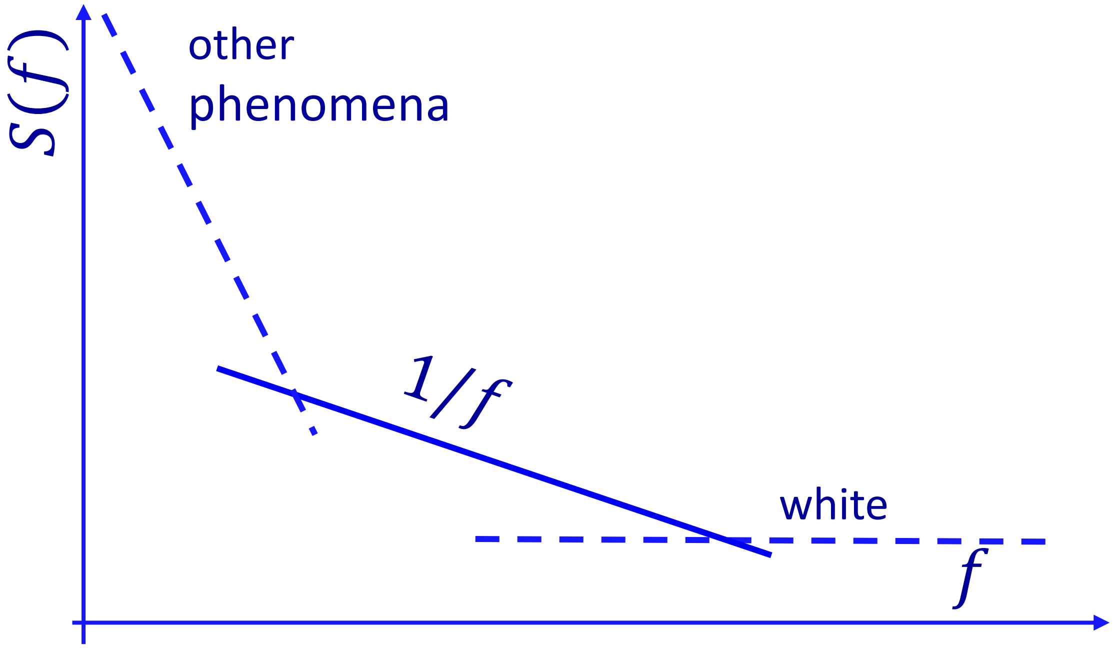
Siebert, B.R.L. and Sommer, K.D. (2010) in *Uncertainty in Handbook of Metrology*, vol. **2** (eds M. Gläser and M. Kochsiek), Wiley-VCH Verlag GmbH, Weinheim, pp. 415–462.

ISBN 978-3-527-40666-1

Flicker $(1/f)$ Noise

Ubiquitous phenomenon in science and technology

Flicker $(1/f)$ noise



- Extremely weak noise phenomenon
- A major issue in some domains of spectral analysis
- Relevant in cryogenic nanodevices and qbits
- Resolution cannot be improved by increasing the measurement time

- Observed in a large variety of phenomena: conductance, electrical contacts, semiconductors, vacuum tubes, music, radio broadcasting, Internet, pulsars, squids, Nile river floods, earthquakes, fractals, etc.
- Observed exact $1/f$ slope up to 8 decades in electronic circuits
- Other fields, $1/f^{\alpha}, \alpha = 0.5 \dots 1.5$
- Discovered by Johnson, 1925
- Studied in carbon microphones and in the fluctuation of resistivity, >1930
- Well explained in some cases (magnetics…)
- No unified theory

Q: Is this Type A or Type B evaluation of uncertainty?

Integrated flicker noise is extremely small

How small the $1/f$ noise can be?

$$\sigma^2 = \int_a^b \frac{1}{f}\, df = \ln\left(\frac{b}{a}\right)$$

Let's consider the crazy-widest frequency range

$$a = \frac{1}{A_U}$$

Age of Universe

$$b = \frac{1}{2\pi\tau_P}$$

Planck time (Gauss)

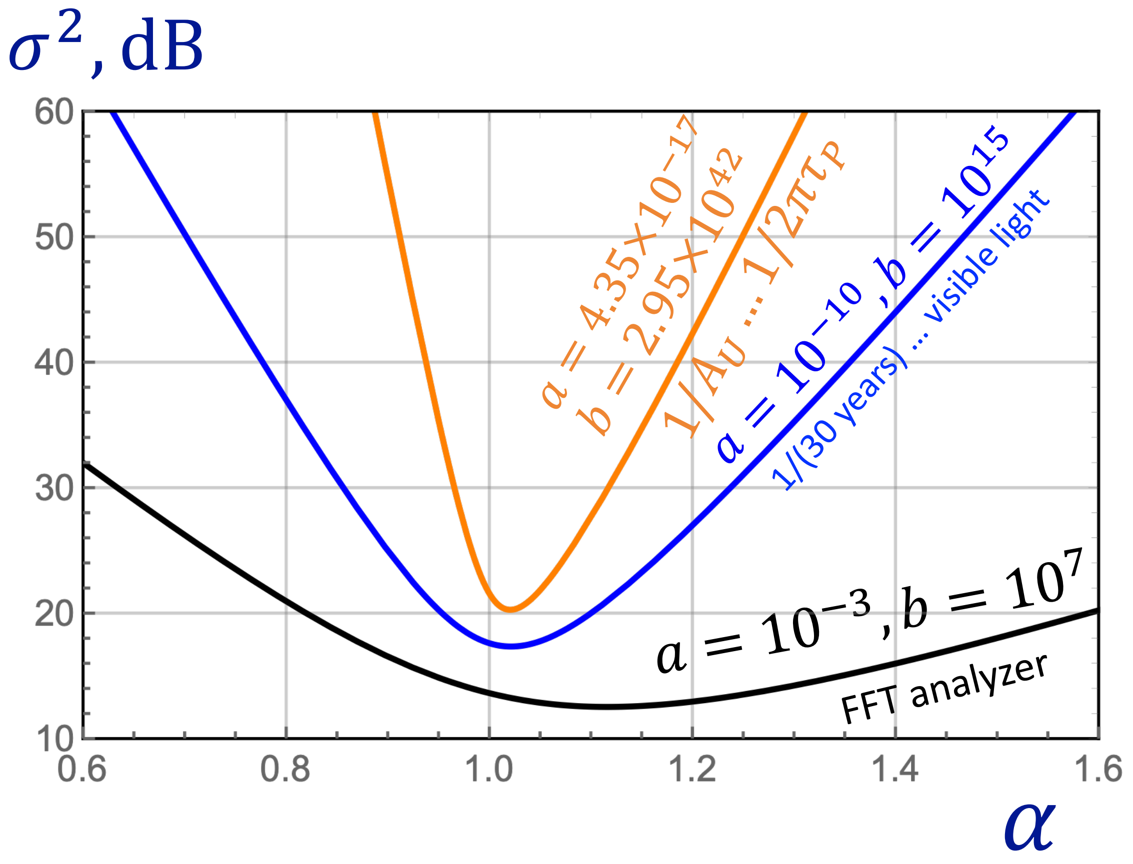
$$A_U = 4.35\times10^{17}\text{s} \;\; (13.8 \text{ By})$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \simeq 5.39\times10^{-44} \text{ s}$$

$$\ln\left(\frac{b}{a}\right) = \ln\left(\frac{1/2\pi t_P}{1/A_U}\right) = 138.4 \qquad (21.4\text{dB})$$

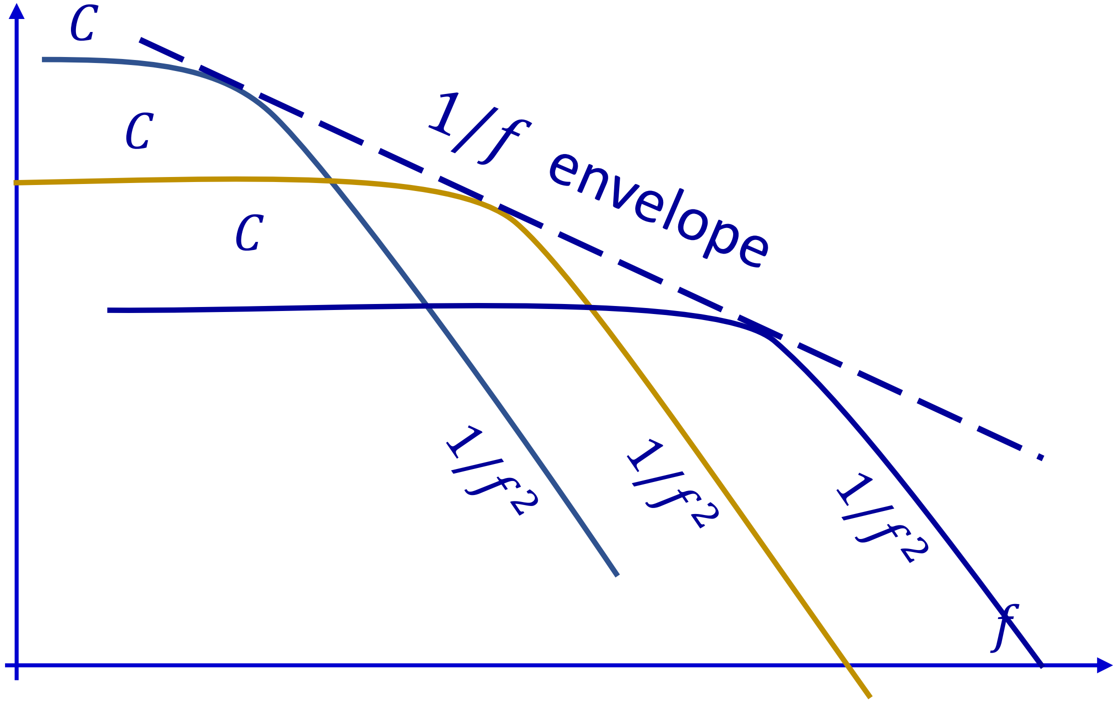
Integrated $1/f^{\alpha}$ noise is small even for $\alpha \neq 1$

$$\sigma^2 = \int_a^b \frac{1}{f^{\alpha}} df$$



Distribution of relaxation times

Uniform (random) distribution of time constants on a log-log scale



Featured reading: E. Milotti, $1/f$ noise, a pedagogical review, arXiv.physics 0204033, April 2002

1/ƒ noise and FD theorem

Flicker ($1/f$) dimensional fluctuation is powered by thermal energy

Debye-Einstein theory for heat capacity

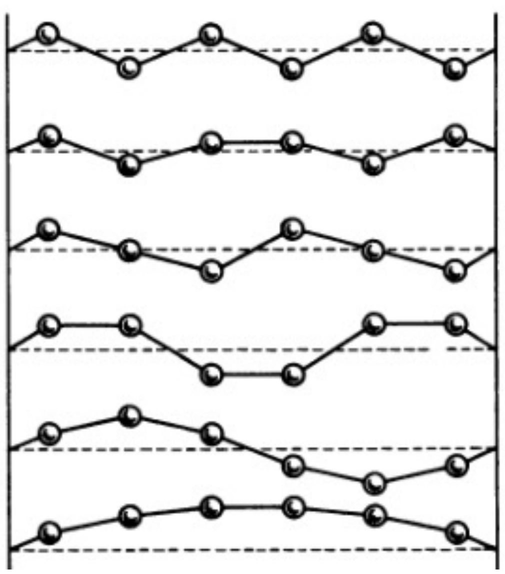


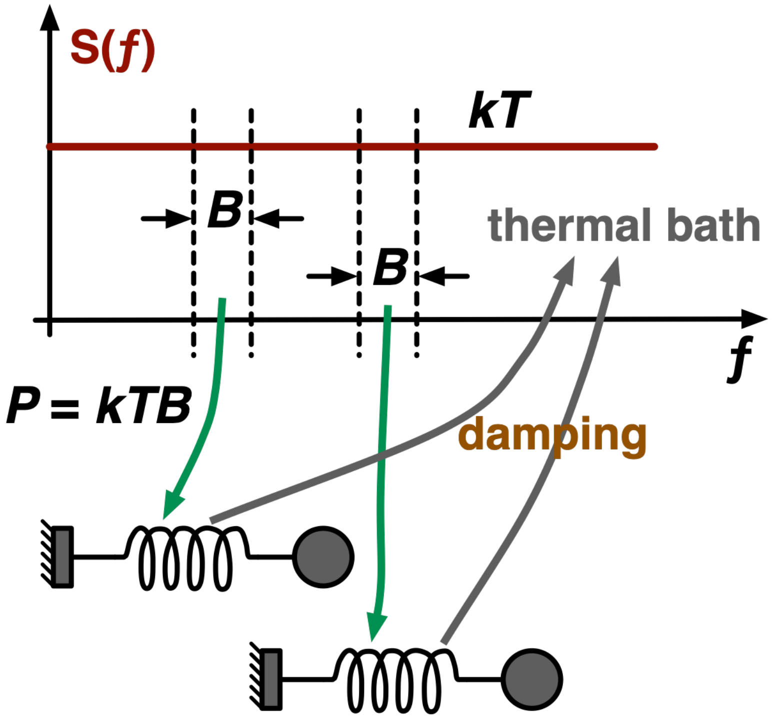
Figure 8-(b) from A. Holden A, The nature of solids, ©1965 Columbia University Press. Dover unabridged, 1992

A single theory explains

- Heat capacity
- Thermal expansion
- Elasticity

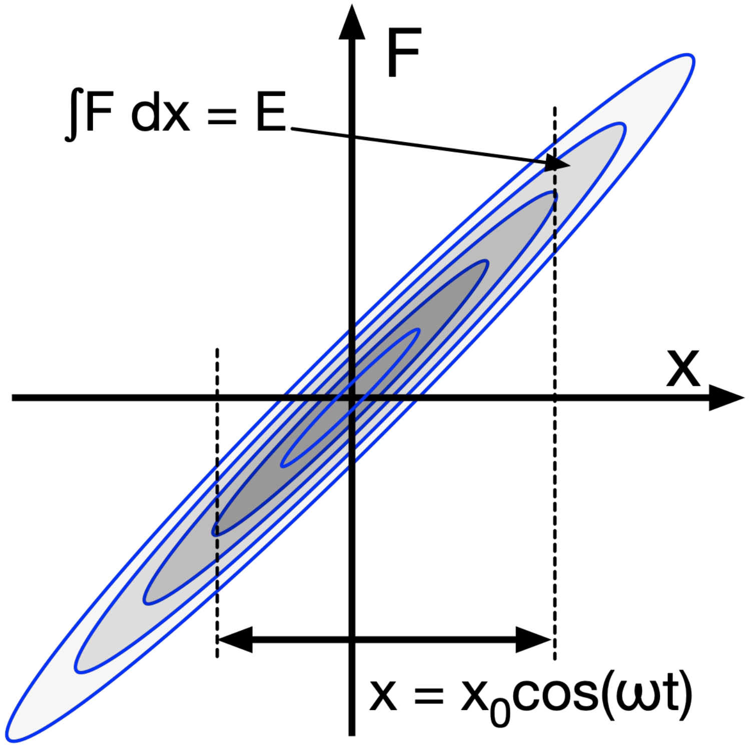
… and their fluctuations

Fluctuation Dissipation theorem in a nutshell



Thermal equilibrium applies to all portions of spectrum

Thermal $1/f$ from structural dissipation



There is no viscous dissipation in solids

Dissipation is structural (hysteresis)

Structural dissipation

micro/nanoscale, instantaneous

Dissipated energy $E = \int F dx$

Small vibrations

The hysteresis cycle keeps the aspect ratio

$E \propto x_0^2$ Energy lost in a cycle

Thermal equilibrium

$P = kT$ in 1 Hz BW

$P \propto kTx_0^2$

$x_0^2 \propto 1/f$ —> flicker

Bibliography about flicker

- C. J. Christiansen, G. L. Pearson, Spontaneous Resistance Fluctuations in Carbon Microphones and Other Granular Resistances, BSTJ 15(2) p.197-223, April 1937 (OA). Arguably, the discovery of flicker.
- F. N. Hooge, $1/f$ noise is no surface effect, Phis Lett 29(3) p.139-140, 21 April 1969 (OA). Classical article.
- D. J. Levitin, P. Chordia, V. Menon, Musical Rhythm Spectra from Bach to Joplin Obey to $1/f$ Power Law, Proc. Nat. Academy of Science 109(10) p.716-3720, February 2012 (OA).
- A. L. McWhorter, $1/f$ Noise and Germanium Surface Properties, Proc. Semiconductor Surface Physics p.207-228, June 1956 (PW). Classical article.
- E. Milotti, $1/f$ noise, a pedagogical review, arXiv.physics 0204033 (OA), April 2002.
- Paladino E et al., $1/f$ Noise, Implications for solid-state quantum information, Rev Modern Phys 86(2) p. 361-418, April-June 2014 (OA)
- Numata K, Kemery A, Camp J, Thermal-noise limit in the frequency stabilization of lasers with rigid cavities, Phys Rev Lett 93(25) 250602, December 2004 (PW).
- P. R. Saulson, Thermal Noise in Mechanical Experiments, Phys Rev D 42(8), October 1990 (PW).
- A. van der Ziel, Unified Presentation of $1/f$ Noise in Electronic Devices: Fundamental $1/f$ sources, Proc IEEE 76(3), March 1988 (PW).
- L. K. J. Vandamme, G. A. Trefan, A review of $1/f$ noise in bipolar transistors, Fluct Noise Lett 1(4) 2001 (PW).
- M. B. Weissman, $1/f$ noise and other slow, nonexponential kinetics in condensed matter, Rev Modern Phys 60(2) p.537-571, April 1988 (PW)

Lecture 1 ends here

Lecture 2

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Flicker noise
- General instrument architecture
- Noise in electronic devices

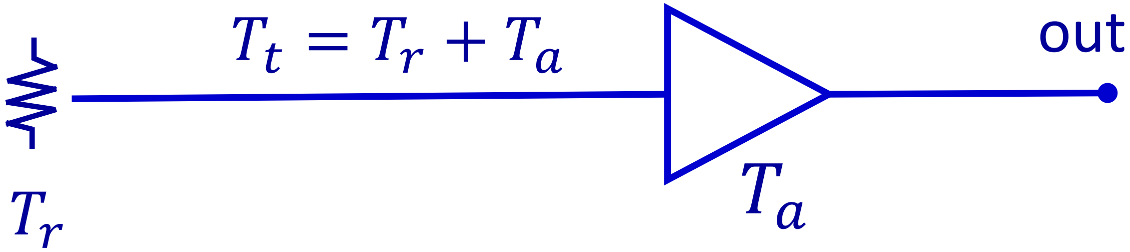
ORCID 0000-0002-5364-1835

home page http://rubiola.org

Equivalent noise temperature

Describe the noise of a device by analogy to thermal noise

Thermal noise $S(\nu) = kT$ constant, for $h\nu \ll kT$



T_a is the equivalent noise temperature of the amplifier, defined in specified conditions (physical temperature and input resistance)

Warning

- The noise temperature a radio-engineering concept
- The physical nature of noise does not matter
- Often misleading in optics: the shot noise contributes to the equivalent temperature

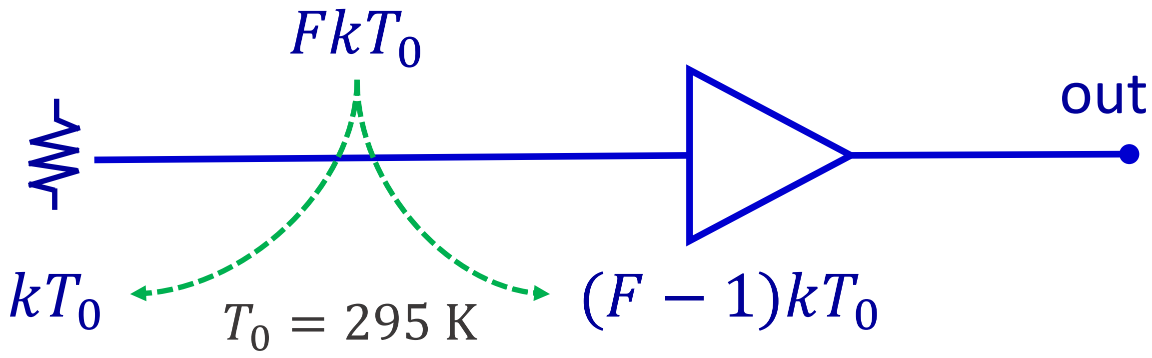
Equivalent temperature T_a is defined by $S(\nu) = k(T_a + T_r)$

Noise factor and noise figure

Noise factor $F = \dfrac{\text{Noise PSD with input termination at 290 K}}{\text{Portion of the above, due to the input termination alone}}$ Definition, IRE 1960

Noise Figure

$\mathrm{NF} = 10\ \log_{10}(F)$



$kT_0 = 4{\times}10^{-21}\mathrm{J}$

−174 dBm/Hz

290 K (17 °C) is a convenient round number

Assume that the whole circuit is at the reference temperature $T_0 = 290\ \mathrm{K}\ \ (17\ ^\circ\mathrm{C})$

The total noise referred to the amplifier input is FkT_0

amplifiers and RF/μw devices

$$FkT_0 = kT_e = k(T_a + T_0), \qquad T_0 = 290\ \mathrm{K}$$

$$F = \frac{(T_a + T_0)}{T_0} \quad \text{and} \quad T_a = (F-1)T_0$$

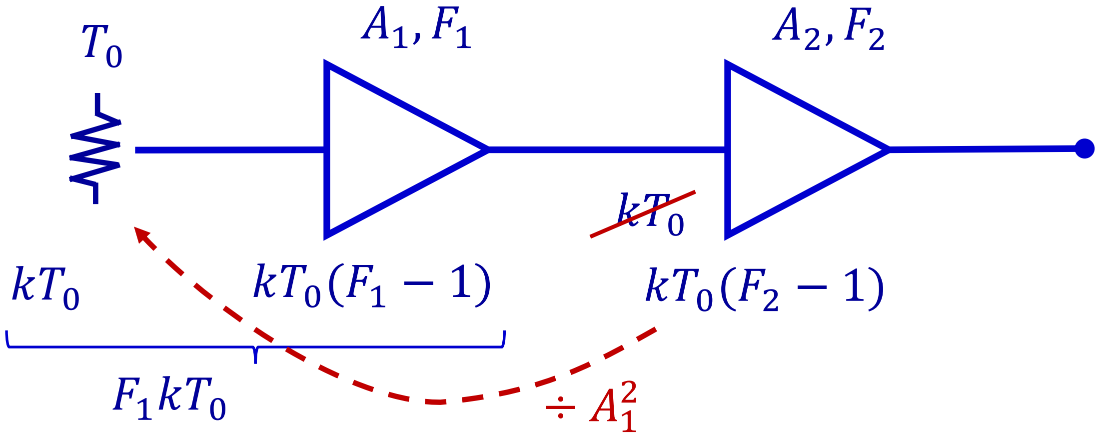
Warning: the noise figure is a radio-engineering concept, may be ***misleading*** in ***optics***

Belongs to PM noise

The Friis formula

H. T. Friis, Noise Figure of Radio Receivers, Proc IRE 32(7) p.419-422, July 1944

A = voltage gain
A^2 = power gain



$$N = F_1 kT_0 + \frac{(F_2 - 1)kT_0}{A_1^2} + \cdots$$

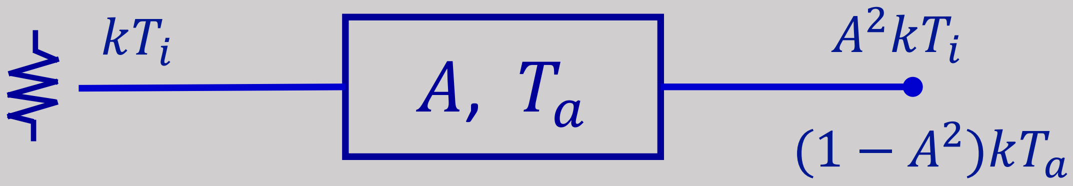
$$F = F_1 + \frac{(F_2 - 1)}{A_1^2} + \frac{(F_3 - 1)}{A_1^2 A_2^2} + \cdots$$

Good design —> main maybe likely negligible

Caveat

- Impedance matching is not included
- Notable conditions
  - Max power transfer
  - Lowest noise
  - Highest SNR

POI – Thermal noise of a dissipative device



$$S(f) = A^2kT_i + (1-A^2)kT_a$$

Describes noise in

- Cables
- Antennas
- Propagation in lossy medium

Arno A. Penzias and Robert W. Wilson (Nobel in Physics, 1978) knew about noise temperature when they measured the background cosmic radiation

Featured readings
A. A. Penzias, R. W. Wilson, A Measurement of Excess Antenna Temperature at 4080 Mc/s, Astrophys J Lett.142(1), p.419-421, 1965
J. D. Kraus, Antennas 2ed, McGraw Hill 1997, ISBN 0-07-035422-7
(The proof is found in Kraus, 1st ed., 1966, Sec.7-2b)

Noise contribution of the input resistor

- The attenuator makes no difference between “noise” and “signal”
- The input signal is “amplified” by a factor $A^2 < 1$

Noise contribution of the attenuator

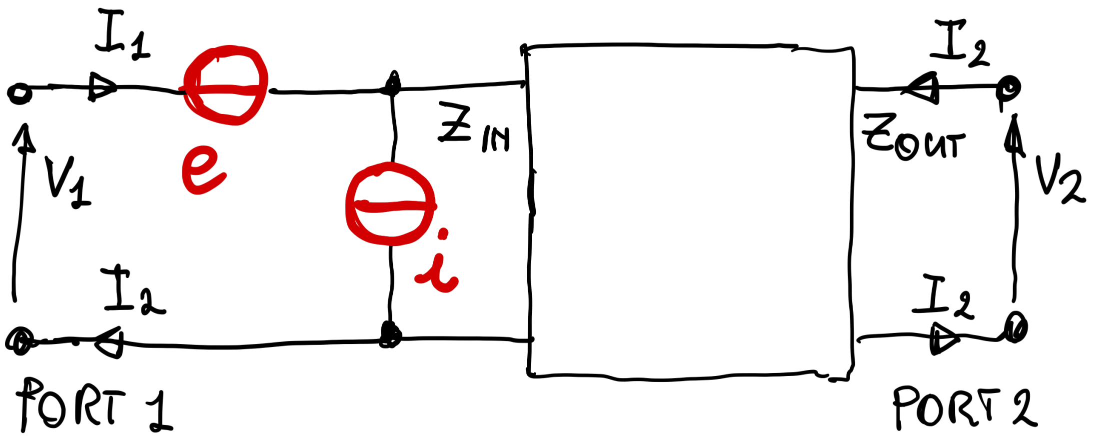
- At uniform temperature T, the sum of the contributions must be kT
- The input contributes A^2kT
- The attenuator contributes the complement $(1-A^2)kT$

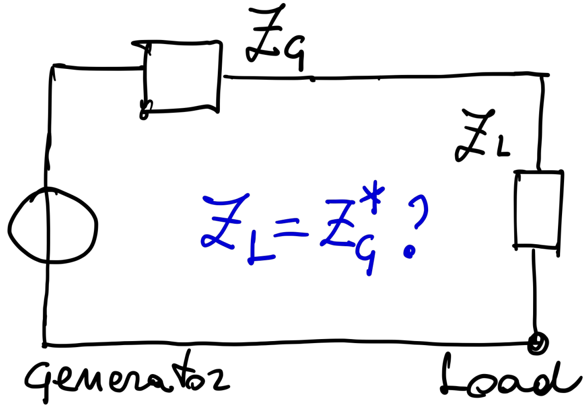
The factors A^2 and $1-A^2$ do not depend on temperature

Belongs to PM noise

The Rothe Dahlke model

e and i are the rms noise in 1 Hz bandwidth





Noise is modeled as a voltage generator $e(t)$ and a current generator $i(t)$

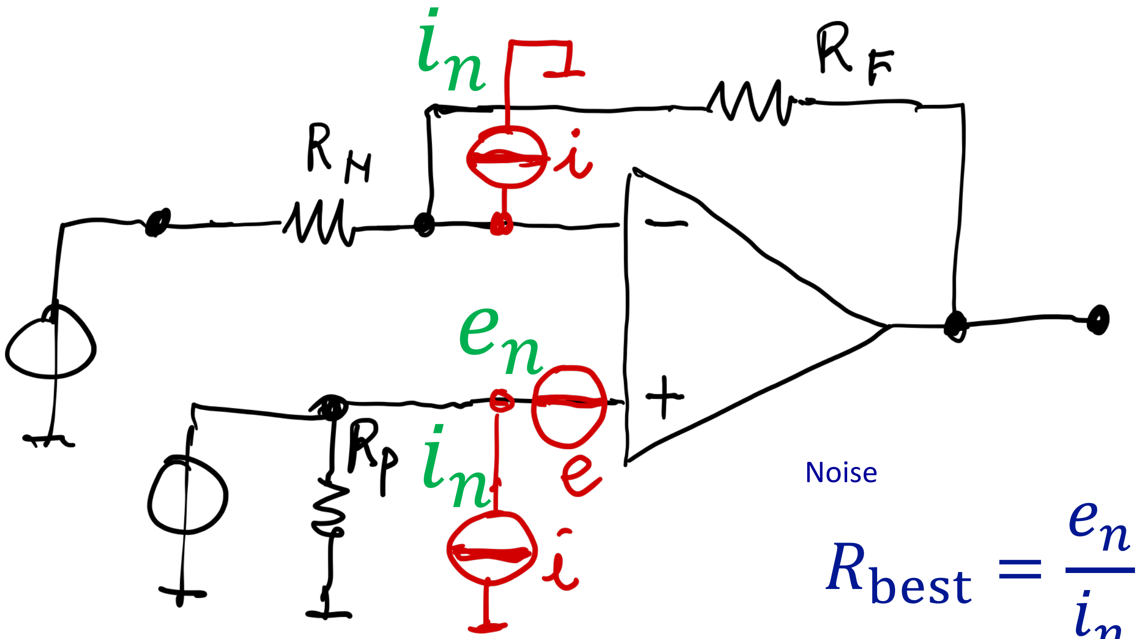
Three different impedance-matching criteria at Port 1 (the device is the load)

- Lowest noise: $Z_G = e_n/i_n$
- Maximum power: $Z_L = Z_G^*$
- Highest Signal-To-Noise Ratio (SNR): something in between

H. Rothe, W. Dahlke, Theory of Noisy Fourpoles, Proc IRE 44(6) p.811-818, June 1956
H. A. Haus, R. B. Adler, Circuit theory of linear noisy networks, John Wiley & Sons 1959

Noise in operational amplifiers

$a \oplus b = (1/a + 1/b)^{-1}$



Noise

$$R_{\text{best}} = \frac{e_n}{i_n}$$

(+thermal)

Noise resistance

$$R_{eq} = R_P + (R_N \oplus R_F)$$

Voltage

$$V = V_{OS} + R_P I_P - (R_N \oplus R_F) I_N$$

Split I_N and I_P into offset and bias, $I_{OS} \pm \frac{1}{2} I_B$

Bias $I_B = \frac{1}{2}(I_P - I_N)$, Offset $\mathrm{I}_{\mathrm{OS}} = I_P - I_N$

Total effect

$$V = V_{OS} +$$
$$+\frac{1}{2}[R_P - (R_F \oplus R_N)] I_B +$$
$$+ [R_P + (R_N \oplus R_F)] I_{OS}$$

Obvious extension to noise

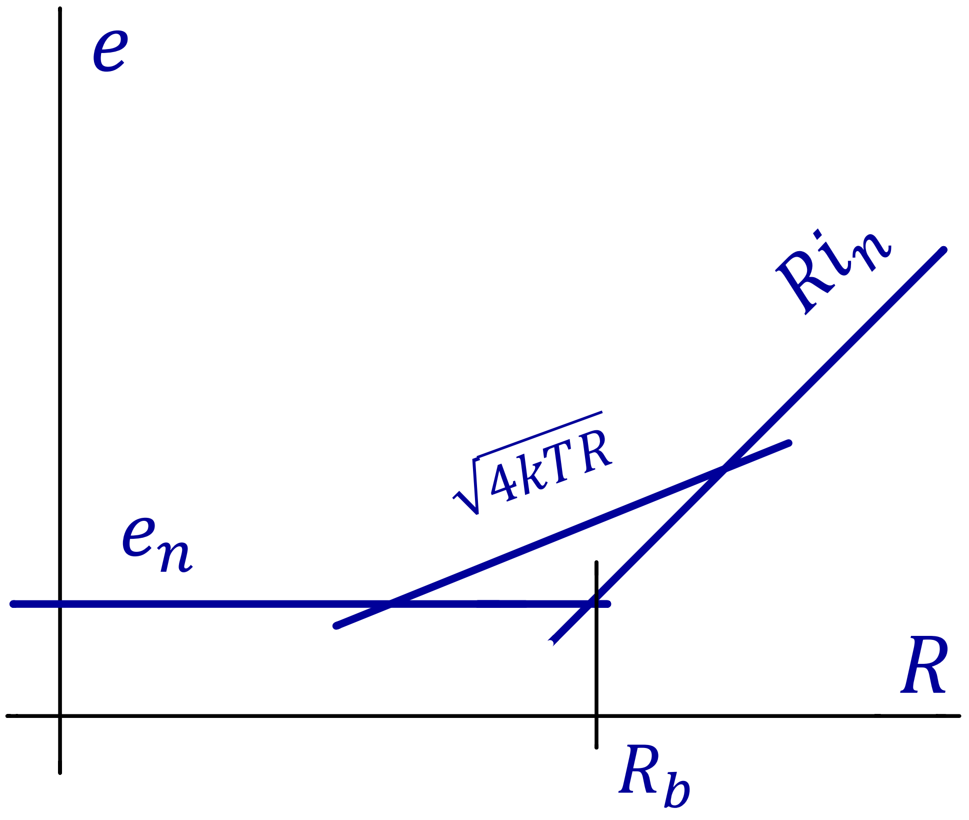
$$V^2 = \sum_i V_i^2$$

Need to design precision electronics?

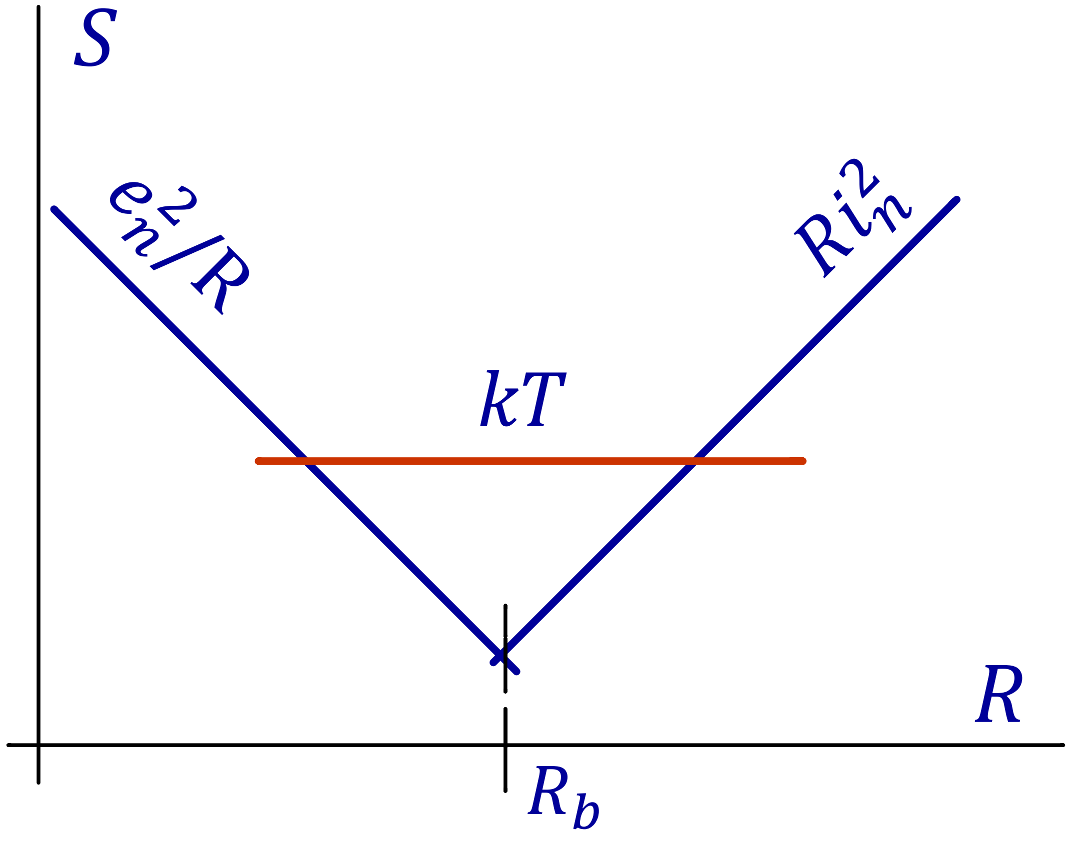
- D. Feucht, Analog Circuit Design Series, 4 volumes, SciTech 2010, ISBN 978-1-891121-XY-Z (Old school but great)
- S. Franco S, Design with operational amplifiers and analog integrated circuits 4ed, McGraw Hill 2015, ISBN 978-0-07-802816-8 (Best for designing with operational amplifiers)
- P. Horowitz, W. Hill, The Art of Electronics 3ed, Cambridge 2015, ISBN 978-0-521-80926-9 (The Bible of instrument design, physical insight)
- Tietze U, Schenk C, Gamm E - Electronic Circuits 2ed - Springer 2007, ISBN 978-3-540-78655-9 (German fashion)

Noise power vs R

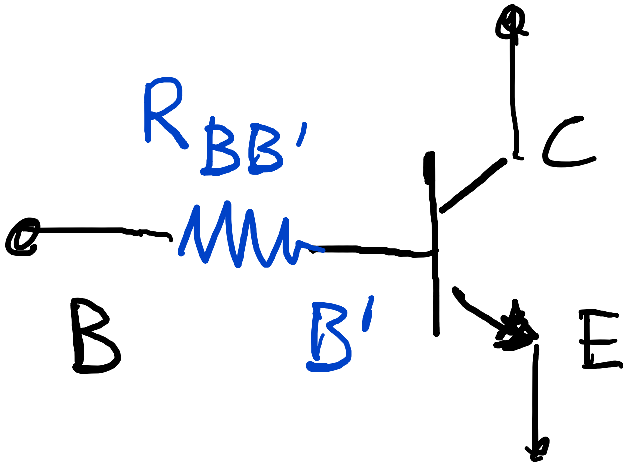
How it is shown

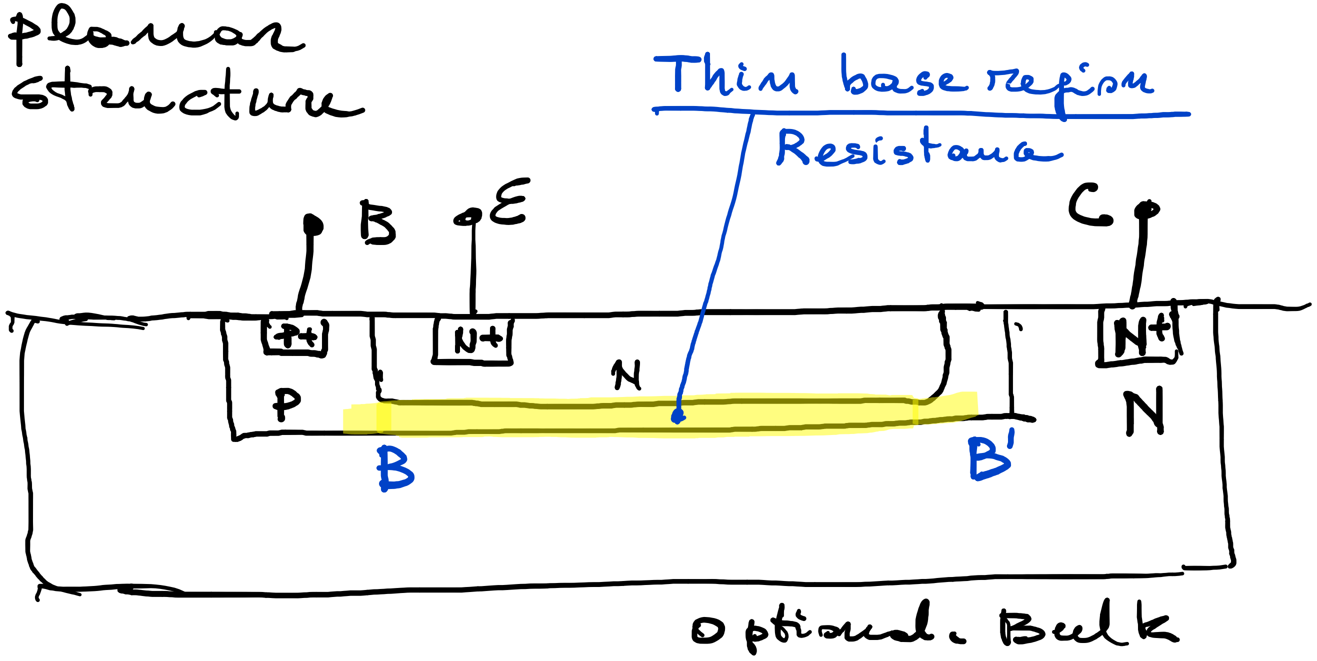


What it means



Noise in bipolar transistors





White noise

e_n –> thermal noise in $R_{BB'}$

$(500\ \Omega \rightarrow 2.9\ \mathrm{nV}/\sqrt{\mathrm{Hz}})$

i_n – shot noise of I_B (note that $I_B \ll I_C$)

$(1\ \mu\mathrm{A} \rightarrow 0.57\ \mathrm{pA}/\sqrt{\mathrm{Hz}})$

Flicker noise

Mainly the $1/f$ of the base current

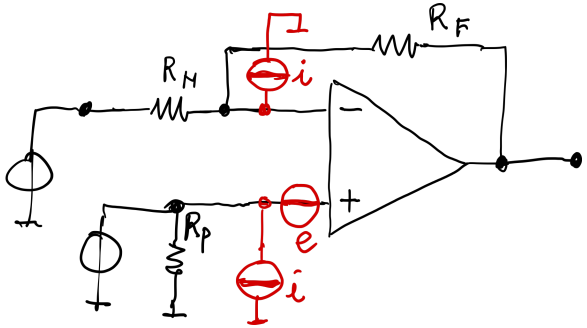
Featured readings

H. K. Gummel and H. C. Poon, "An integral charge control model of bipolar transistors", Bell Syst. Tech. J. 49, pp. 827-852, 1970

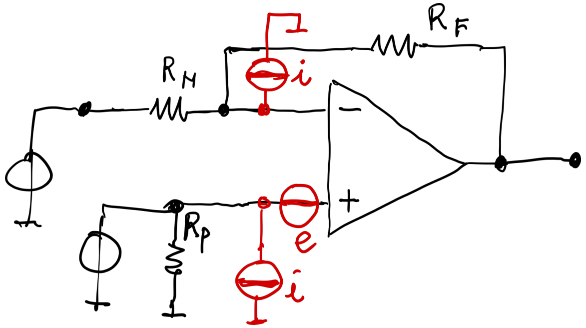
Horowitz P, Hill W, *The Art of Electronics* 3rd ed (2015), §8.3

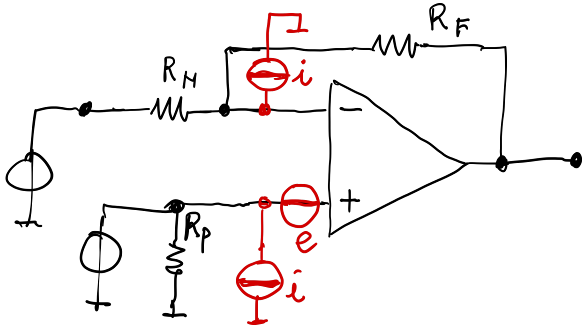
The Enrico’s low-level near-DC design

II – Lowest $1/f$ noise

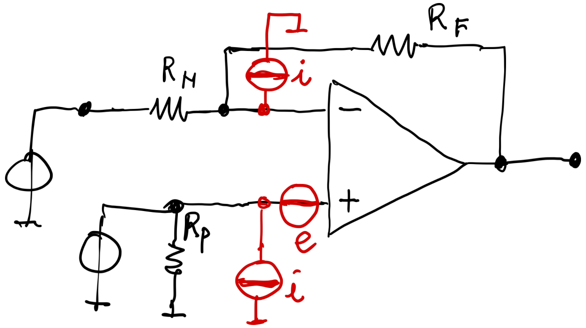


I – Lowest white noise





III – Lowest 1-K thermal drift



IV– Lowest aging

- Try a few designs based on different criteria
- Give a score to each feature
- Don’t look down at not-so-important parameters
- Let beginners believe that only a small number of parts are important in precision electronics

Featured reading, low white noise and low $1/f$ noise design

E. Rubiola, F. Lardet-Vieudrin, Low flicker-noise amplifier for 50 Ω sources, Rev. Scientific Instruments 75(5) p.1323-1326, May 2004

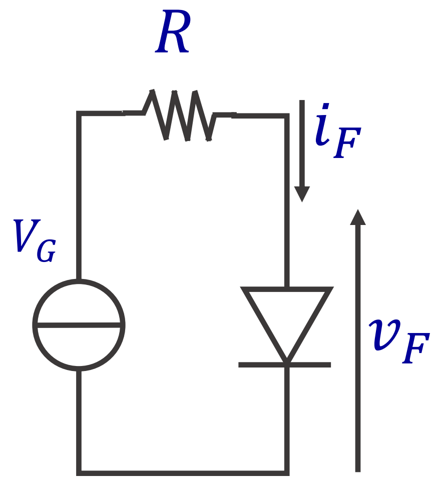
Featured reading, random walk and aging

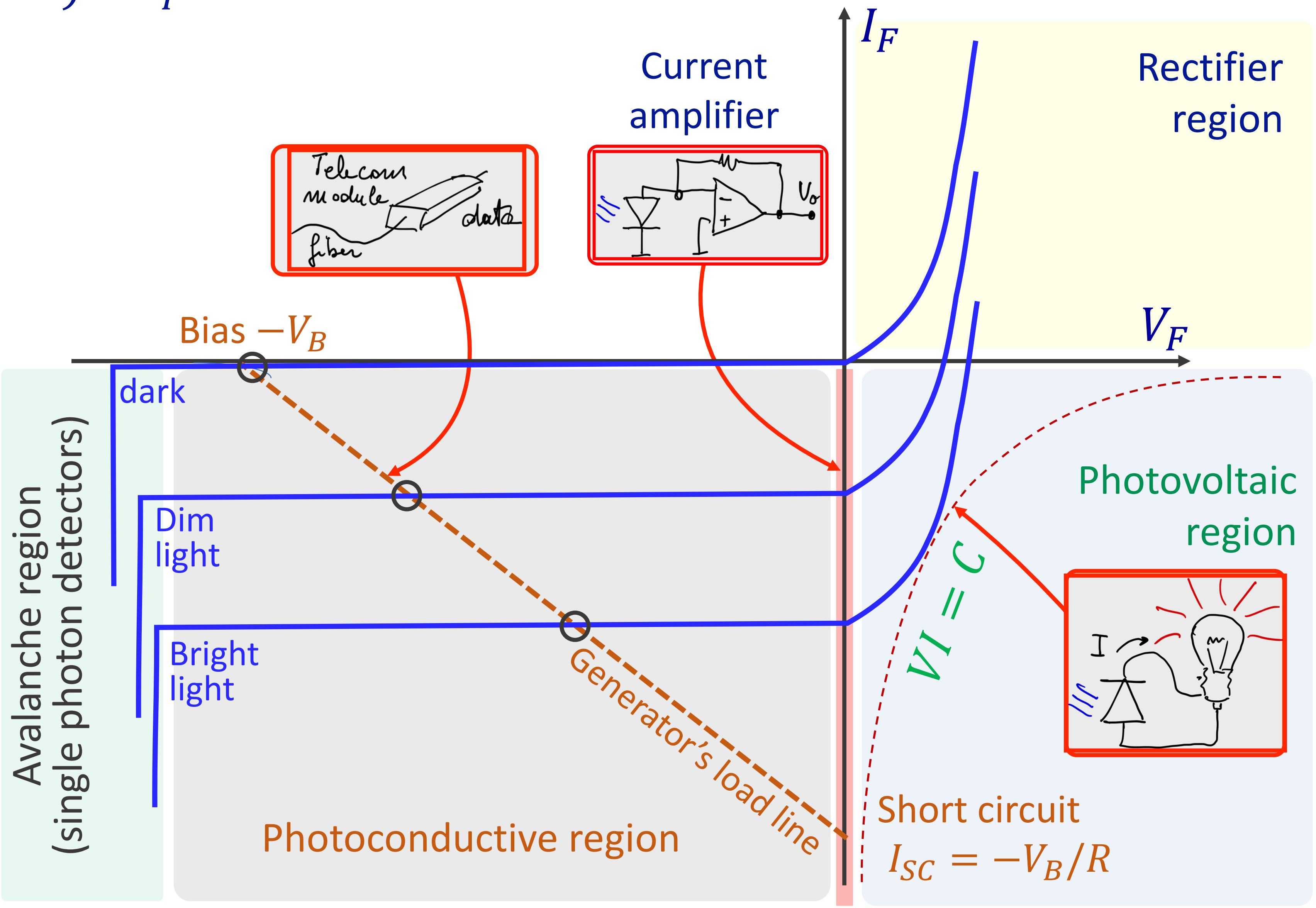
E. Rubiola, C. Francese, A. De Marchi, Long-Term Behavior of Operational Amplifiers, IEEE T IM 50(1) p.89-94, February 2001

Photodiode

$$I_F = I_s(e^{V_F/V_T} - 1) - I_P$$

$$V_T = kT/e$$





Current amplifier

Rectifier region

Bias $-V_B$

dark

Dim light

Bright light

Avalanche region (single photon detectors)

Photoconductive region

Generator's load line

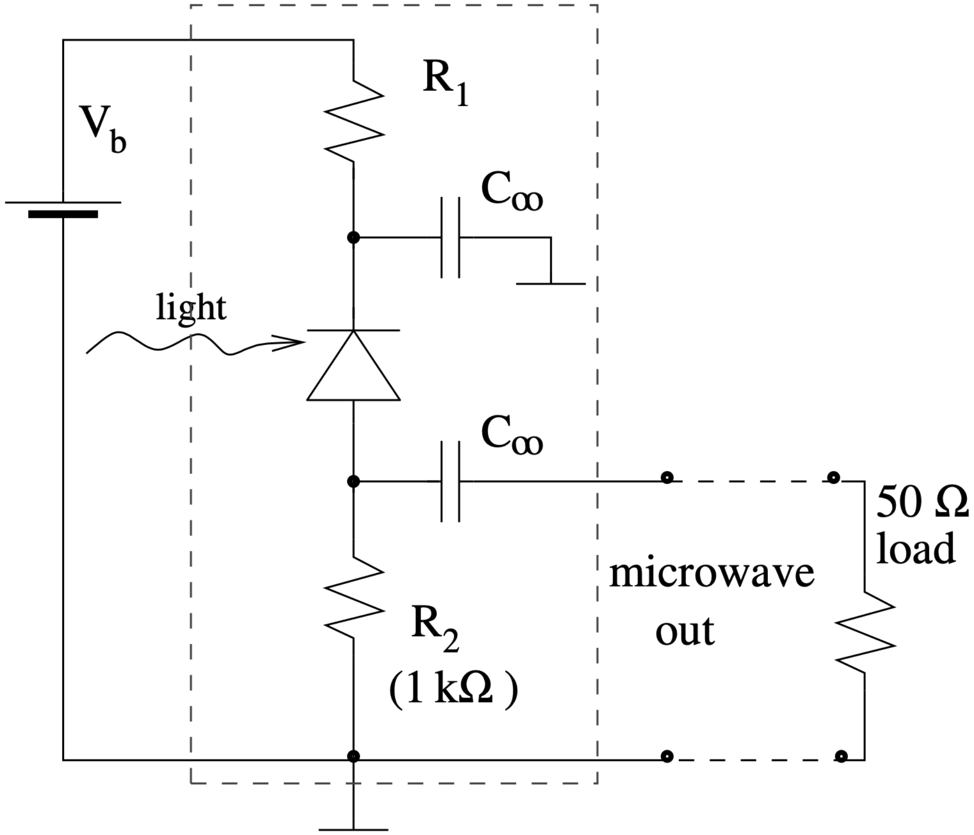
Photovoltaic region

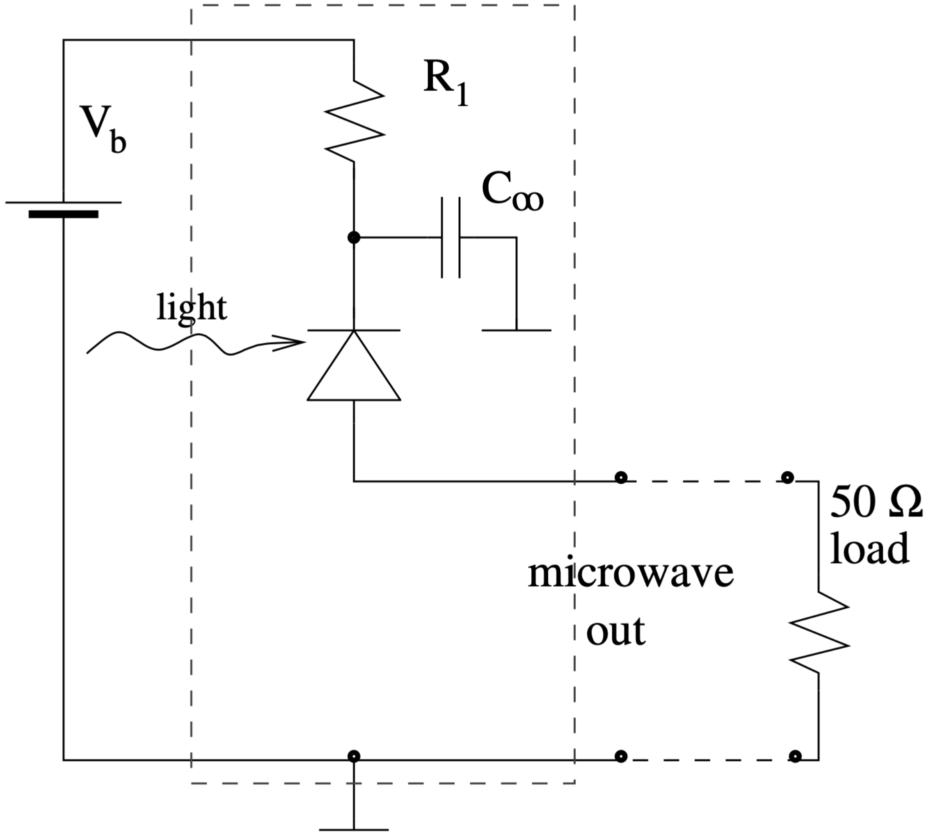
$VI = C$

Short circuit

$I_{SC} = -V_B/R$

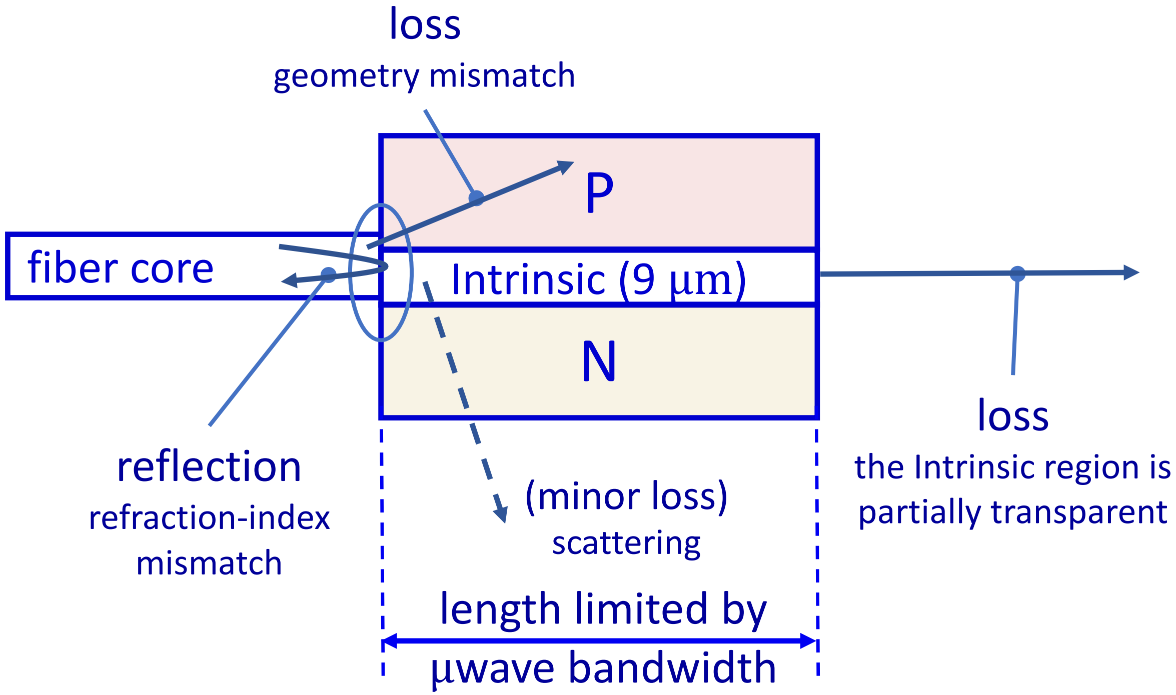
Fast photodiodes





Quantum efficiency

High-speed PIN photodetector



ϕ photons / s

$\eta\phi$ are detected

$(1-\eta)\phi$ are lost

$$\phi = \frac{P}{h\nu}$$

Photodetector signal and noise

Shot Noise

$$I = \eta e \phi = \frac{\eta e}{h\nu} P$$ Current

$$S_I(f) = 2e \frac{\eta e}{h\nu} P \qquad \mathrm{A}^2/\mathrm{Hz}$$ Current PSD

$$N_s = 2e \frac{\eta e}{h\nu} R \qquad \mathrm{W}/\mathrm{Hz}$$ Output-power PSD

Total noise: shot, dark current, thermal in the load

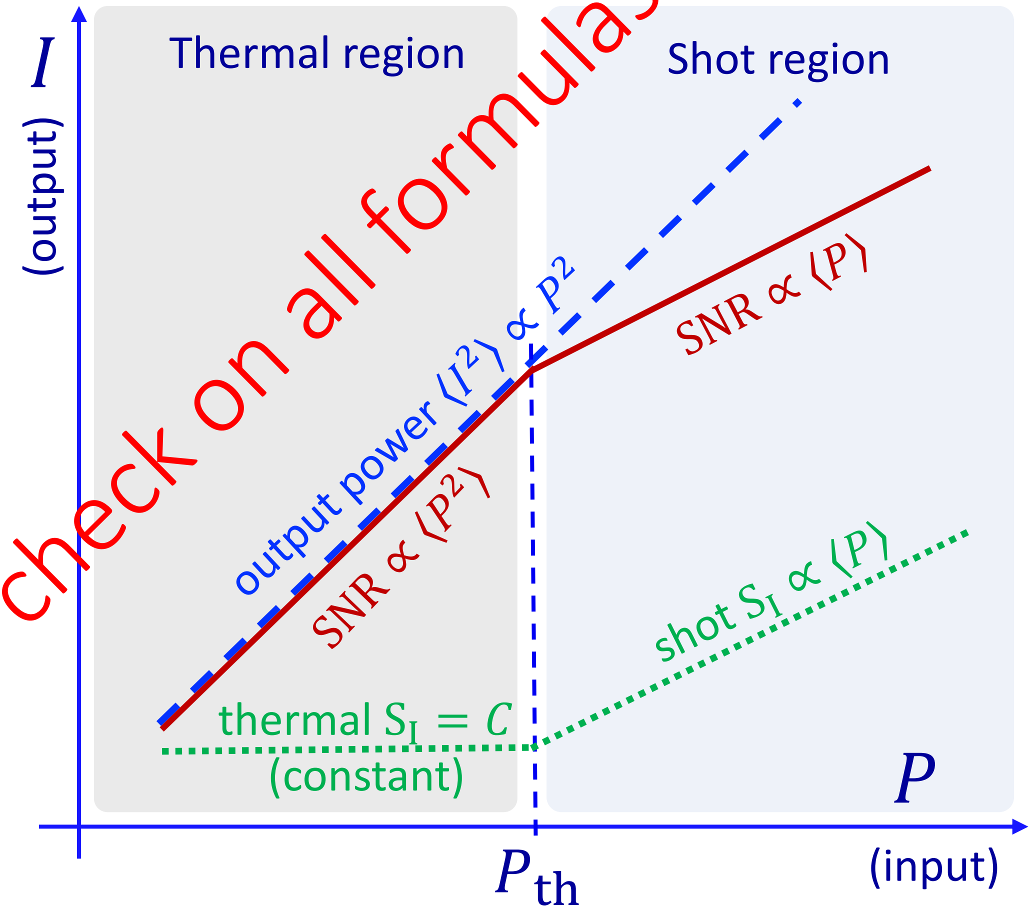
$$S_I = 2e(I_s + I_d) + \frac{4kT}{R} \qquad [\mathrm{A}^2/\mathrm{Hz}]$$

Shot is dominant at high P, thermal at low P

Threshold $S_{\mathrm{sh}} = S_{\mathrm{th}}$

$$P_{\mathrm{th}} = 2 \frac{h\nu}{e^2 \eta} \frac{kT}{R}$$

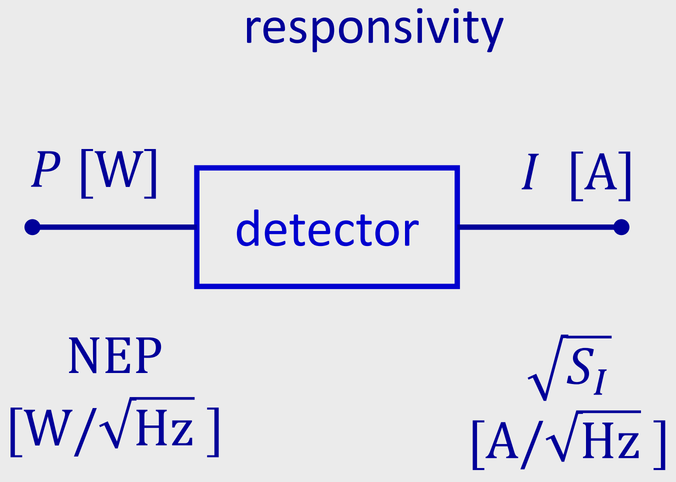
Thumb rule: $P_{\mathrm{th}} \approx 1\ \mathrm{mW}$ at 1.5 µm, 50 Ω, 300 K



Double check on all formulas

Note: intensity modulation affects P_{th}

Noise Equivalent Power (NEP)



responsivity

P [W]

detector

I [A]

NEP

$[\mathrm{W}/\sqrt{\mathrm{Hz}}\,]$

$\sqrt{S_I}$

$[\mathrm{A}/\sqrt{\mathrm{Hz}}\,]$

The output can be I, V, or any other quantity (including a number at the output of an ADC)

Don't mistake

optical power P at the input

signal power σ_s^2 at the output

noise power σ_n^2 at the output

- Radiometric concept
- Applies to quantum detectors, bolometers, and any other radiation detector

The NEP is the input power in that gives $\mathrm{SNR} = 1$ (Signal-to-Noise Ratio) in 1 Hz bandwidth

output

$$\mathrm{NEP}^2 = \frac{P^2}{\Delta f} \quad \text{at} \quad \sigma_n^2 = \sigma_s^2$$

Example: Photodiode

$$\sigma_s^2 = I^2 = \left(\frac{e\eta P}{h\nu}\right)^2 \qquad \sigma_n^2 = S_I(f)\Delta f$$

Featured reading: S. Leclerq, Discussion about Noise Equivalent Power and its use for photon noise calculation, March 2007

Available: http://www.iram.fr/~leclercq/Reports/About_NEP_photon_noise.pdf (retrieved April 2020)

Also: P. L. Richards, Bolometers for infrared and millimeter waves, J Appl Phys 76(1) p.1-25, 1 July 1994

NEP in photodetectors

Double check on all formulas

Low power, thermal region

$$S_I = \frac{4kT}{R}$$

(signal)^2 = (noise)^2 in Δf

$$\left(\frac{e\eta P}{h\nu}\right)^2 = \frac{4kT}{R}\Delta f$$

$$\frac{P^2}{\Delta f} = \frac{h^2\nu^2}{e^2\eta^2}\frac{4kT}{R}$$

$$\mathrm{NEP} = 2\frac{h\nu}{e\eta}\sqrt{kT/R}$$

Thumb rule:

NEP = 1.8x10^{-11} W/√Hz,

1.5μm, 50Ω, 300K, η=0.8

High power, shot region

$$S_I = 2\frac{e^2\eta P}{h\nu}$$

(signal)^2 = (noise)^2 in Δf

$$\left(\frac{e\eta P}{h\nu}\right)^2 = 2\frac{e^2\eta P}{h\nu}\Delta f$$

do not simplify *P*

$$\frac{P^2}{\Delta f} = \frac{h^2\nu^2}{e^2\eta^2}2\frac{e^2\eta P}{h\nu}$$

$$\mathrm{NEP} = \sqrt{2\frac{h\nu}{\eta}P}$$

Thumb rule:

NEP = 1.8x10^{-11} W/√Hz,

1.5μm, η=0.8, P=1mW

Experimental techniques

Special cases

Extremely low current

- Charge amplifier (AD549, bias ≈100 e/s rms)
- Don’t assume that insulators do insulate
- Prevent leakage with layout rules and guarding
- Narrow bandwidth only
- Polymers take in vibes (piezoelectricity)

Extremely low voltages

- Chopper (switching) amplifier (AD8628 ≈2 nV/K thermal)
- Bandwidth limited by the chopper frequency
- Thermocouples (Seebeck effect) are everywhere (soldering alloy, O_2 in Cu cables)
- Polymers take in vibes (electrostriction/piezoelectricity)

Highest gain accuracy

- Use Vishay resistor pairs (thermally compensated ratio)
- Unsuspected effects
    - Common mode rejection extremely critical
    - Open loop gain of OAs affects the accuracy
    - Thermal feedback inside OAs due to the power dissipated in the output stage
    - …and others

Lowest noise

- The choice of all resistances depends on e_n and i_n
- Bipolar transistor are better than field-effect transistors
- The design for lowest white or lowest 1/ƒ is not the same
- PNP amplifiers feature lower 1/ƒ noise

Photodiode signal

- The photodiode has high output impedance (current generator with a capacitance in parallel)
- Special design rules (Read J. G. Graeme, Photodiode amplifiers, McGraw Hill 1995, ISBN 0-07-024247-X)

Highest speed (video amplifier)

- Current feedback amplifiers (CFA, the bandwidth does not decrease with the gain)
- Higher noise

Highest speed (video amplifier) without CFAs

- Takes OPAs with extremely high gain-bandwidth product
- Self oscillations difficult to prevent (simulation must include L and C associated to the PCB

Featured reading: P. Horowitz, W. Hill, The Art of Electronics, 3rd ed, Cambridge 2015

Low-frequency electromagnetic shielding

Electric shielding is poor

- Skin effect

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}} \quad \text{for } \omega \ll 1/\rho\epsilon$$

In Copper

9.2 mm at 50 Hz

2.06 mm at 1 kHz

ω = angular frequency

ρ = resistivity

μ = magnetic permeability

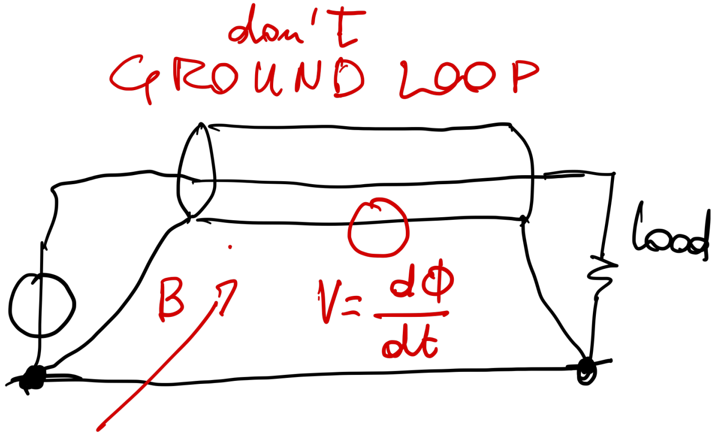
ϵ = electric permittivity

Magnetic shield is effective

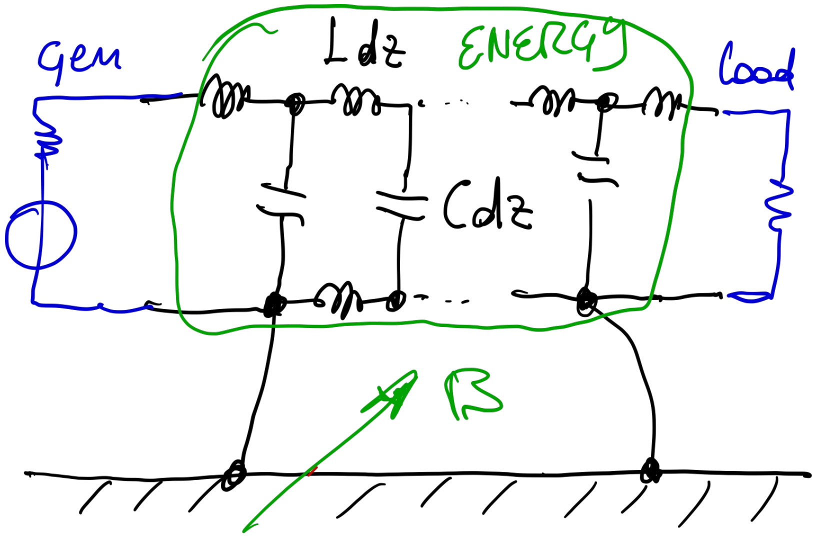
- Mumetal
  - Various compositions, about Ni 77%, Fe 16%, Cu 5%, Cr 2%
  - Ductile/malleable
- Permalloy
  - Ni 80%, Fe 20%,
- $\mu_r = 10^5$
- Require annealing
- Suffer shocks/acceleration

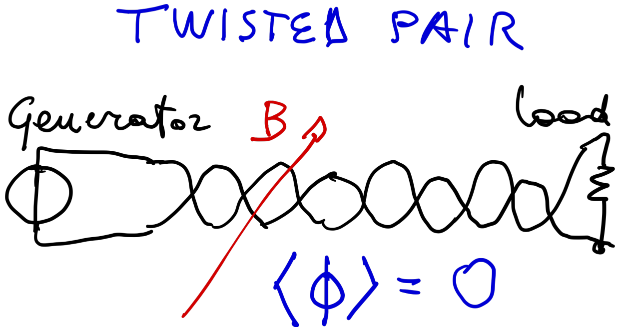
Superconductors are perfect (and impractical) electric and magnetic shields (Meissner effect)

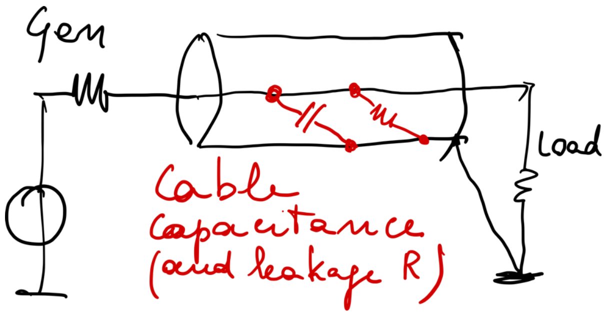
Cables and guarding



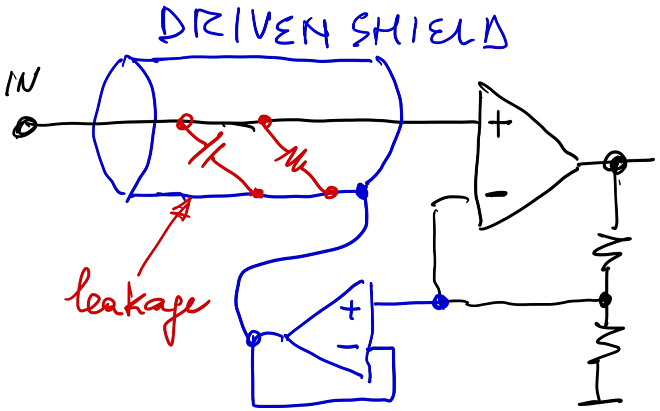
RF in cables & twisted pairs propagates as a field
cutoff frequency f_c = 2…10 kHz
ground loops allowed (far) beyond f_c

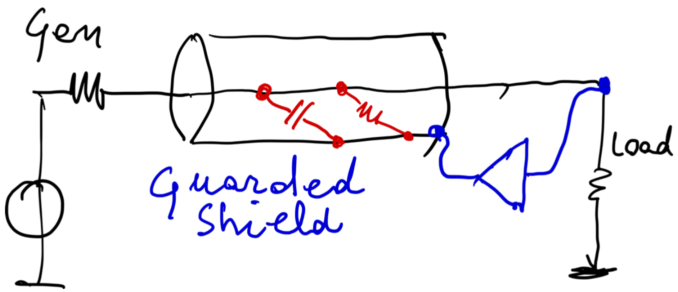


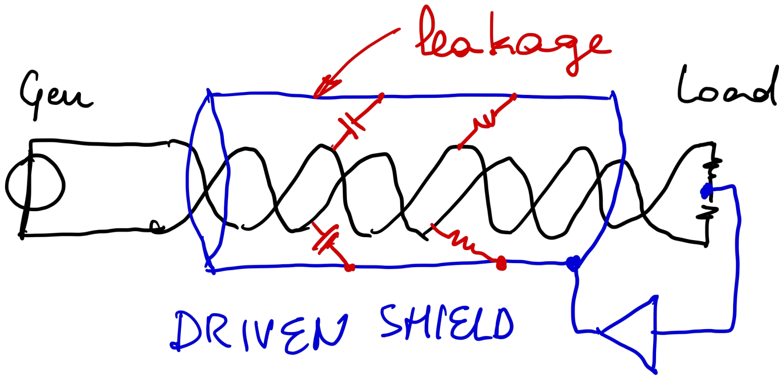




Featured readings
H. W. Ott, Electromagnetic Compatibility Engineering, Wiley 2009, ISBN 978-0-470-18930-6
C. R. Paul, Introduction to Electromagnetic Compatibility, Wiley 2006, ISBN 978-0-471-75500-5

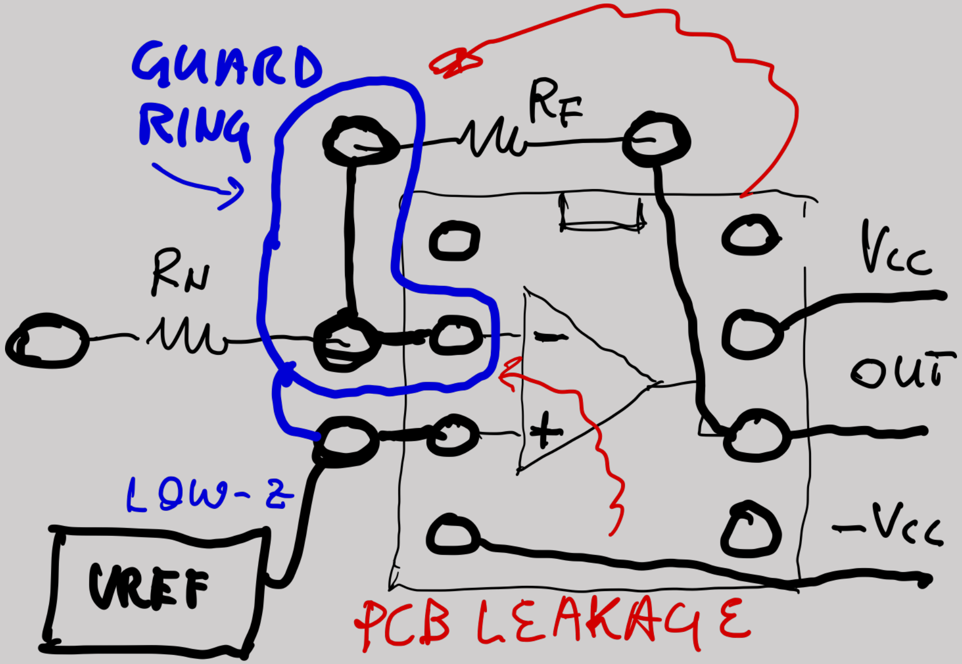






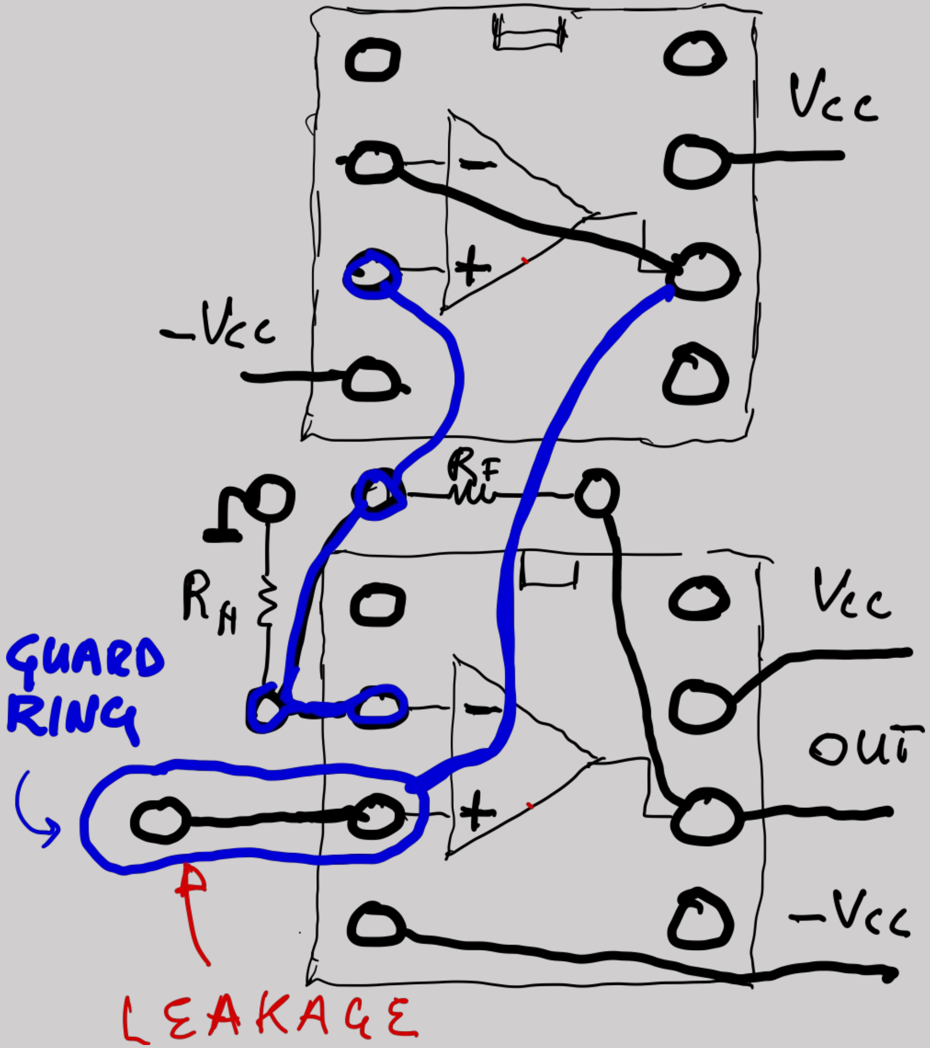
Guarding on printed circuit boards

Inverting amplifier



Standard operational amplifier,
8-pin DIL package, top view

Non inverting amplifier



Homework

- Work out the noise temperature of the operational amplifier at $R_{\mathrm{best}} = e_n/i_n$
- Calculate T_{eq} for the OP27 and the LT1028
- You should find almost the same T_{eq} , despite the fact that the noise of the two amplifier is so different.
- Can you figure out why?

Power Spectral Density (PSD) and its Estimation

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

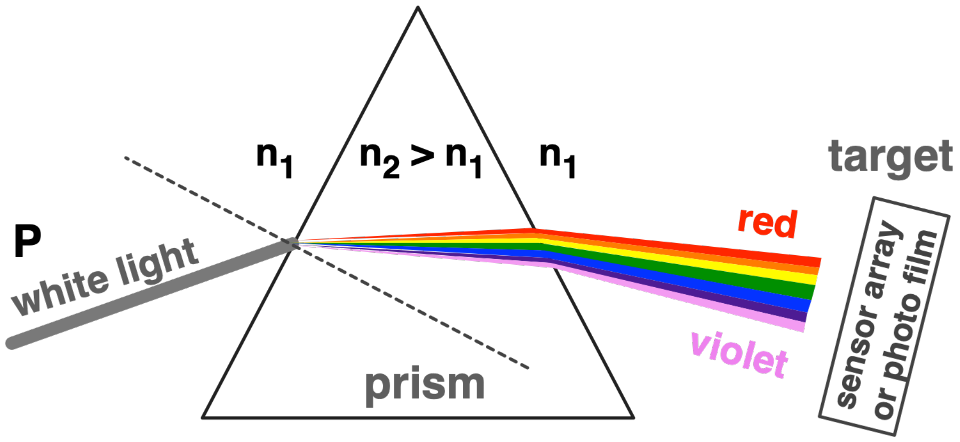
INRiM, Torino, Italy

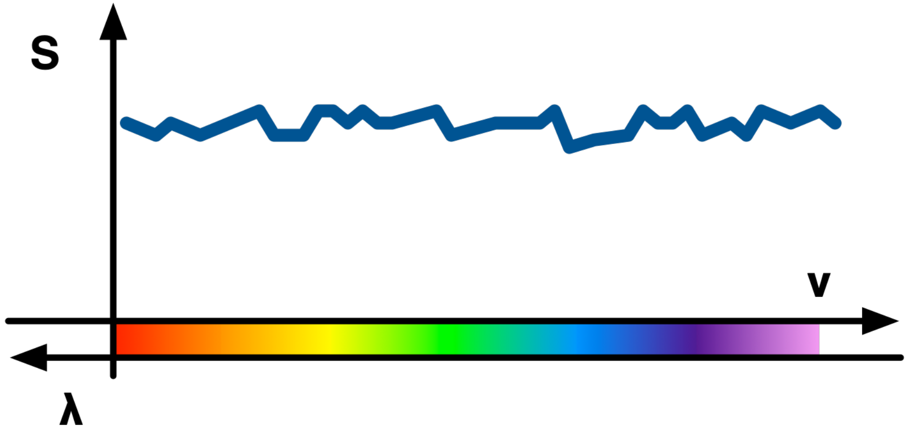
Featured reading: E. Rubiola, F. Vernotte
The cross-spectrum experimental method

home page http://rubiola.org

Physical concept of spectrum

More precisely, Power Spectral Density





- The PSD is the distribution of power vs. frequency (power in 1-Hz bandwidth)
- The PS is the distribution of energy vs. frequency (energy in 1-Hz bandwidth)
- Power (energy) in physics is a square (integrated) quantity
- PSD –> W/Hz, or V^2/Hz, A^2/Hz, rad^2/Hz etc.

$$S_v(f) = \frac{\langle v_B^2(f) \rangle}{B}$$

Discrete: Δf is the resolution
Continuous: $\Delta f \to df$

$$\frac{\text{average power in the bandwidth } B \text{ centered at } f}{\text{bandwidth } B}$$

The power spectral density

for stationary random process

$$\mathcal{C}_x(\tau) = \mathbb{E}\{[\mathrm{x}(t) - \mu][\mathrm{x}(t-\tau) - \mu]^*\}$$

Autocovariance function
Improperly referred to as the correlation, denoted with $R_{xx}(\tau)$

$$\mu = \mathbb{E}\{\mathrm{x}\}$$

$$S(\omega) = \mathcal{F}\{\mathcal{C}(\tau)\} = \int_{-\infty}^{\infty} \mathcal{C}(\tau)\, e^{-i\omega\tau}\ d\tau$$

PSD (two-sided)

Fourier transform of the autocovariance

$$\mathcal{C}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [x(t) - \mu]\,[x(t-\tau) - \mu]^*\, dt$$

For ergodic process, interchange ensemble and time average
process $\mathrm{x}(t)$ → realization $x(t)$

$$S_x^{II}(\omega) = \lim_{T\to\infty} \frac{1}{T} X_T(\omega) X_T^*(\omega) = \lim_{T\to\infty} \frac{1}{T} |X_T(\omega)|^2$$

Truncated

Wiener Khinchin theorem, if the process is stationary and ergodic, $S_x(f)$ can be calculated from the Fourier transform of a realization

Experiments –> finite time, and single-sided PSD averaged on m realizations

$$S^I(f) = 2S^{II}(\omega/2\pi)$$
$$f > 0$$

$$\boxed{S_x(f) = \frac{2}{T} \langle X_T(f) X_T^*(f) \rangle_m}$$

Lecture 2 ends here

Lecture 3

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Noise in RF/microwave devices (cont)
- Photodetectors
- Analog-to-digital and digital-to-analog conversion

ORCID 0000-0002-5364-1835

home page http://rubiola.org

DFT, FFT, FFTW, SFFT

The Discrete Fourier Transform (DFT) approximates the (continuous) FT

$$X\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} x(kT) e^{i2\pi nk/N}$$

$T =$ sampling interval, $f_s = 1/T$
$N = 0 \ldots N - 1$ integer frequency, $f = n/NT$

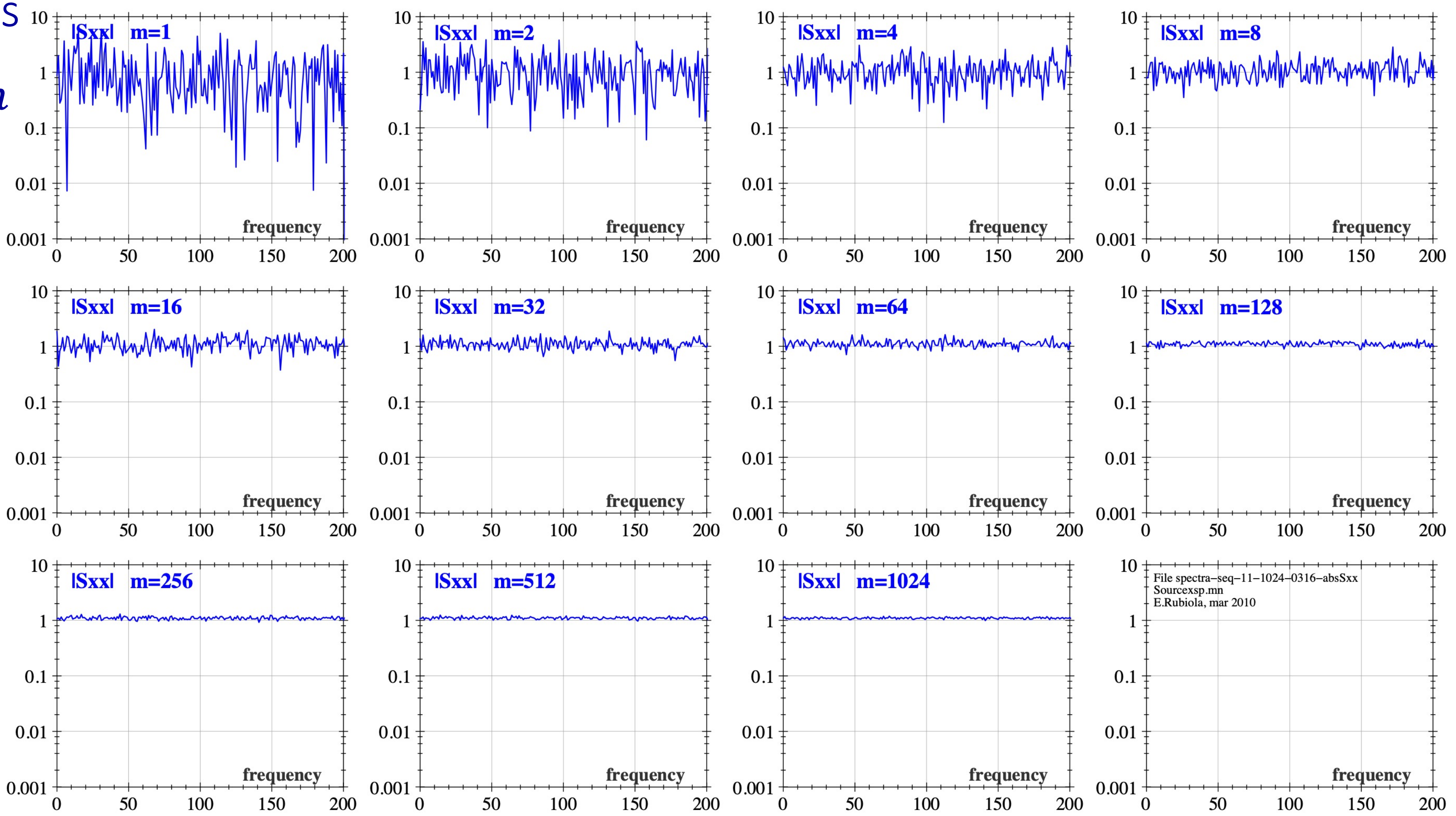
- The direct computation of the DFT takes $\approx N^2$ multiplications
- The FFT is an algorithm for Fast computation of the DFT that takes $\approx N \log(N)$ multiplications
- The FFTW, “the Fastest Fourier Transform in the West,” is an even faster. Still $\approx N \log(N)$ multiplications (M. Frigo, S.G. Johnson, MIT) See http://fftw.org/
- SFFT “faster-than-fast” Sparse FFT (D.Katabi, P.Indyk, MIT) See http://groups.csail.mit.edu/netmit/sFFT/
- The difference between DFT, FFT, and FFTW is (at most) computing time (you don’t want to implement your FT algorithm, do you?)

Estimation of $|S_{xx}(f)|$

Average on m realizations

$$\widehat{S_{xx}}(f) = \langle S_{xx}(f) \rangle_m$$

Running the measurement, m increases and $\widehat{S_{xx}}(f)$ shrinks => better confidence level



Power spectral density $S_{xx}(f)$, noise only

$x(t) \leftrightarrow X(f)$ is white Gaussian noise

Take one frequency, $S(f) \rightarrow S$

Same applies to all frequencies

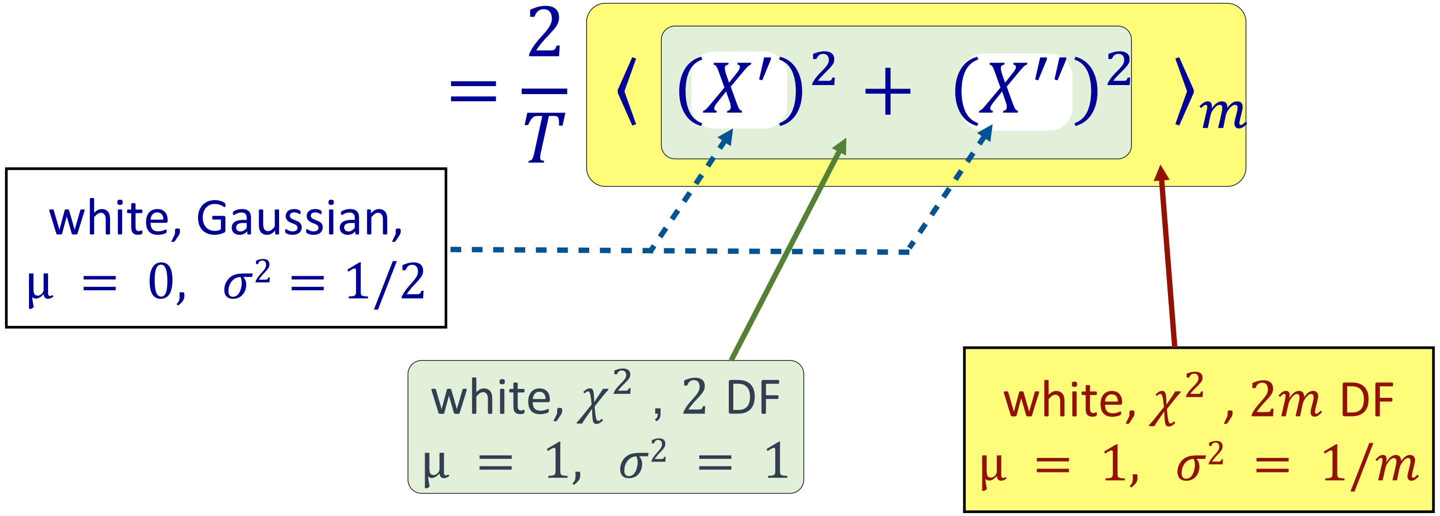
Normalization: power in 1 Hz bandwidth

$\mathbb{V}\{X\} = 1$, equally split between $\Re\{\}$ and $\Im\{\}$

thus $\mathbb{V}\{X'\} = \mathbb{V}\{X''\} = 1/2$

$$\langle S_{xx} \rangle_m = \frac{2}{T}\langle X\,X^* \rangle_m$$

$$= \frac{2}{T}\langle (X' + iX'')\times(X' - iX'') \rangle_m$$



$$= \frac{2}{T}\langle\ (X')^2 + (X'')^2\ \rangle_m$$

white, Gaussian,
$\mu = 0,\ \ \sigma^2 = 1/2$

white, χ^2, 2 DF
$\mu = 1,\ \ \sigma^2 = 1$

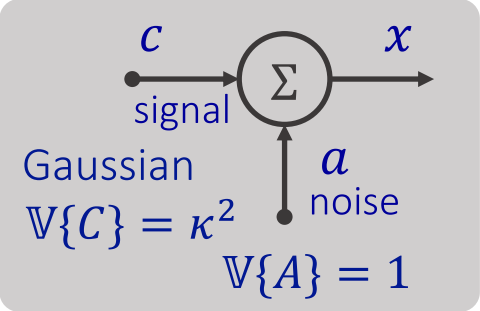
white, χ^2, $2m$ DF
$\mu = 1,\ \ \sigma^2 = 1/m$

Conclusion

$$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{1}{m}}$$

the $S_{xx}(f)$ track shrinks as $1/\sqrt{m}$

PSD $S_{xx}(f)$, signal and noise



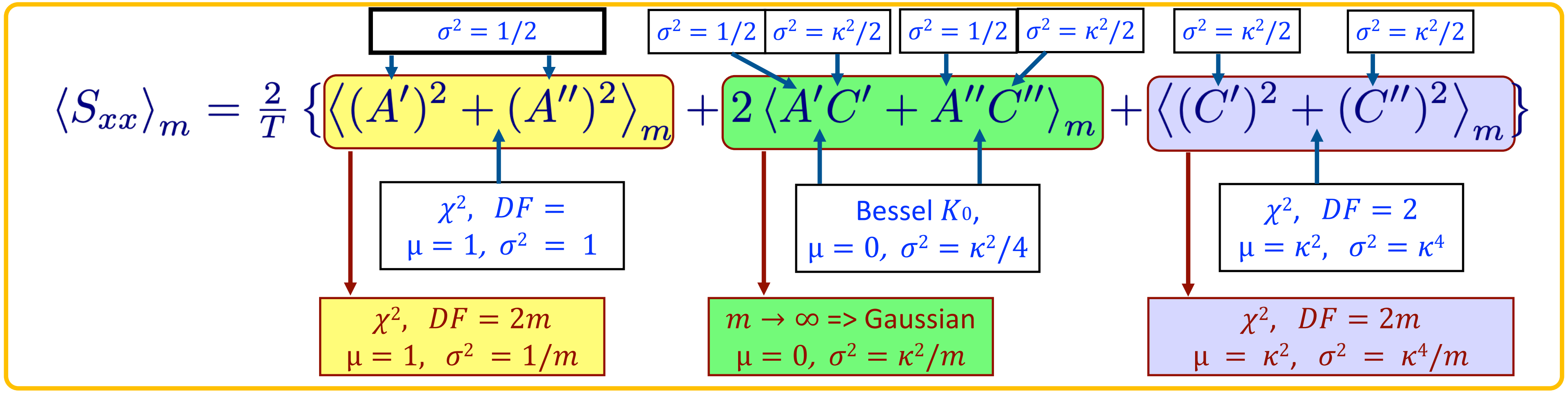
Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\} = 1,\ \mathbb{V}\{C\} = \kappa^2$
$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = 1/2$ and $\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$

$$\langle S_{xx}\rangle_m = \frac{2}{T}\langle XX^*\rangle_m = \frac{2}{T}\langle (X'+iX'')\times(X'-iX'')\rangle_m$$

$$X = (C'+iC'')+(A'+iA'')$$

$\Re\{|S_{xx}|\} \rightarrow$

$\Im\{|S_{xx}|\} = 0$



$$\langle S_{xx}\rangle_m = \frac{2}{T}\left\{\left\langle (A')^2+(A'')^2\right\rangle_m + 2\left\langle A'C'+A''C''\right\rangle_m + \left\langle (C')^2+(C'')^2\right\rangle_m\right\}$$

$\sigma^2 = 1/2$

$\sigma^2 = 1/2$ | $\sigma^2 = \kappa^2/2$ | $\sigma^2 = 1/2$ | $\sigma^2 = \kappa^2/2$

$\sigma^2 = \kappa^2/2$ | $\sigma^2 = \kappa^2/2$

χ^2, $DF =$
$\mu = 1$, $\sigma^2 = 1$

Bessel K_0,
$\mu = 0$, $\sigma^2 = \kappa^2/4$

χ^2, $DF = 2$
$\mu = \kappa^2$, $\sigma^2 = \kappa^4$

χ^2, $DF = 2m$
$\mu = 1$, $\sigma^2 = 1/m$

$m \rightarrow \infty$ => Gaussian
$\mu = 0$, $\sigma^2 = \kappa^2/m$

χ^2, $DF = 2m$
$\mu = \kappa^2$, $\sigma^2 = \kappa^4/m$

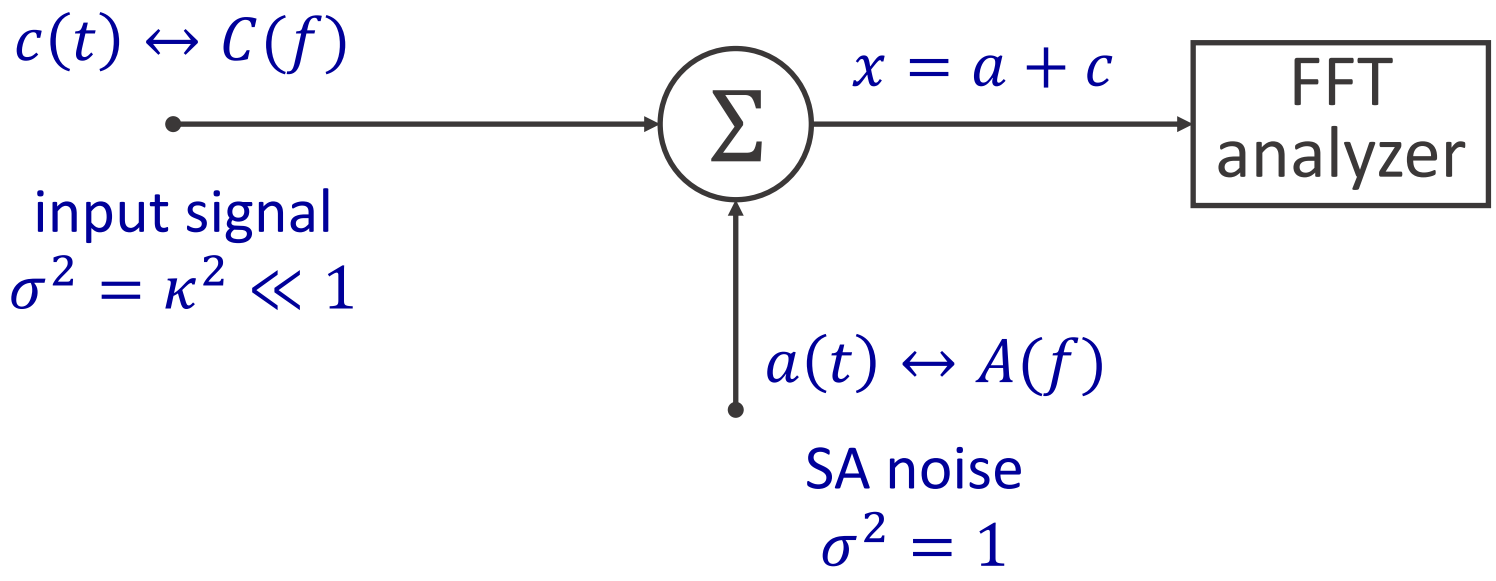
$$\mu = 1+\kappa^2 \qquad \sigma^2 = \frac{1+\kappa^2+\kappa^4}{m} \qquad \frac{\sigma}{\mu} = \sqrt{\frac{1+\kappa^2+\kappa^4}{m}}\,\frac{1}{1+\kappa^2}$$

$$\frac{\sigma}{\mu} \simeq \frac{1}{\sqrt{m}}\left[1-\frac{\kappa^2}{2}\right], \quad \kappa \ll 1$$

the track
shrinks as $1/\sqrt{m}$

$$\frac{\sigma}{\mu} \simeq \frac{1}{\sqrt{m}}\left[1-\frac{1}{2\kappa^2}\right], \quad \kappa \gg 1$$

Measurement of a small signal



$c(t) \leftrightarrow C(f)$

input signal

$\sigma^2 = \kappa^2 \ll 1$

Σ

$x = a + c$

FFT analyzer

$a(t) \leftrightarrow A(f)$

SA noise

$\sigma^2 = 1$

$C = 0$	$C \neq 0$
$\mu = 1$	$\mu = 1 + \kappa^2$
$\sigma^2 = \dfrac{1}{m}$	$\sigma^2 = \dfrac{1 + \kappa^2 + \kappa^4}{m}$

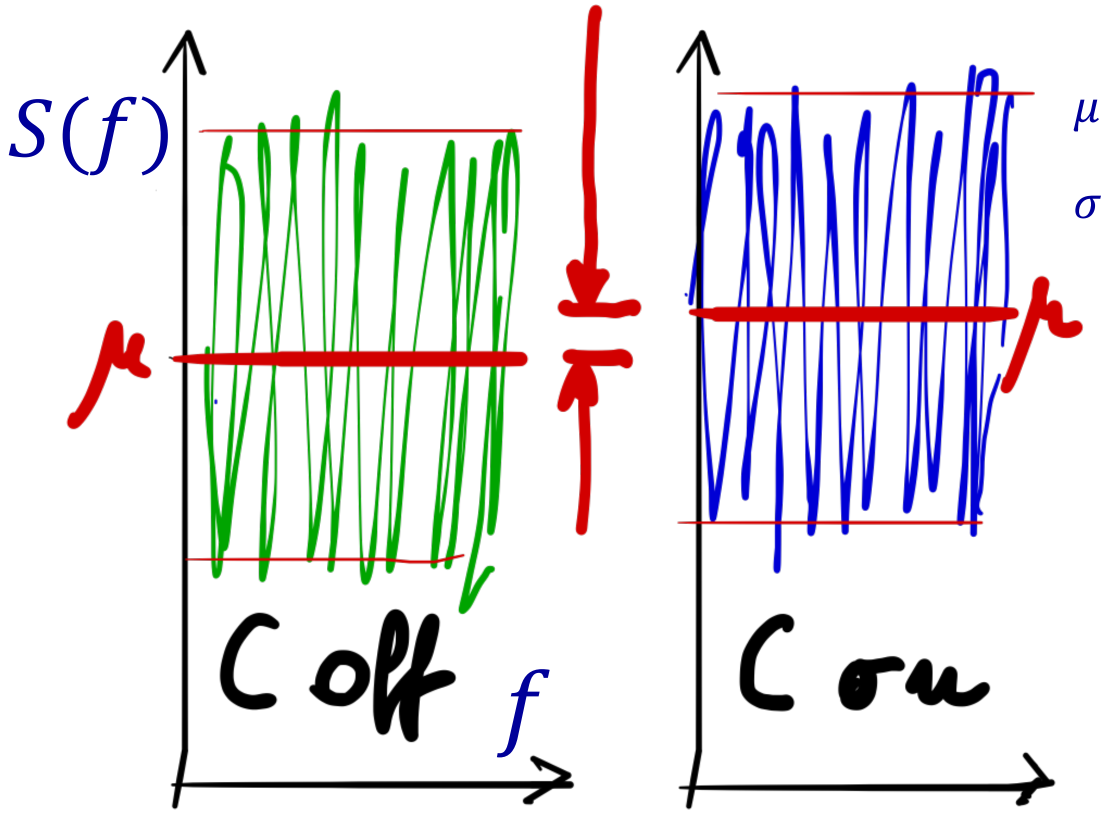
The Dicke radiometer

Small m

Contrast not detected

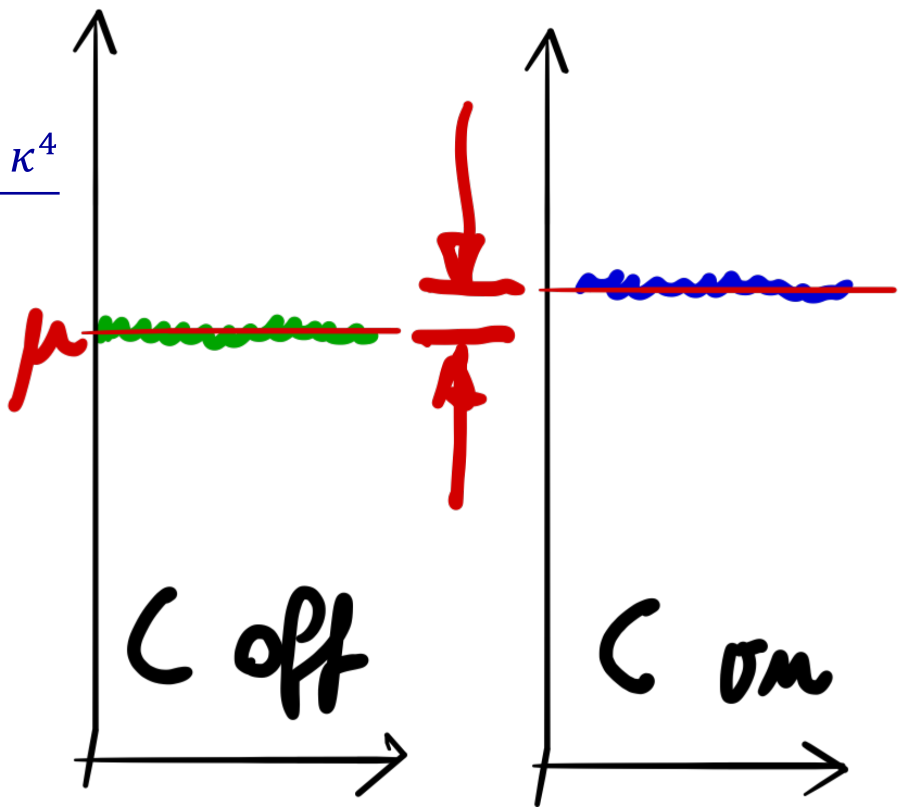
Large m

Contrast well detected



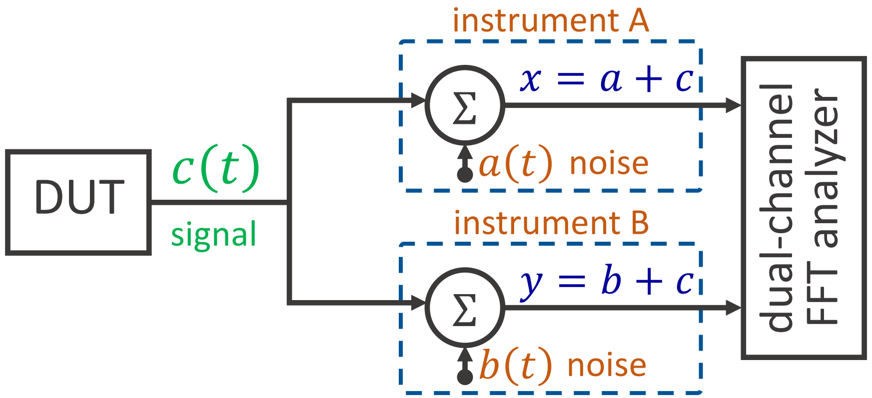
$$\mu = 1 + \kappa^2$$

$$\sigma^2 = \frac{1 + \kappa^2 + \kappa^4}{m}$$

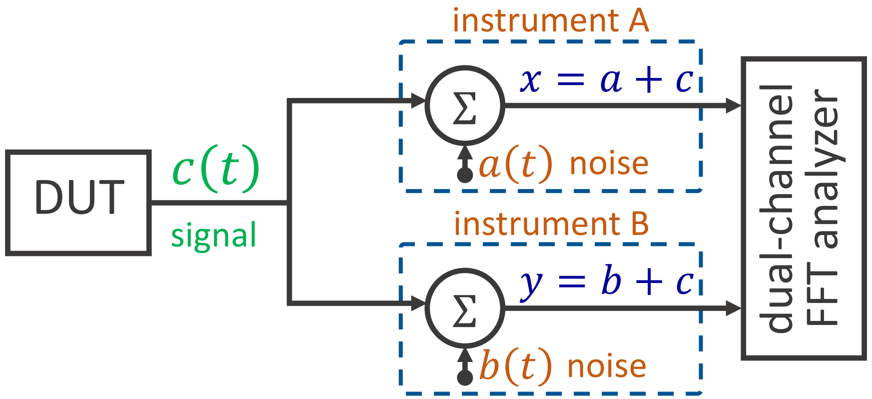


Historical reading: R. H. Dicke, The Measurement of Thermal Radiation at Microwave Frequencies, RSI 17(7) p.268-275, July 1946

Cross Spectrum Theory



Correlation Measurements



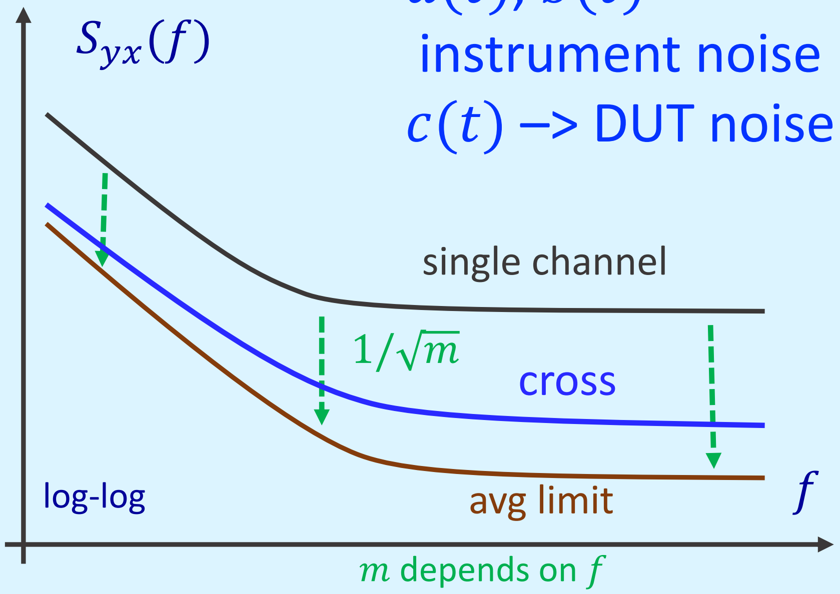
also crosstalk d(t)

Read the tutorial

E. Rubiola, F. Vernotte, The cross-spectrum experimental method, February 2010, arXiv:1003.0113 [physics.ins-det]

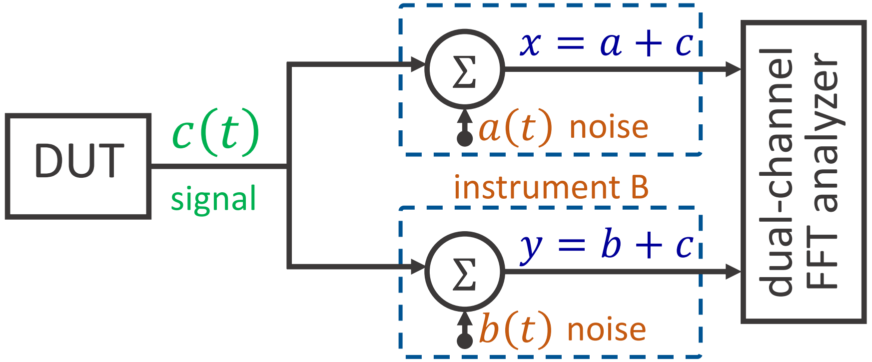
Two separate instruments measure the same DUT.
Only the DUT noise is common

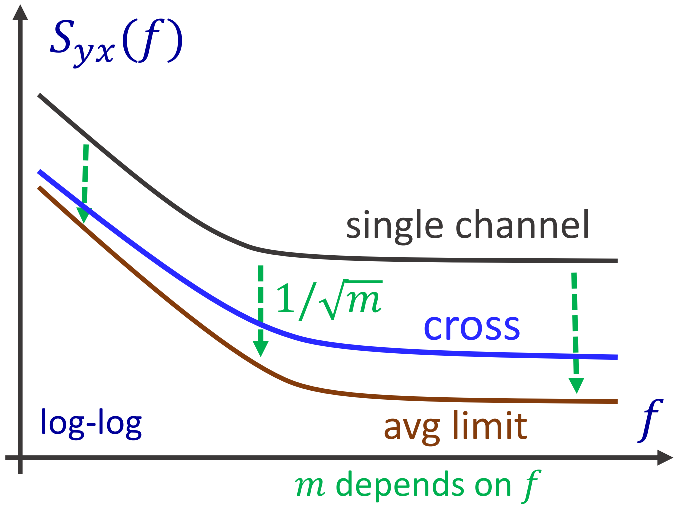
noise measurements		
DUT noise, normal use	a, b, c	instrument noise, DUT noise
background, ideal case	a, b $c = 0$	instrument noise, no DUT
background, real case	a, b $d \neq 0$	c is the correlated instrument noise Zero DUT noise



$a(t), b(t)$ –> instrument noise
$c(t)$ –> DUT noise

Cross PSD $S_{yx}(f)$ – Simplified





$$S_{yx} = \frac{2}{T}\langle (B+C)(A+C)^*\rangle_m$$

$$= \frac{2}{T}\langle \cancel{BA^*} + \cancel{BC^*} + \cancel{CA^*} + CC^*\rangle_m$$

rejected ∝ $1/\sqrt{m}$

$$\mathbb{E}\{S_{yx}\} = \frac{2}{T}\langle CC^*\rangle_m = S_c \qquad S_c \in \mathbb{R}$$

$$\mathbb{V}\left\{\langle S_{yx}\rangle_m\right\} = \frac{1}{m}$$

The $\widehat{S_{yx}} = |S_{yx}|$ estimator takes in the full noise

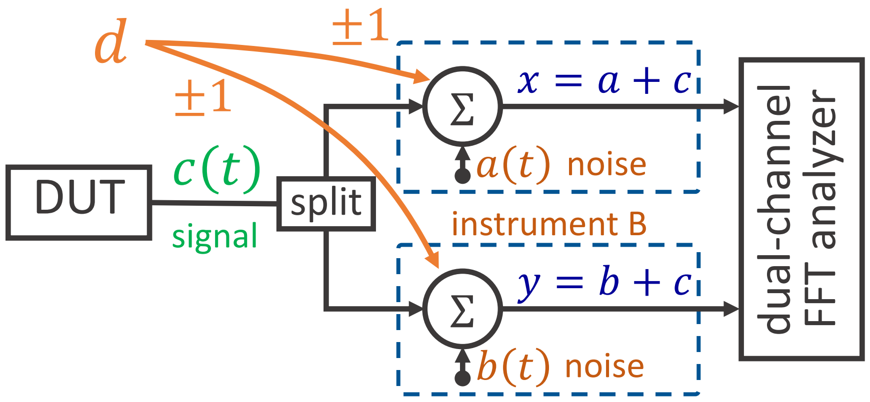
$$\mathbb{V}\left\{\langle \Re\{S_{yx}\}\rangle_m\right\} = \frac{1}{2m}$$

The $\widehat{S_{yx}} = \Re\{S_{yx}\}$ estimator takes in half the noise

Read the tutorial

E. Rubiola, F. Vernotte, The cross-spectrum experimental method, February 2010, arXiv:1003.0113 [physics.ins-det]

A correlated disturbing term



Same role of $c(t)$, but for the sign ς

$$S_{yx} = \frac{2}{T}[B + C + \varsigma_y D][A + C + \varsigma_x D]^*$$

After averaging

$$S_{yx} \to \underbrace{S_c}_{\text{DUT spectrum}} + \underbrace{\overset{\pm 1}{\varsigma} S_d}_{\text{bias}}$$

Also $\Re\{S_{yx}\} \to S_c + \varsigma S_d$ and $\Im\{S_{yx}\} \to 0$

$\varsigma > 0$ –> noise over-estimation

- We may accept this

$\varsigma < 0$ –> noise under-estimation

- May be embarrassing

$S_c + \varsigma S_d < 0$ –> nonsense

- The disturbing term prevail

The common superstition that

- The instrument adds its own noise
- Over-estimation of the DUT noise

is wrong in the case of cross spectrum (and covariances)

$S_{yx}(f)$ with a correlated term

A, B —> instrument background
C —> DUT noise
channel 1 $X = A + C$
channel 2 $Y = B + C$
A, B, C are independent Gaussian processes
$\Re\{\}$ and $\Im\{\}$ are independent Gaussian processes

Normalization: in 1 Hz bandwidth

$$\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1$$
$$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2$$
$$\mathbb{V}\{C\} = \kappa^2$$
$$\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$$

Cross-Spectrum

$$\langle S_{yx}\rangle_m = \frac{2}{T}\langle YX^*\rangle_m = \frac{2}{T}\langle (Y' + iY'') \times (X' - iX'')\rangle_m$$

Expand using

$$X = (A' + iA'') + (C' + iC'') \quad \text{and} \quad Y = (B' + iB'') + (C' + iC'')$$

Split Syx into three sets

$$\langle S_{yx}\rangle_m = \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{instr}} + \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{mixed}} + \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{DUT}}$$

background only

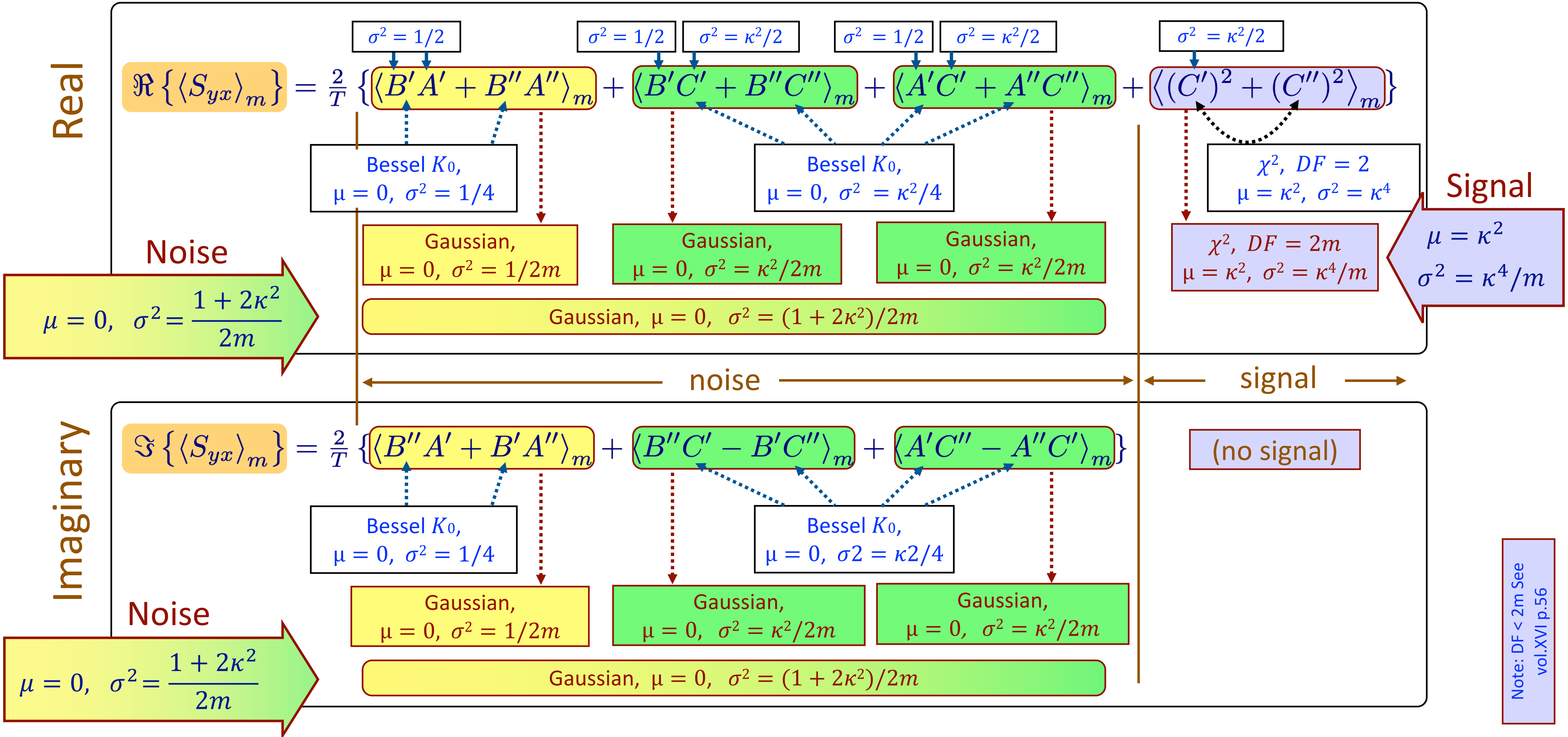
background and DUT noise

DUT noise only

... and work it out !!!

S_{yx} with correlated term $\kappa \neq 0$

All the DUT signal goes in $\Re\{S_{yx}\}$, while $\Im\{S_{yx}\}$ contains only noise



Real

$\sigma^2 = 1/2$

$\sigma^2 = 1/2$ $\sigma^2 = \kappa^2/2$

$\sigma^2 = 1/2$ $\sigma^2 = \kappa^2/2$

$\sigma^2 = \kappa^2/2$

$$\Re\left\{\langle S_{yx}\rangle_m\right\} = \frac{2}{T}\left\{\langle B'A' + B''A''\rangle_m + \langle B'C' + B''C''\rangle_m + \langle A'C' + A''C''\rangle_m + \left\langle (C')^2 + (C'')^2\right\rangle_m\right\}$$

Bessel K_0, $\mu = 0,\ \sigma^2 = 1/4$

Bessel K_0, $\mu = 0,\ \sigma^2 = \kappa^2/4$

$\chi^2,\ DF = 2$ $\mu = \kappa^2,\ \sigma^2 = \kappa^4$

Gaussian, $\mu = 0,\ \sigma^2 = 1/2m$

Gaussian, $\mu = 0,\ \sigma^2 = \kappa^2/2m$

Gaussian, $\mu = 0,\ \sigma^2 = \kappa^2/2m$

$\chi^2,\ DF = 2m$ $\mu = \kappa^2,\ \sigma^2 = \kappa^4/m$

Gaussian, $\mu = 0,\ \sigma^2 = (1+2\kappa^2)/2m$

Noise

$\mu = 0,\ \sigma^2 = \dfrac{1+2\kappa^2}{2m}$

Signal

$\mu = \kappa^2$

$\sigma^2 = \kappa^4/m$

noise signal

Imaginary

$$\Im\left\{\langle S_{yx}\rangle_m\right\} = \frac{2}{T}\left\{\langle B''A' + B'A''\rangle_m + \langle B''C' - B'C''\rangle_m + \langle A'C'' - A''C'\rangle_m\right\}$$

(no signal)

Bessel K_0, $\mu = 0,\ \sigma^2 = 1/4$

Bessel K_0, $\mu = 0,\ \sigma 2 = \kappa 2/4$

Gaussian, $\mu = 0,\ \sigma^2 = 1/2m$

Gaussian, $\mu = 0,\ \sigma^2 = \kappa^2/2m$

Gaussian, $\mu = 0,\ \sigma^2 = \kappa^2/2m$

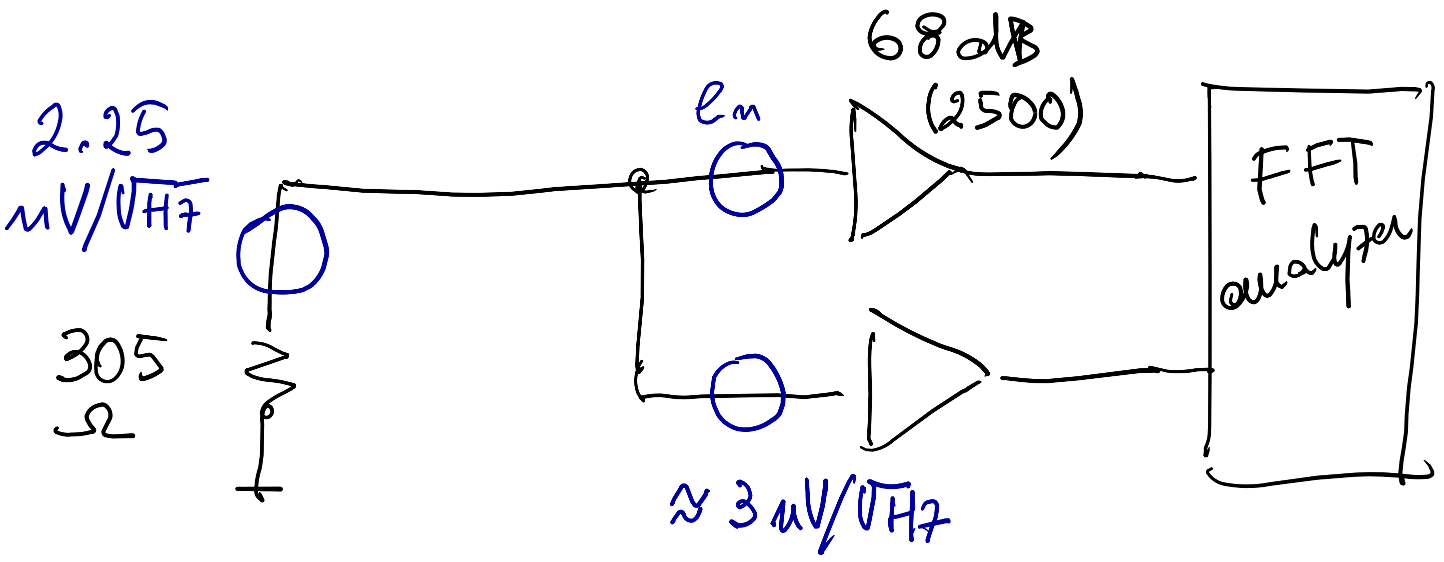
Gaussian, $\mu = 0,\ \sigma^2 = (1+2\kappa^2)/2m$

Noise

$\mu = 0,\ \sigma^2 = \dfrac{1+2\kappa^2}{2m}$

Note: DF < 2m See vol.XVI p.56

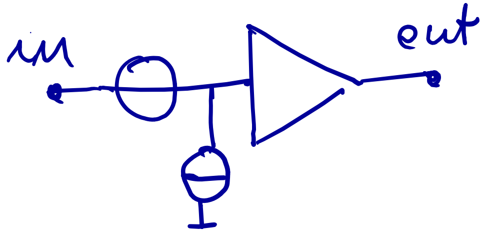
Example / Experiment



Rothe Dahlke model

e_n statistically indep.

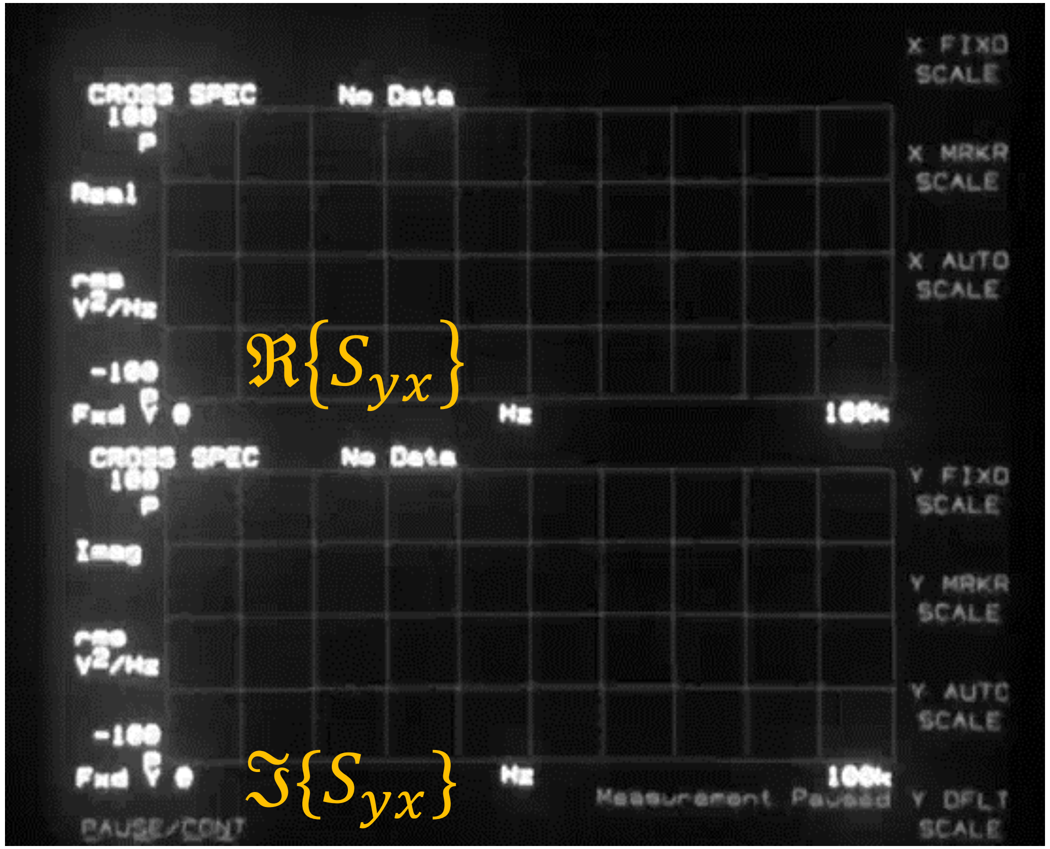
$R\, i_n \to 0$ JFET amplifier



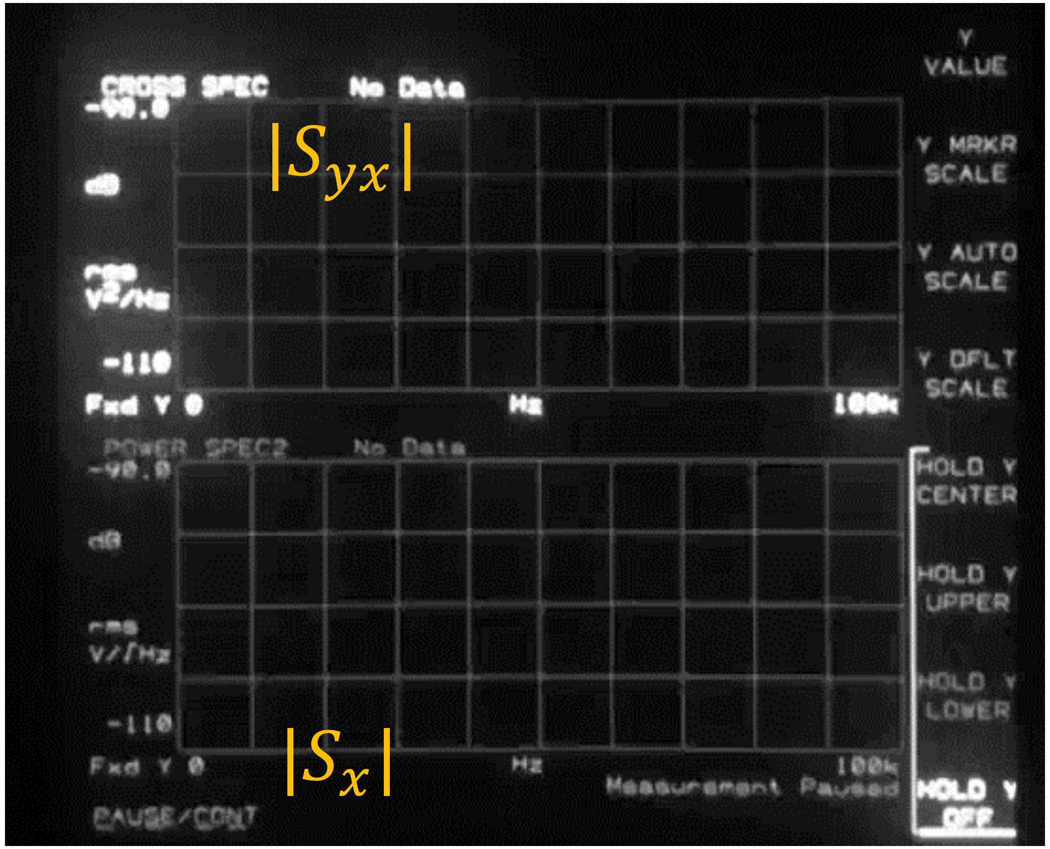
	single ch.	cross	
input	4.6	2.25	nV/√Hz
output	5.6	11.5	μV/√Hz

Experiment – Noise of a 305 Ω resistor

Estimator $\widehat{S_{yx}} = \Re\{S_{yx}\}$, and $\Im\{S_{yx}\}$



Estimator $\widehat{S_{yx}} = |S_{yx}|$, and $|S_x|$



Focus on $\mathbb{E}$ and $\mathbb{V}$

	Term	$\mathbb{E}$	$\mathbb{V}$	PDF	Note
$\mathfrak{R}$	$\langle B'A' + B''A'' + B'C' + B''C'' + C'A' + C''A''\rangle_m$ Bessel K_0, $\mu = 0,\ \sigma^2 = \kappa^2/4$ Bessel K_0, $\mu = 0,\ \sigma^2 = \kappa^2/4$	0	$\dfrac{1+2\kappa^2}{2m}$	Bessel K	average of zero-mean Gaussian processes
$\mathfrak{I}$	$\langle B''A' + B'A'' + B''C' + B'C'' + C''A' + C'A''\rangle_m$	0	$\dfrac{1+2\kappa^2}{2m}$	Bessel K	average of zero-mean Gaussian processes
$\mathfrak{R}$	$\langle C'^2 + C''^2\rangle_m$ white, χ^2, $2\ DF$ $\mu = \kappa^2,\ \sigma^2 = \kappa^4$	κ^2	κ^4/m	χ^2 $r = 2m$	average of χ^2 processes

Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1,\ \mathbb{V}\{C\} = \kappa^2$

$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2,$ and $\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$

Estimator $\hat{S}_{yx} = \Re\{\langle S_{yx}\rangle_m\}$

Best (unbiased) estimator

$$\boxed{\frac{T}{2}\Re\{\langle S_{yx}\rangle_m\}} = \langle B'A' + B''A'' + B'C' + B''C'' + C'A' + C''A''\rangle_m + \langle C'^2 + C''^2\rangle_m$$

$$\mathbb{E} = 0,\ \mathbb{V} = (1+2\kappa^2)/(2m)$$

Noise

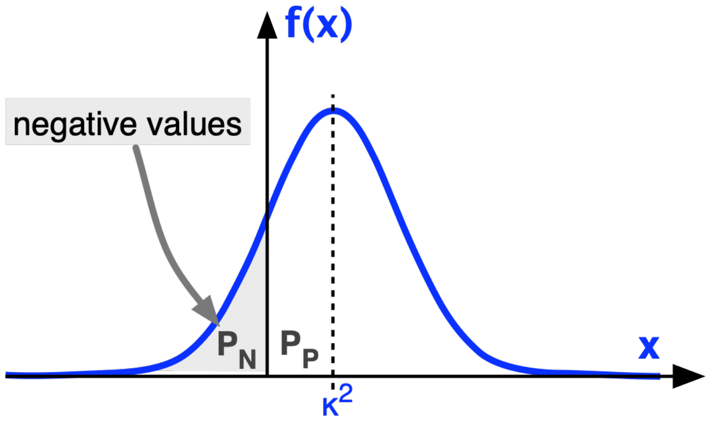
$$\mathbb{E} = \kappa^2,\ \mathbb{V} = \kappa^4/m$$

Signal

$$\mathbb{E}\{\} = \kappa^2$$

$$\sqrt{\mathbb{V}\{\}} = \sqrt{\frac{1+2\kappa^2+2\kappa^4}{2m}} \simeq \frac{1+\kappa^2}{\sqrt{2m}}$$

$$\boxed{\frac{\sqrt{\mathbb{V}\{\}}}{\mathbb{E}}} = \frac{\sqrt{1+2\kappa^2+2\kappa^4}}{\kappa^2\sqrt{2m}} \simeq \boxed{\frac{1+\kappa^2}{\kappa^2\sqrt{2m}}}$$



$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\kappa^2}{\sqrt{2}\,\sigma}\right)$$

0 dB SNR requires that $m = 1/2\kappa^4$.
Example $\kappa = 0.1$ (DUT noise 20 dB lower than single-channel background).
Averaging on $5{\times}10^3$ spectra is necessary to get SNR = 0 dB.

Estimator $\hat{S}_{yx} = |\langle S_{yx} \rangle_m|\,,\ \kappa \to 0$

The default of
most instruments

$$|\langle S_{yx} \rangle_m| = \frac{2}{T} \sqrt{[\Re\{\langle YX^* \rangle_m\}]^2 + [\Im\{\langle YX^* \rangle_m\}]^2}$$

noise, Re

signal

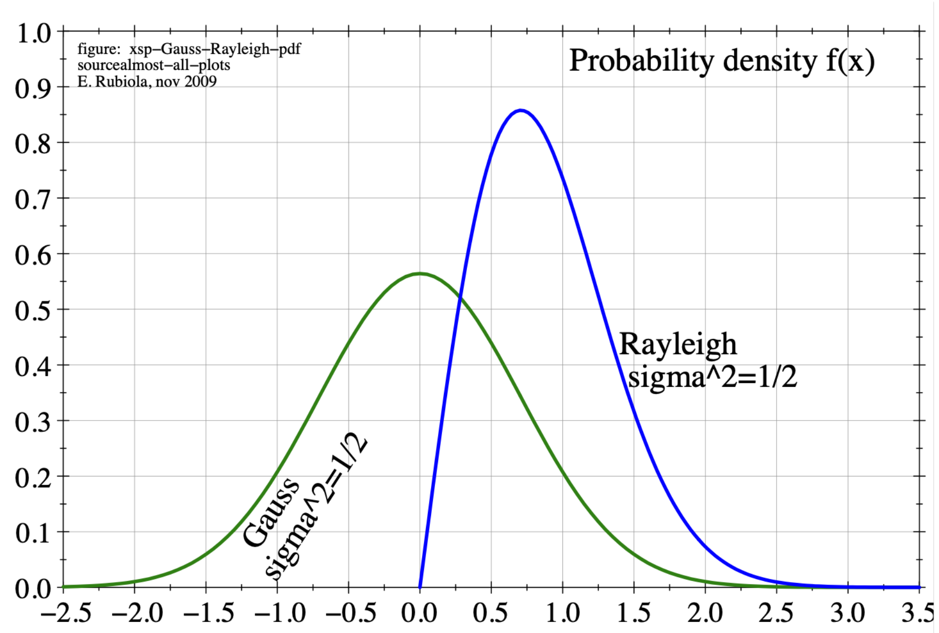
noise, Im

$\kappa \to 0$ Rayleigh distribution

$$\frac{T}{2}\mathbb{E}\left\{\left|\langle S_{yx} \rangle_m\right|\right\} = \sqrt{\frac{\pi}{4m}} = \frac{0.886}{\sqrt{m}}$$

$$\frac{T}{2}\mathbb{V}\left\{\left|\langle S_{yx} \rangle_m\right|\right\} = \frac{1}{m}\left(1 - \frac{\pi}{4}\right) = \frac{0.215}{m}$$

$$\frac{\text{dev}\left\{\left|\langle S_{yx} \rangle_m\right|\right\}}{\mathbb{E}\left\{\left|\langle S_{yx} \rangle_m\right|\right\}} = \sqrt{\frac{4}{\pi} - 1} = 0.523$$

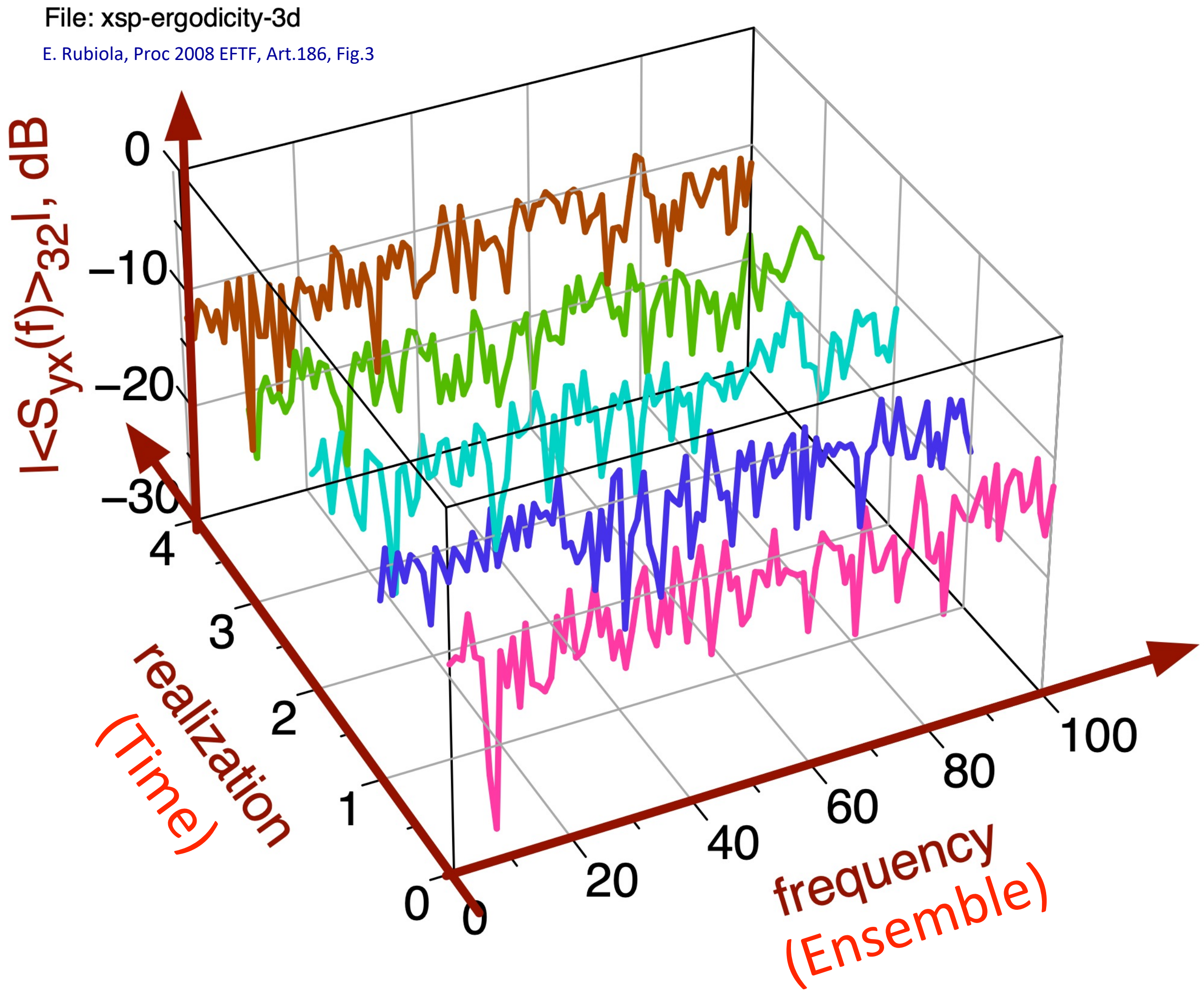


Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1,\ \mathbb{V}\{C\} = \kappa^2$
$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2,$ and $\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$

Ergodicity

Let's collect a sequence of spectra

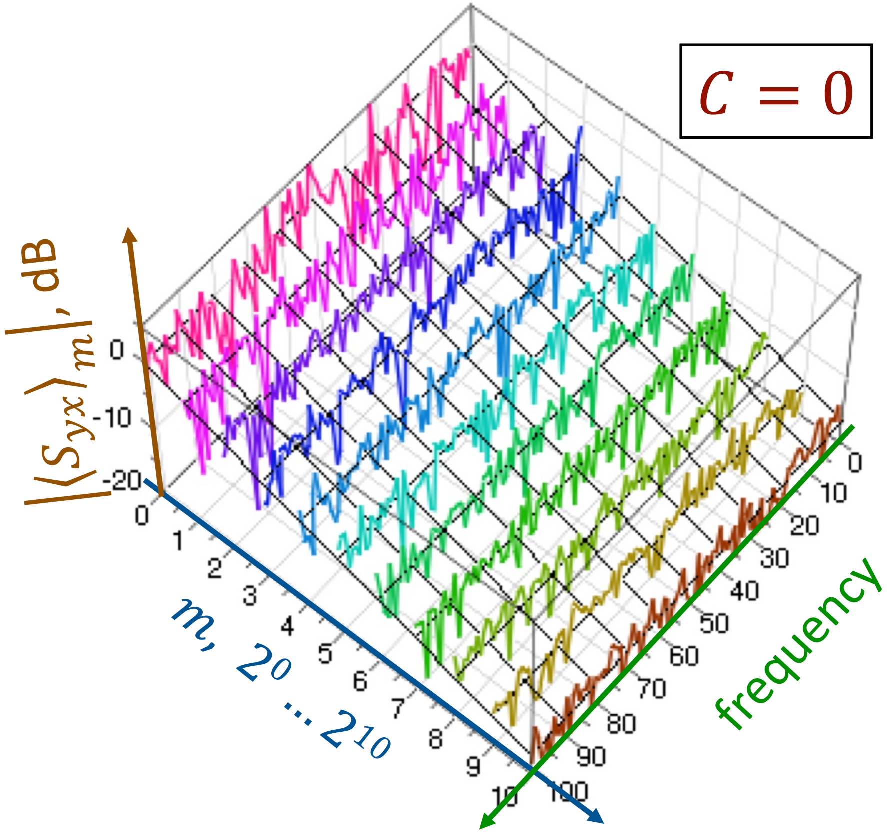
- Ergodicity —> Interchange
    - time /ensemble statistics
    - sequence-index i and frequency f.
- Same average and the deviation on
    - frequency axis
    - sequence of spectra

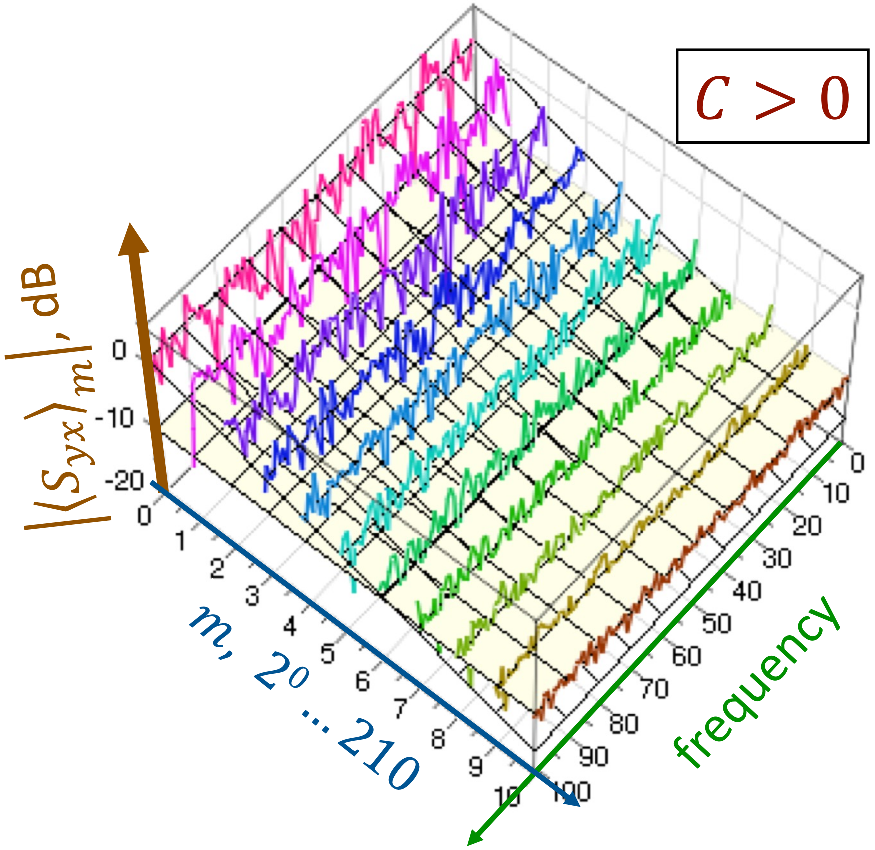

File: xsp-ergodicity-3d

E. Rubiola, Proc 2008 EFTF, Art.186, Fig.3

E. Rubiola, The Magic of Cross Correlation in Measurements from DC to Optics, Proc EFTF, Art n.186, April 2008
E. Rubiola, F. Vernotte, The cross-spectrum experimental method, arXiv:1003.0113 [physics.ins-det], 27 Feb 2010

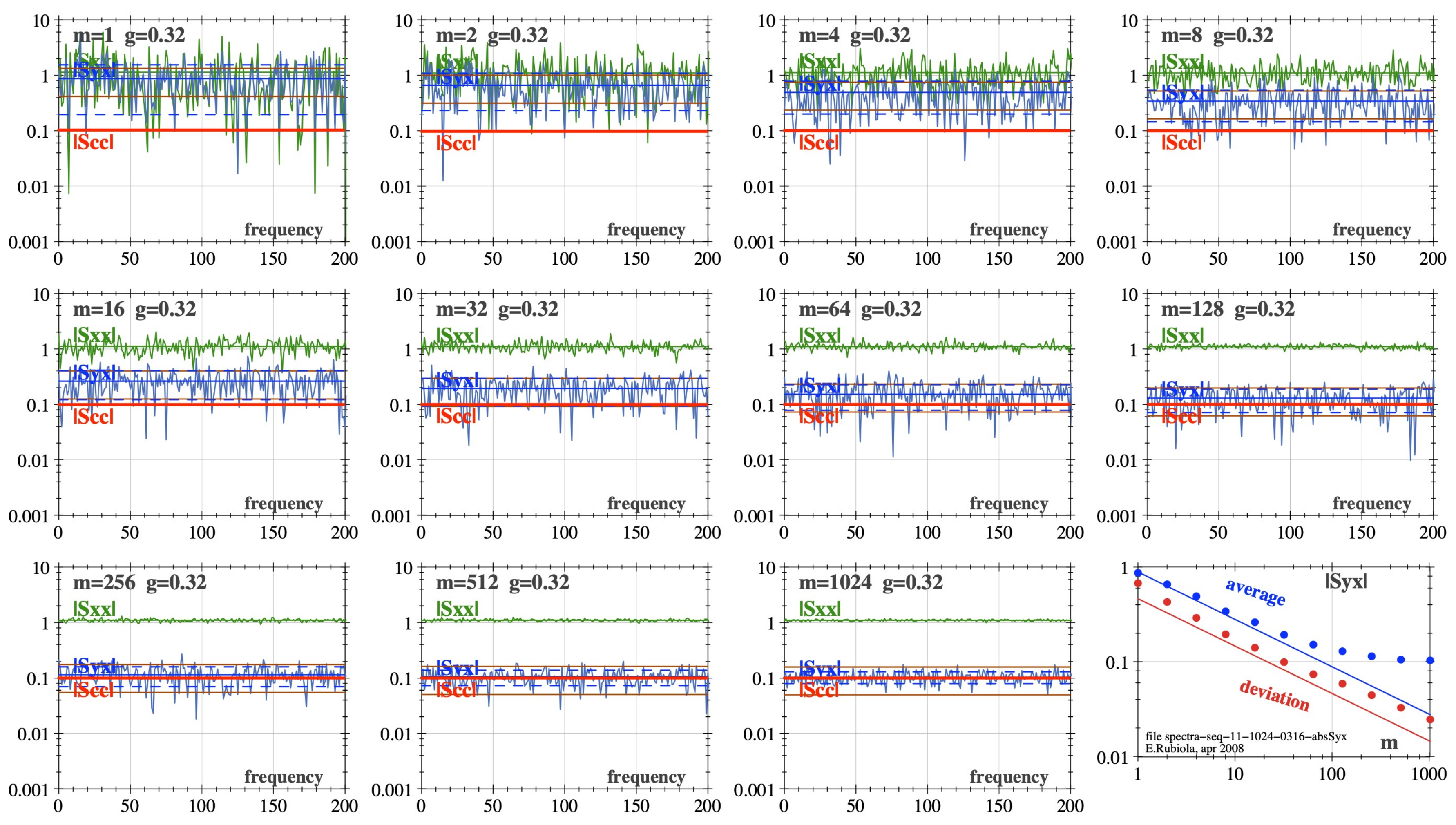
Example: $|S_{yx}|$


$C = 0$
$|\langle S_{yx} \rangle_m|$, dB
0
-10
-20
0
1
2
3
4
5
6
7
8
9
10
m, $2^0 \dots 2^{10}$
0
10
20
30
40
50
60
70
80
90
100
frequency


$C > 0$
$|\langle S_{yx} \rangle_m|$, dB
0
-10
-20
0
1
2
3
4
5
6
7
8
9
10
m, $2^0 \dots 210$
0
10
20
30
40
50
60
70
80
90
100
frequency

Measurement of $|S_{yx}|$ with $\kappa > 0$

video –> skip



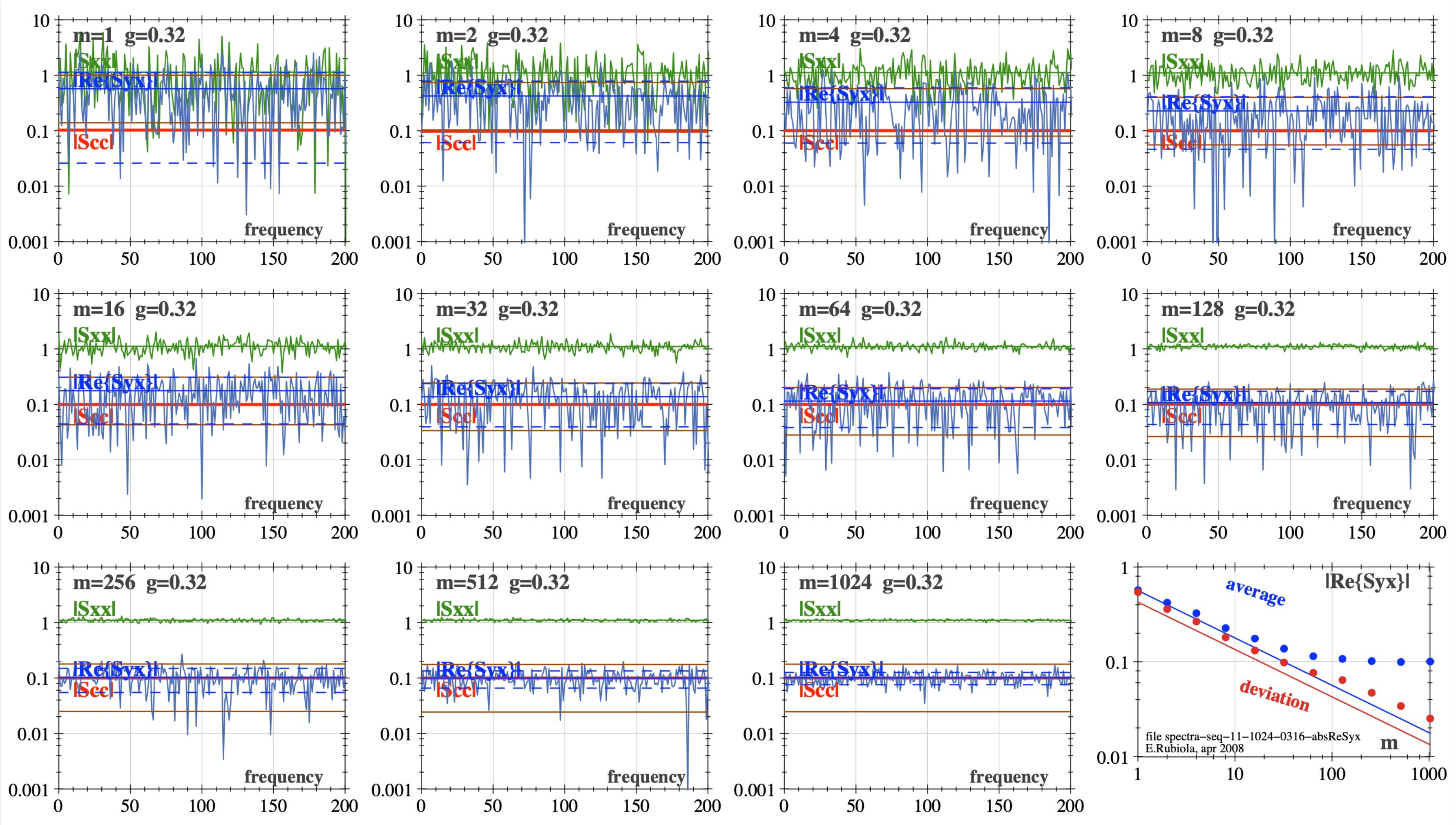
Running the measurement, m increases

S_{xx} shrinks => better confidence level

S_{yx} decreases => higher single-channel noise rejection

Measurement of $\Re\{S_{yx}\}$ with $\kappa > 0$

video -> skip



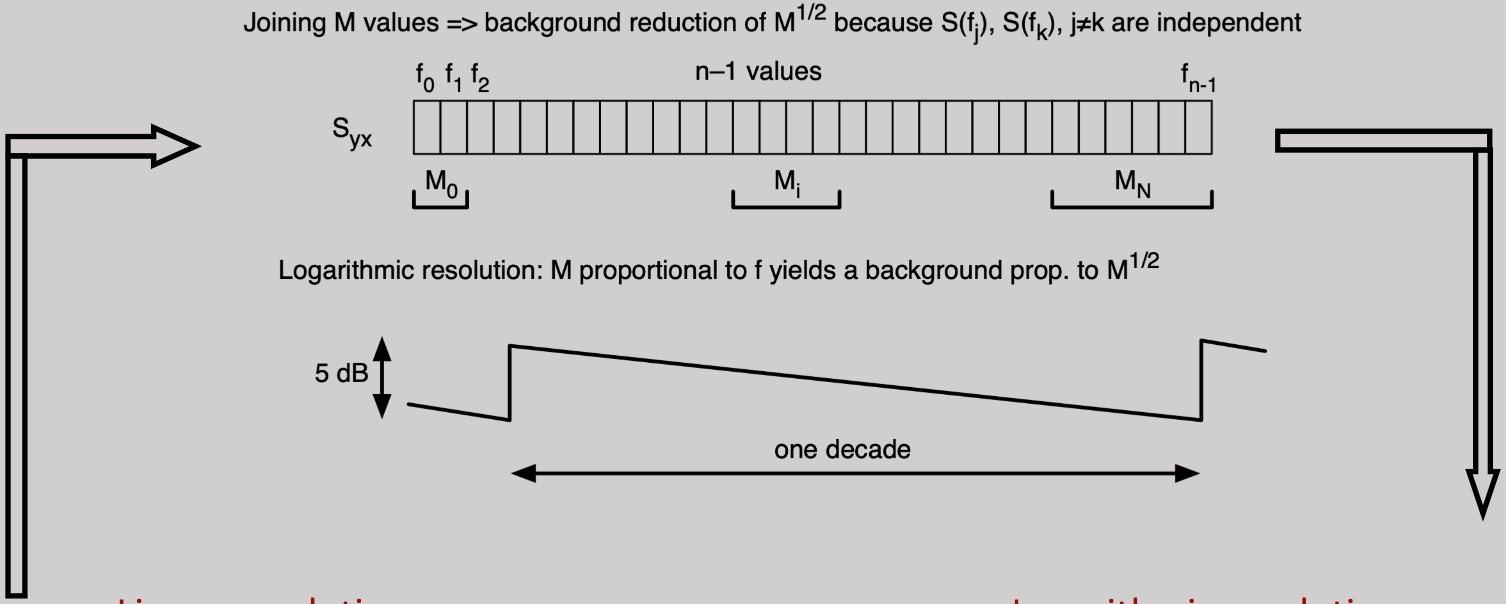
Running the measurement, m increases

S_{xx} shrinks => better confidence level

S_{yx} decreases => higher single-channel noise rejection

Linear vs logarithmic resolution

skip



Joining M values => background reduction of $M^{1/2}$ because $S(f_j)$, $S(f_k)$, $j \neq k$ are independent

f_0 f_1 f_2

n–1 values

f_{n-1}

S_{yx}

M_0

M_i

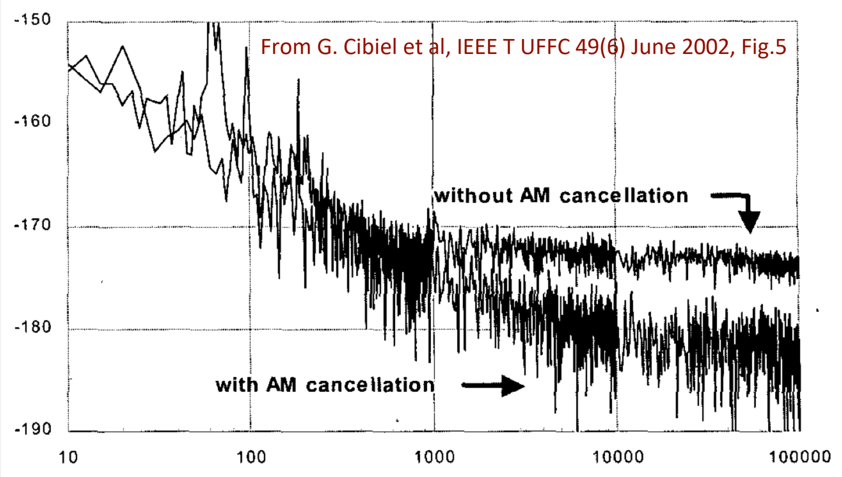
M_N

Logarithmic resolution: M proportional to f yields a background prop. to $M^{1/2}$

5 dB

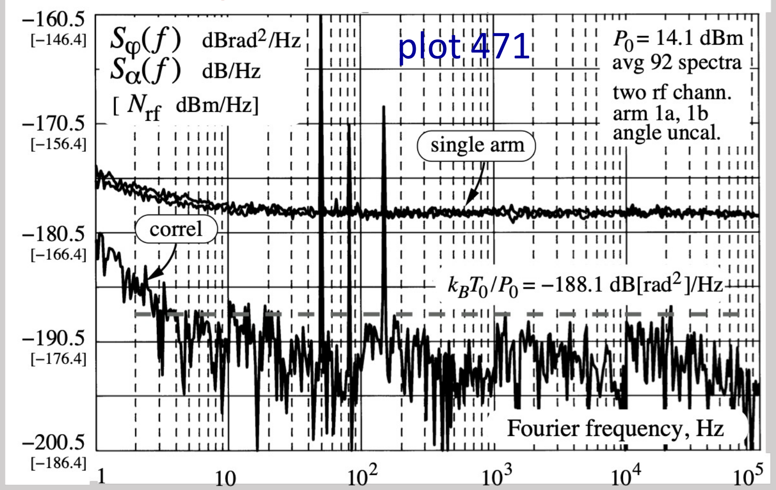
one decade

Linear resolution



From G. Cibiel et al, IEEE T UFFC 49(6) June 2002, Fig.5

Logarithmic resolution



From E. Rubiola, V. Giordano, RSI 73(6) jun 2002, Fig.7

Conclusions

- Rejection of the instrument noise
- AM noise, RIN, etc. –> validation of the instrument without a reference low-noise source
- Display quantities

  $\left\langle \Re\{S_{yx}\} \right\rangle_m$ is the best estimator, fast and accurate

  $\left\langle \Im\{S_{yx}\} \right\rangle_m$ gives the background noise

  $\left| \left\langle S_{yx} \right\rangle_m \right|$ is a poor choice: biased, and 4-fold measurement time
- Applications in many fields of metrology

The cross spectrum method is magic

Correlated noise makes magic difficult

home page http://rubiola.org

Appendix: Statistics

Boring but necessary exercises

Vocabulary of statistics

- A random process $\mathbf{x}(t)$ is defined through a random experiment e that associates a function $x_{\mathrm{e}}(t)$ to each outcome e.
  - The set of all the possible $x_{\mathrm{e}}(t)$ is called ensemble
  - The function $x_{\mathrm{e}}(t)$ is called realization or sample function.
  - The ensemble average is called mathematical expectation $\mathbb{E}\{\}$
- A random process is said stationary if its statistical properties are independent of time.
  - Often we restrict the attention to some statistical properties.
  - Broadly similar to the physical concept of repeatability.
- A random process $\mathbf{x}(t)$ said ergodic if a realization observed in time has the statistical properties of the ensemble.
  - Ergodicity makes sense only for stationary processes.
  - Often we restrict the attention to some statistical properties.
  - Broadly similar to the physical concept of reproducibility

Example: thermal noise of a resistor of value R

- The experiment e is the random choice of a resistor e
- The realization $x_{\mathrm{e}}(t)$ is the noise waveform measured across the resistor e
- We always measure $\langle x^2 \rangle = 4kTR\Delta f$, so the process is stationary
- After measuring many resistors, we conclude that $\langle x^2 \rangle = 4kTR\Delta f$ always holds. The process is ergodic.

A relevant property of noise

A theorem states that

there is no a-priori relation between PDF[1] and PSD

For example, white noise can originate from

- Poisson process (emission of a particle at random time)
- Random telegraph (random switch between two level)
- Thermal noise (Gaussian)

(1) PDF = Probability Density Function

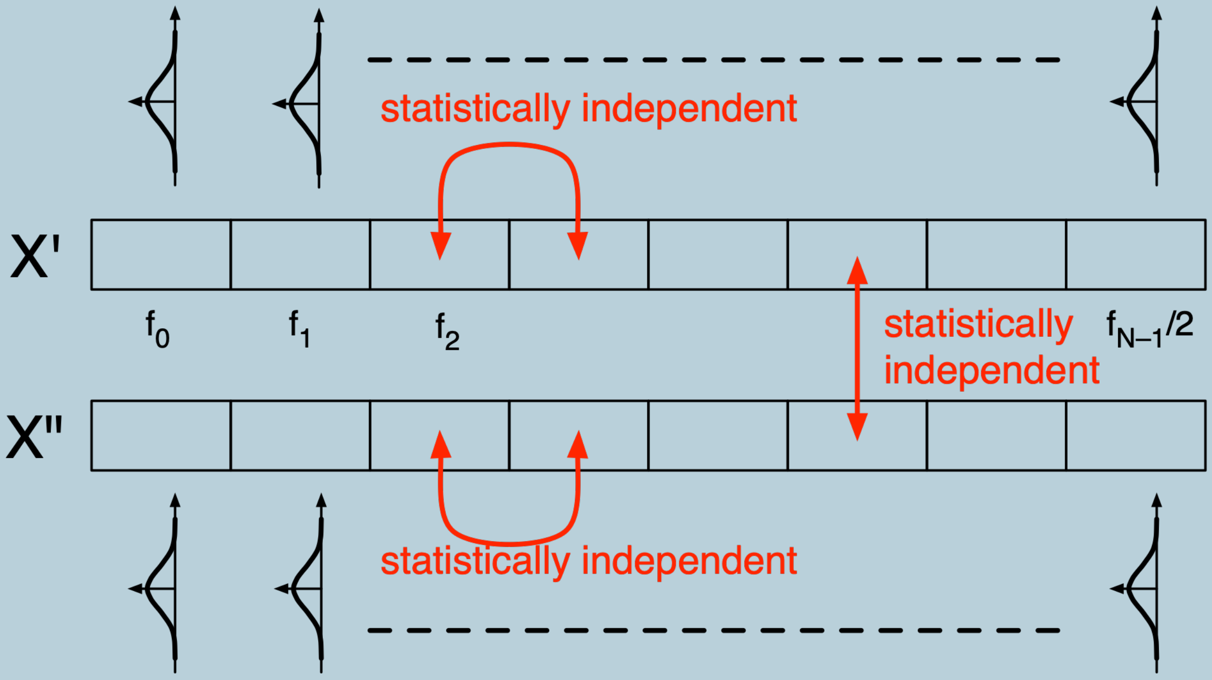
Why white Gaussian noise?

- Whenever randomness occurs at microscopic level, noise tends to be Gaussian (central-limit theorem)
- Most environmental effects are not “noise” in strict sense (often, they are more *disturbing* than noise)
- Colored noise types ($1/f$, $1/f^2$, etc.) can be whitened, analyzed, and un-whitened
- Of course, WG noise is easy to understand

Zero-mean white Gaussian noise

$$x(t) \leftrightarrow X(f) = X'(f) + iX''(f)$$

1. Both $x(t) \leftrightarrow X(f)$ are Gaussian
2. $X(f_1)$ and $X(f_2)$, $f_1 \neq f_2$
   1. are statistically independent,
   2. $\mathbb{V}\{X(f_1)\} = \mathbb{V}\{X(f_2)\}$
3. real and imaginary part:
   1. X' and X'' are statistically independent
   2. $\mathbb{V}\{X'\} = \mathbb{V}\{X''\} = \frac{1}{2}\mathbb{V}\{X\}$
4. $Y = X_1 + X_2$
   1. Y is Gaussian
   2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\} + \mathbb{V}\{X_2\}$
5. $Y = X_1 X_2$
   1. Y is Bessel K_0
   2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\}\mathbb{V}\{X_2\}$



N degrees of freedom

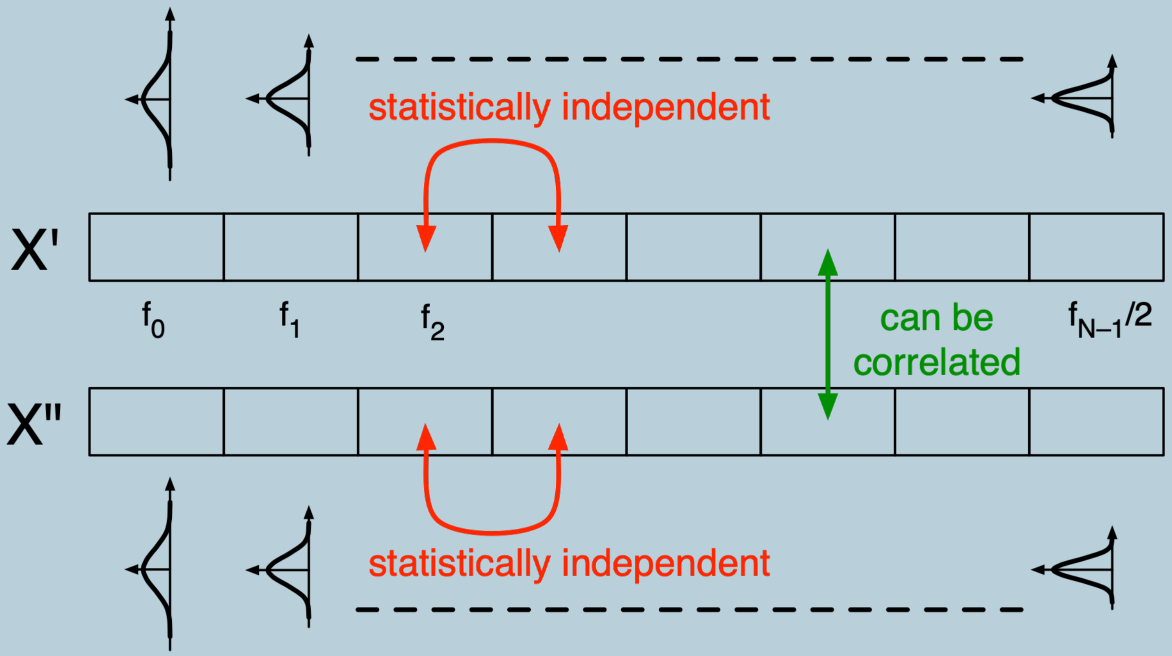
Properties of parametric noise

$$x(t) \leftrightarrow X(f) = X'(f) + iX''(f)$$

1. Pair $x(t) \leftrightarrow X(f)$
    1. there is no a-priori relation between the distribution of $x(t)$ and $X(f)$ (theorem)
    2. Central limit theorem: $x(t)$ and $X(f)$ end up to be Gaussian
2. $X(f_1)$ and $X(f_2)$
    1. generally, statistically independent
    2. $\mathbb{V}\{X(f_1)\} \neq \mathbb{V}\{X(f_2)\}$ in general
3. Real and imaginary part, same frequency
    1. $X'(f)$ and $X''(f)$ can be correlated
    2. in general, $\mathbb{V}\{X'\} \neq \mathbb{V}\{X''\}$
4. $Y = X_1 + X_2$, zero-mean independent Gaussian

$$\mathbb{V}\{Y\} = \mathbb{V}\{X_1\} + \mathbb{V}\{X_2\}$$

5. If X_1 and X_2 are zero-mean independent Gaussian
    1. $Y = X_1 X_2$ is zero-mean Bessel K
    2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\}\mathbb{V}\{X_2\}$



N degrees of freedom

Gaussian (normal) distribution

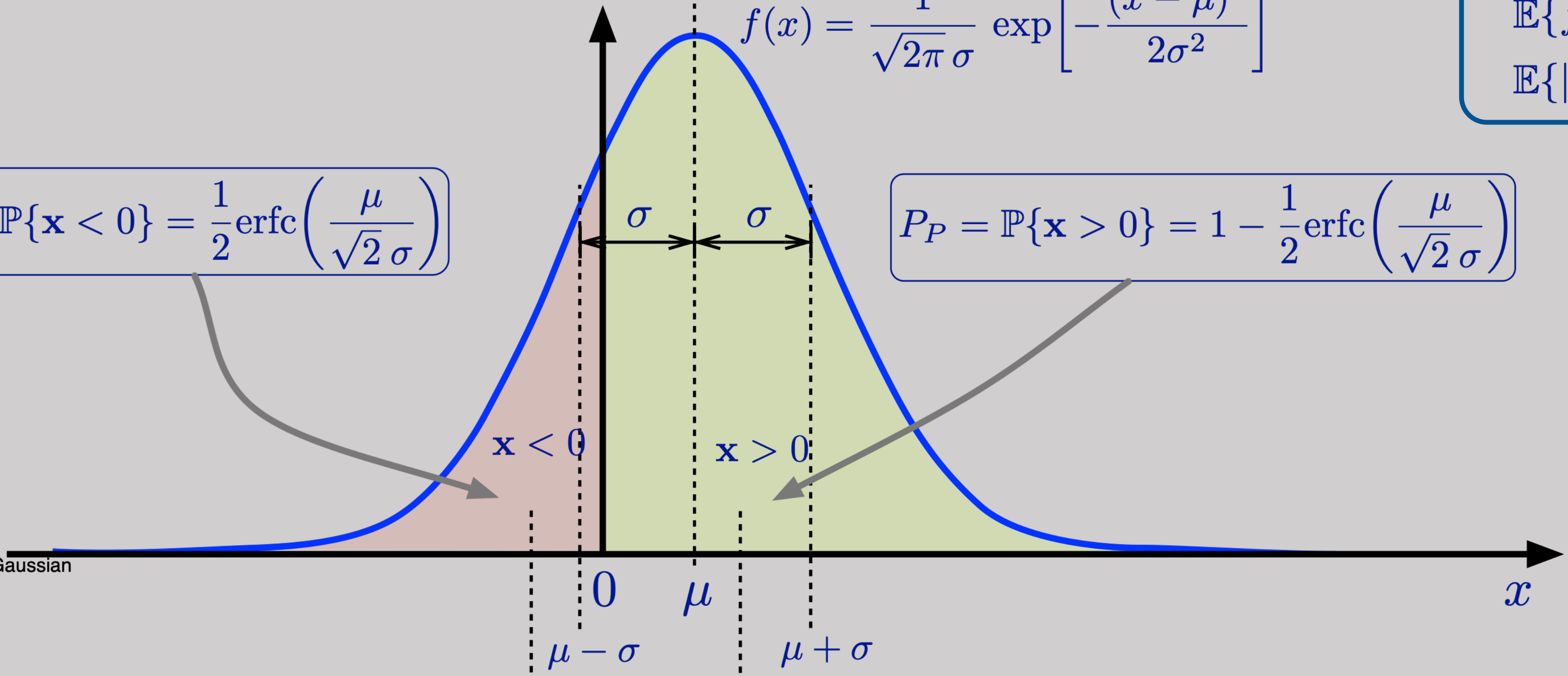
x is normal distributed with mean μ and variance σ^2

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\mathbb{E}\{f(x)\} = \mu$$

$$\mathbb{E}\{f^2(x)\} = \mu^2 + \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



File: xsp-Gaussian

$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)$$

$$P_P = \mathbb{P}\{\mathbf{x} > 0\} = 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)$$

$$\mu_N = \mu - \frac{1}{\frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)}\,\frac{\sigma}{\sqrt{2\pi\exp(\mu^2/\sigma^2)}}$$

$$\mu_P = \mu + \frac{1}{1 - \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)}\,\frac{\sigma}{\sqrt{2\pi\exp(\mu^2/\sigma^2)}}$$

Sum and average of random variables

1. The central limit theorem states that

   For large m, the PDF of the sum of m statistically independent processes tends to a Gaussian distribution

2. Let $X = X_1 + X_2 + \cdots + X_m$ be the sum of m processes of mean $\mu_1, \mu_2 \ldots \mu_m$ and variance $\sigma_1^2, \sigma_2^2, \ldots \sigma_m^2$. The process X tends to Gaussian PDF, expectation

   Expectation $\mathbb{E}\{X\} = \mu_1 + \mu_2 + \cdots + \mu_m$

   Variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2$

3. The average $\langle X \rangle_m = \frac{1}{m}(X_1 + X_2 + \cdots + X_m)$ has Gaussian PDF,

$$\mathbb{E}\{X\} = \frac{1}{m}(\mu_1 + \mu_2 + \cdots + \mu_m), \text{ and}$$

$$\sigma^2 = \frac{1}{m}(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2)$$

Since white noise and flicker noise arise from the sum of a large number of small-scale phenomena, they are Gaussian distributed

PDF = Probability Density Function

Children of the Gaussian distribution

Chi-square

$$\chi^2 = \sum_i x_i^2$$

Bessel K_0

$$x = x_1 x_2$$

Rayleigh

$$x = \sqrt{x_1^2 + x_2^2}$$

One-Sided Gaussian

Chi-square (χ^2) distribution

Definition

DF = degrees of freedom

x_i are normal distributed variables

zero mean, and variance σ^2

$$\chi^2 = \sum_{i=1}^{r} x_i^2$$

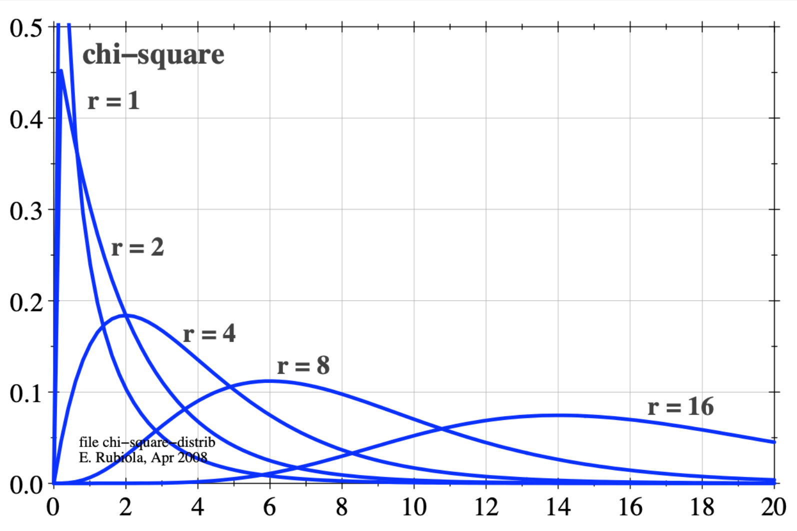
is χ^2 distributed with r DF

Sum

The sum of m χ^2-distributed variables

$$\chi^2 = \sum_{j=1}^{m} \chi_j^2 \;, \qquad r = \sum_{j=1}^{m} r_j$$

has χ^2 distribution with $r = m$ DF



$$f(x) = \frac{x^{\frac{r}{2}-1}\, e^{-\frac{x^2}{2}}}{\Gamma\left(\frac{1}{2}r\right) 2^{\frac{r}{2}}} \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sigma^2 r$$

$$\mathbb{E}\{[f(x)]^2\} = \sigma^4 r(r+2)$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = 2\sigma^4 r$$

$z! = \Gamma(z+1), \quad z \in \mathbb{N}$

Averaging m complex χ^2 variables

averaging m variables$|X|^2$, complex $X = X' + iX''$,
yields a χ^2 distribution with $r = 2m$

$$\frac{1}{m}\,\chi^2 = \frac{1}{m}\sum_{j=1}^{m}(X_j')^2 + (X_j'')^2$$

$$\mathbb{E}\left\{\frac{1}{m}\,f(x)\right\} = \frac{\sigma^2 r}{m} = 2\sigma^2$$

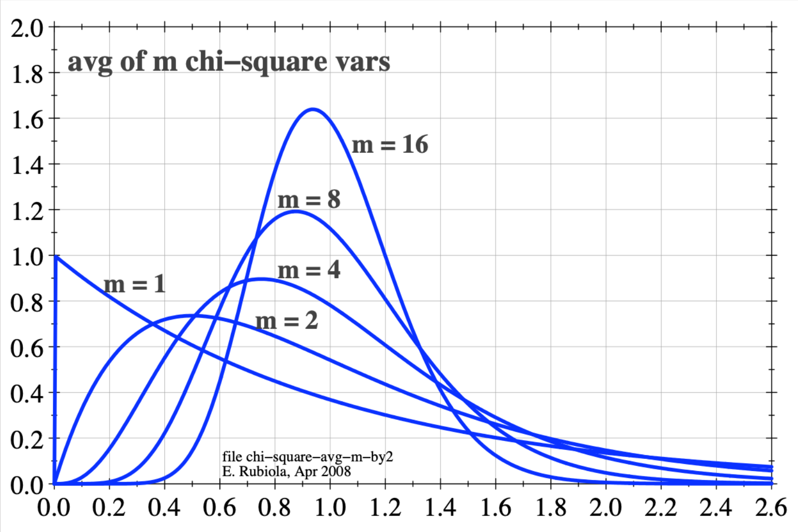
$$\mathbb{E}\left\{\left|\frac{1}{m}\,f(x) - \mathbb{E}\left\{\frac{1}{m}\,f(x)\right\}\right|^2\right\} = \frac{2\sigma^4 r}{m^2} = \frac{4\sigma^4}{m}$$

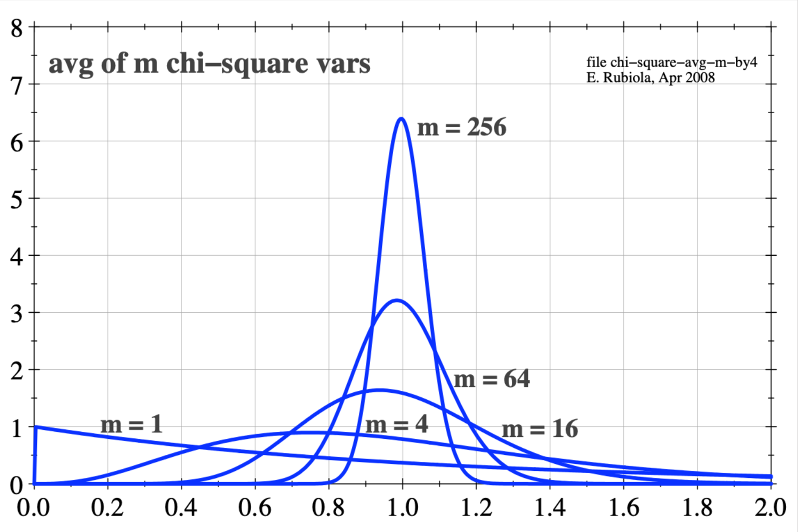
$$\frac{\text{dev}}{\text{avg}} = \frac{1}{\sqrt{m}}$$

relevant case: $\sigma^2 = 1/2$

$\text{avg} = 1$

$\text{dev} = \dfrac{1}{\sqrt{m}}$





Product of independent zero-mean Gaussian random variables

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

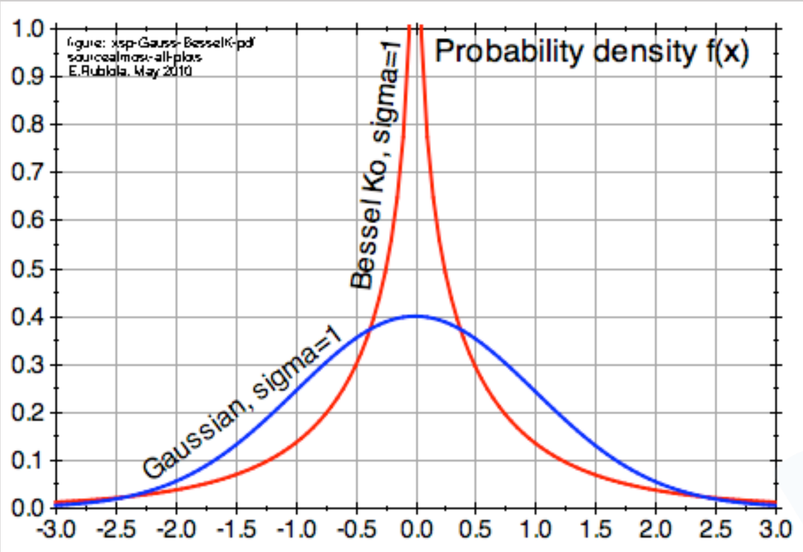
$$x = x_1 x_2$$

x has Bessel K_0 distribution with variance $\sigma^2 = \sigma_1^2 \sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma} K_0\left(-\frac{|x|}{\sigma}\right)$$

$$\mathbb{E}\{f(x)\} = 0$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



Bessel K_0 distribution

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

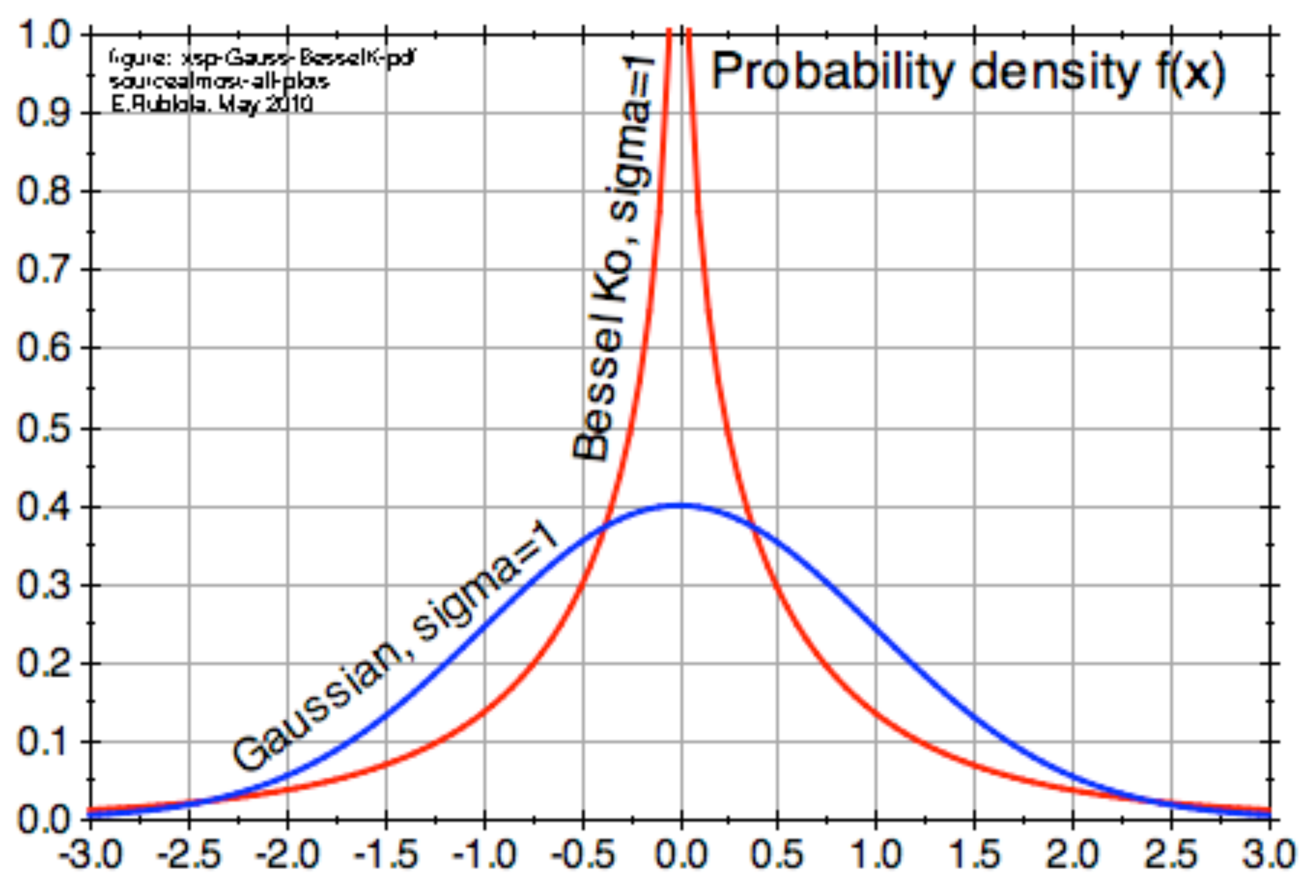
$$x = x_1\, x_2$$

x has Bessel K_0 distribution with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma}\, K_0\left(-\frac{|x|}{\sigma}\right)$$

$$\mathbb{E}\{f(x)\} = 0$$

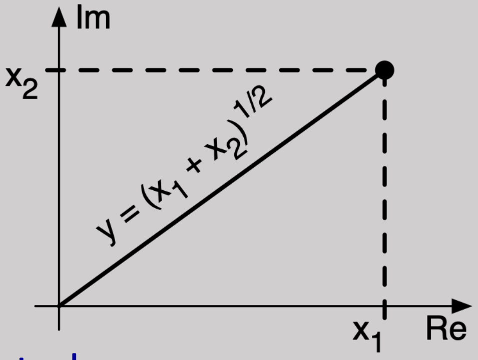
$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



Rayleigh distribution

x_1 and x_2 are normal distributed with zero mean and equal variance σ^2

$$x = \sqrt{x_1^2 + x_2^2}$$



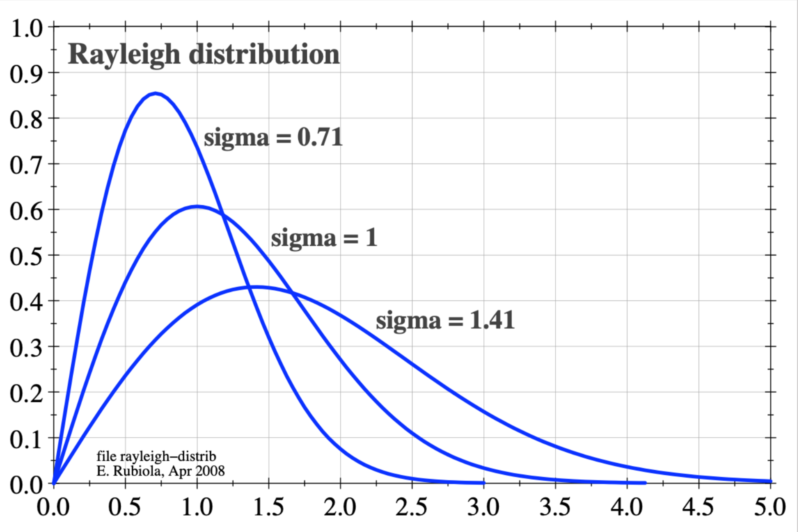
x is Rayleigh-distributed

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x \ge 0$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{\pi}{2}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = 2\sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \frac{4-\pi}{2}\sigma^2$$



Rayleigh distribution with $\sigma^2 = 1/2$

quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
$\text{average} = \sqrt{\frac{\pi}{4}}$	0.886 [−0.525]
$\text{deviation} = \sqrt{1 - \frac{\pi}{4}}$	0.463 [−3.34]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{4}{\pi} - 1}$	0.523 [−2.82]
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{4}{\pi} - 1}$	1.523 [+1.83]
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{4}{\pi} - 1}$	0.477 [−3.21]
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{4/\pi - 1}}{1 - \sqrt{4/\pi - 1}}$	3.19 [5.04]

One-sided Gaussian distribution

x is normal distributed with zero mean and variance σ^2

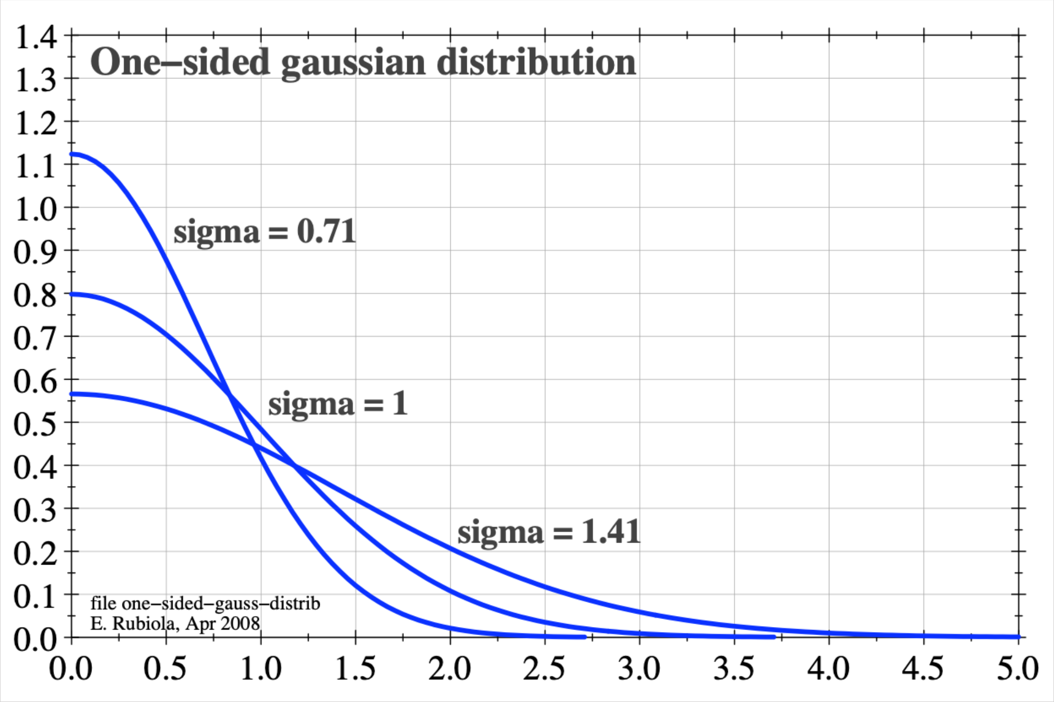
$$y = |x|$$

$$f(x) = 2\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = \sigma^2$$

$$\mathbb{E}\{|f(x)-\mathbb{E}\{f(x)\}|^2\} = \left(1-\frac{2}{\pi}\right)\sigma^2$$



one-sided Gaussian distribution with $\sigma^2 = 1/2$	
quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
average $= \sqrt{\frac{1}{\pi}}$	0.564 [−2.49]
deviation $= \sqrt{\frac{1}{2}-\frac{1}{\pi}}$	0.426 [−3.70]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{\pi}{2}-1}$	0.756 [−1.22]
$\frac{\text{avg}+\text{dev}}{\text{avg}} = 1+\sqrt{\frac{1}{2}-\frac{1}{\pi}}$	1.756 [+2.44]
$\frac{\text{avg}-\text{dev}}{\text{avg}} = 1-\sqrt{\frac{1}{2}-\frac{1}{\pi}}$	0.244 [−6.12]
$\frac{\text{avg}+\text{dev}}{\text{avg}-\text{dev}} = \frac{1+\sqrt{1/2-1/\pi}}{1-\sqrt{1/2-1/\pi}}$	7.18 [8.56]

Applications of the Cross Spectrum Measurement

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

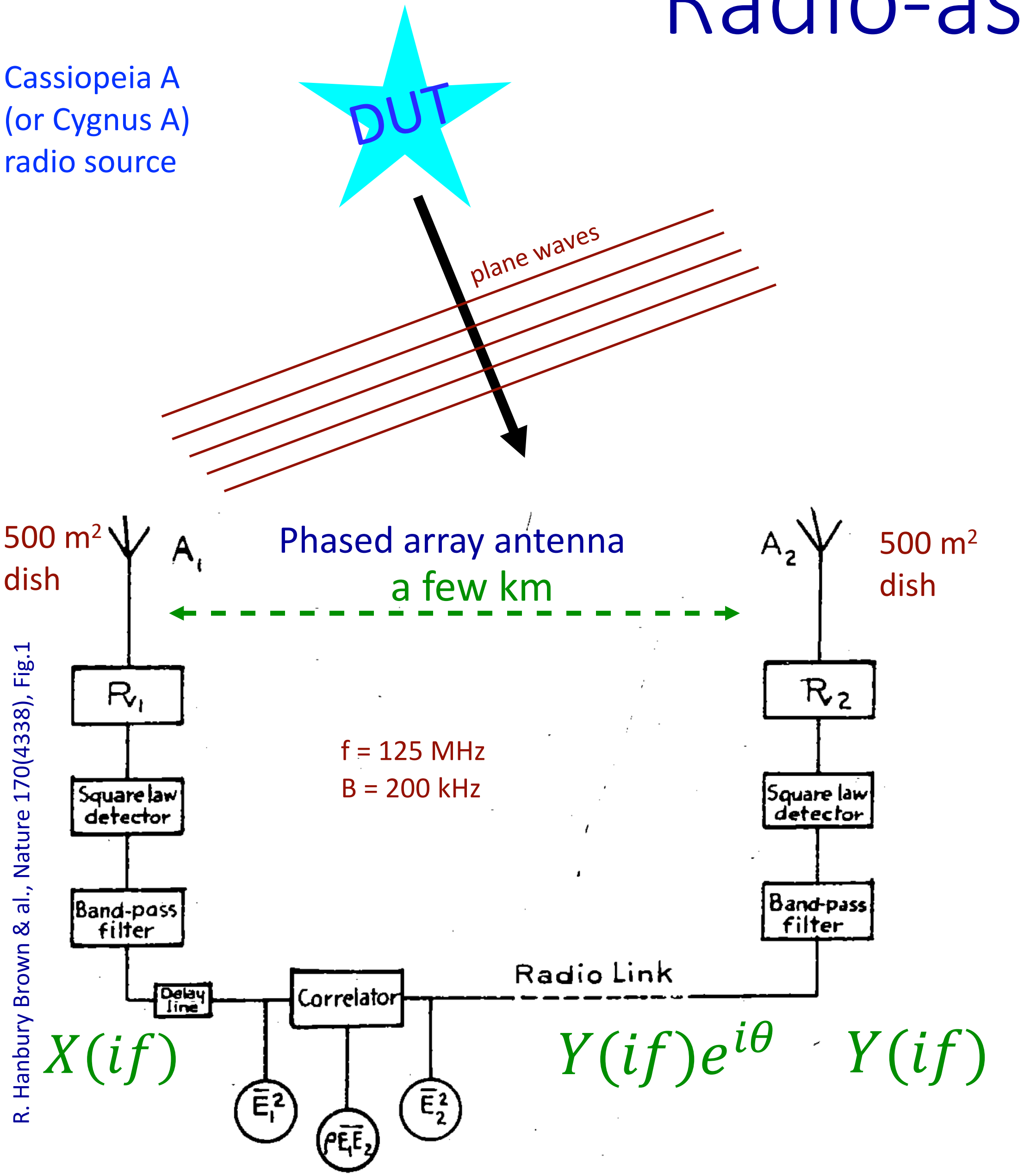
home page http://rubiola.org

Summary

- Radio-astronomy (Hanbury-Brown, 1952)
- Early implementations
- Radiometry (Allred, 1962)
- Noise calibration (Spietz, 2003)
- Frequency noise (Vessot 1964)
- Phase noise (Walls 1976)
- Dual delay line system (Lance, 1982)
- Phase noise (Rubiola 2000 & 2002)
- Effect of amplitude noise (Rubiola, 2007)
- Frequency stability of a resonator (Rubiola)
- Dual-mixer time-domain instrument (Allan 1975, Stein 1983)
- Amplitude noise & laser RIN (Rubiola 2006)
- Noise of a power detector (Grop & Rubiola)
- Noise in chemical batteries (Walls 195)
- Semiconductors (Sampietro RSI 1999)
- Electromigration in thin films (Stoll 1989)
- Fundamental definition of temperature
- Hanbury Brown - Twiss effect (Hanbury-Brown & Twiss 1956, Glattli 2004)

The real fun starts here

Radio-astronomy



Cassiopeia A
(or Cygnus A)
radio source

DUT

plane waves

500 m^2 dish

Phased array antenna

a few km

500 m^2 dish

f = 125 MHz
B = 200 kHz

$X(if)$

$Y(if)e^{i\theta}$

$Y(if)$

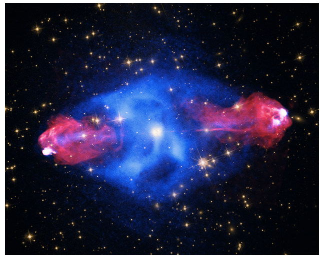
R. Hanbury Brown & al., Nature 170(4338), Fig.1

Measurement of the apparent angular size of stellar radio sources

Jodrell Bank, Manchester, UK

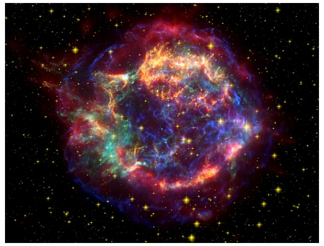
- The radio link breaks the hypothesis of symmetry of the two channels, introducing a phase θ
- The cross spectrum is complex
- The antenna directivity results from the phase relationships
- The phase of the cross spectrum indicates the direction of the radio source

α Cigni (Deneb)



NASA

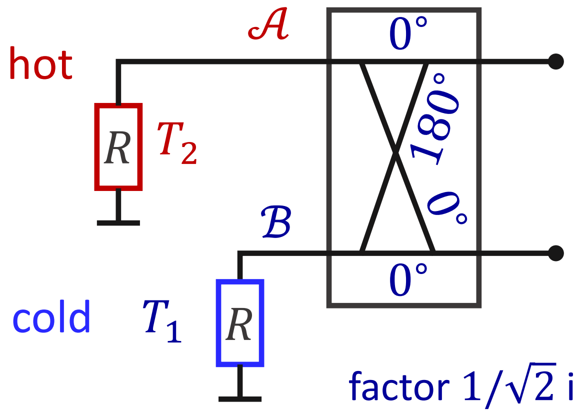
α Cassiopeiae (Schedar)



NASA

R. Hanbury Brown & al., Nature 170(4338) p.1061-1063, 20 Dec 1952
R. Hanbury Brown, R. Q. Twiss, Phyl. Mag. ser.7 no.366 p.663-682

Radiometry & Johnson thermometry



$X = \frac{1}{\sqrt{2}}(\mathcal{A} + \mathcal{B})$

$Y = \frac{1}{\sqrt{2}}(\mathcal{A} - \mathcal{B})$

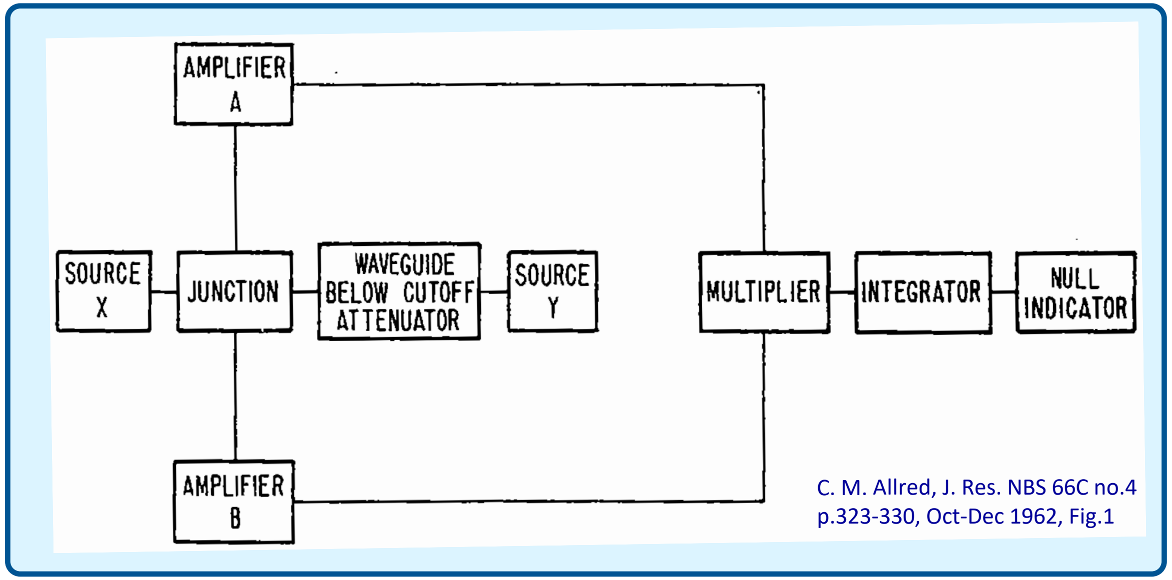
factor $1/\sqrt{2}$ is energy conservation

Temperature difference

$$S_{yx} = \tfrac{1}{2}k(T_2 - T_1)$$

$$T_2 - T_2 < 0 \;\Rightarrow S_{yx} < 0$$

See also E.Rubiola, V.Giordano, RSI 73(6), June 2002



C. M. Allred, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962, Fig.1

noise comparator

C. M. Allred, A precision noise spectral density comparator, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962

Article made publicly available by NIST, https://nvlpubs.nist.gov/nistpubs/jres/66C/jresv66Cn4p323_A1b.pdf

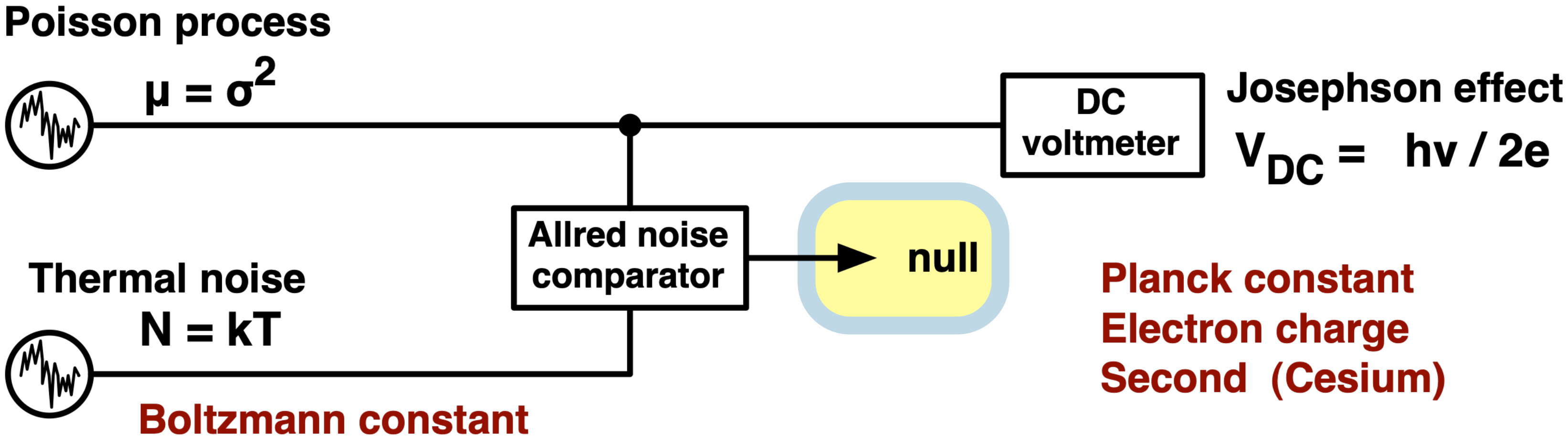
Conceptual implementation of the Kelvin

Boltzmann constant $k = 1.380649 \times 10^{-23}$ J/K exact (≥20 May 2019)

thermal noise $S = kT$

shot noise $S = 2eIR$

high accuracy of I
with a dc instrument



Poisson process

$\mu = \sigma^2$

DC voltmeter

Josephson effect

$V_{DC} = h\nu / 2e$

Allred noise comparator

null

Thermal noise

$N = kT$

Boltzmann constant

Planck constant
Electron charge
Second (Cesium)

Property of the Poisson process

$$\mu = \sigma^2$$

Noise calibration

thermal noise $S = kT$

shot noise $S = 2eIR$

high accuracy of I
with a dc instrument

Compare shot and thermal noise with a noise bridge

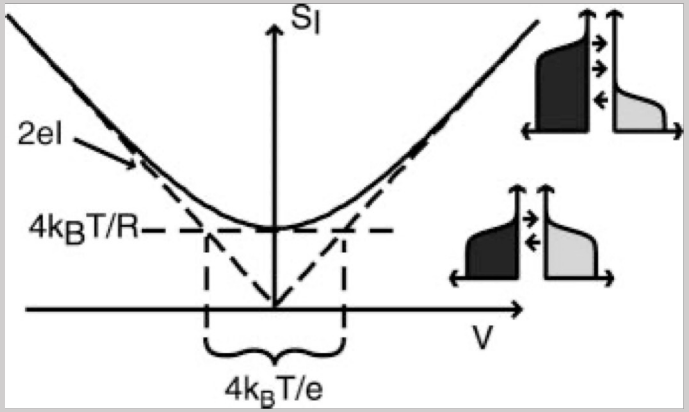


Fig. 1. Theoretical plot of current spectral density of a tunnel junction (Eq. 3) as a function of dc bias voltage. The diagonal dashed lines indicate the shot noise limit, and the horizontal dashed line indicates the Johnson noise limit. The voltage span of the intersection of these limits is $4k_BT/e$ and is indicated by vertical dashed lines. The bottom inset depicts the occupancies of the states in the electrodes in the equilibrium case, and the top inset depicts the out-of-equilibrium case where $eV \gg k_BT$.

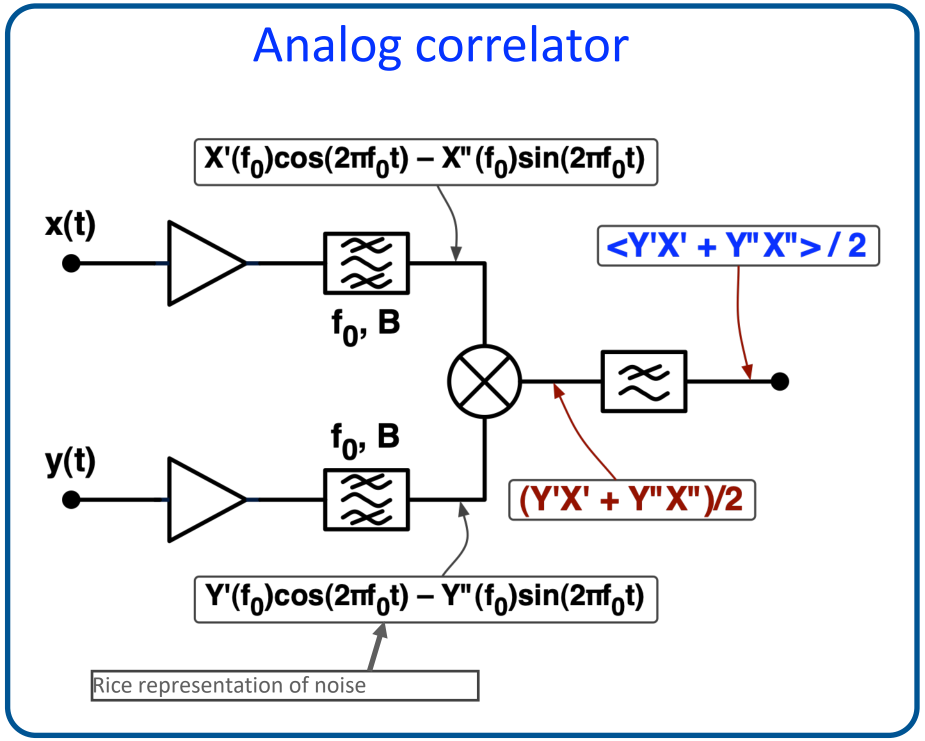
L. Spietz & al., Science 300(20) p. 1929-1932, jun 2003

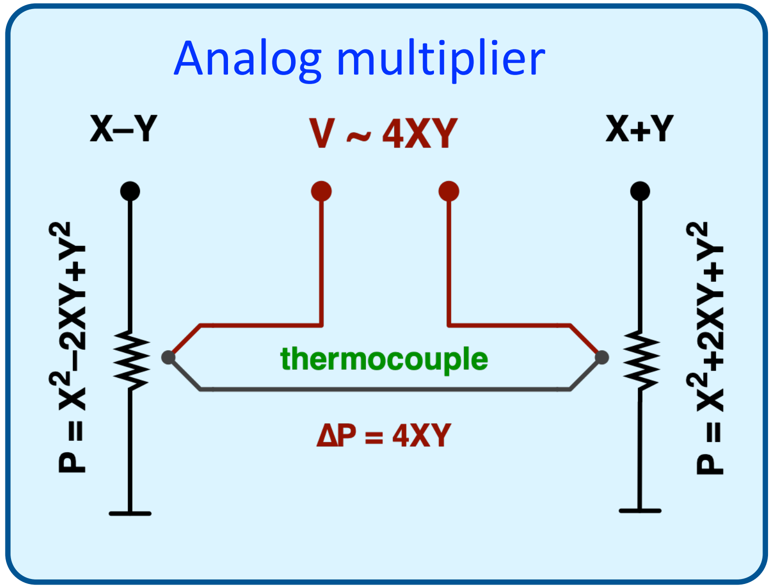
In a tunnel junction, theory predicts the amount of shot and thermal noise

L. Spietz & al., Primary electronic thermometry using the shot noise of a tunnel junction, Science 300(20) p. 1929-1932, jun 2003

Early implementations

1940-1950 technology





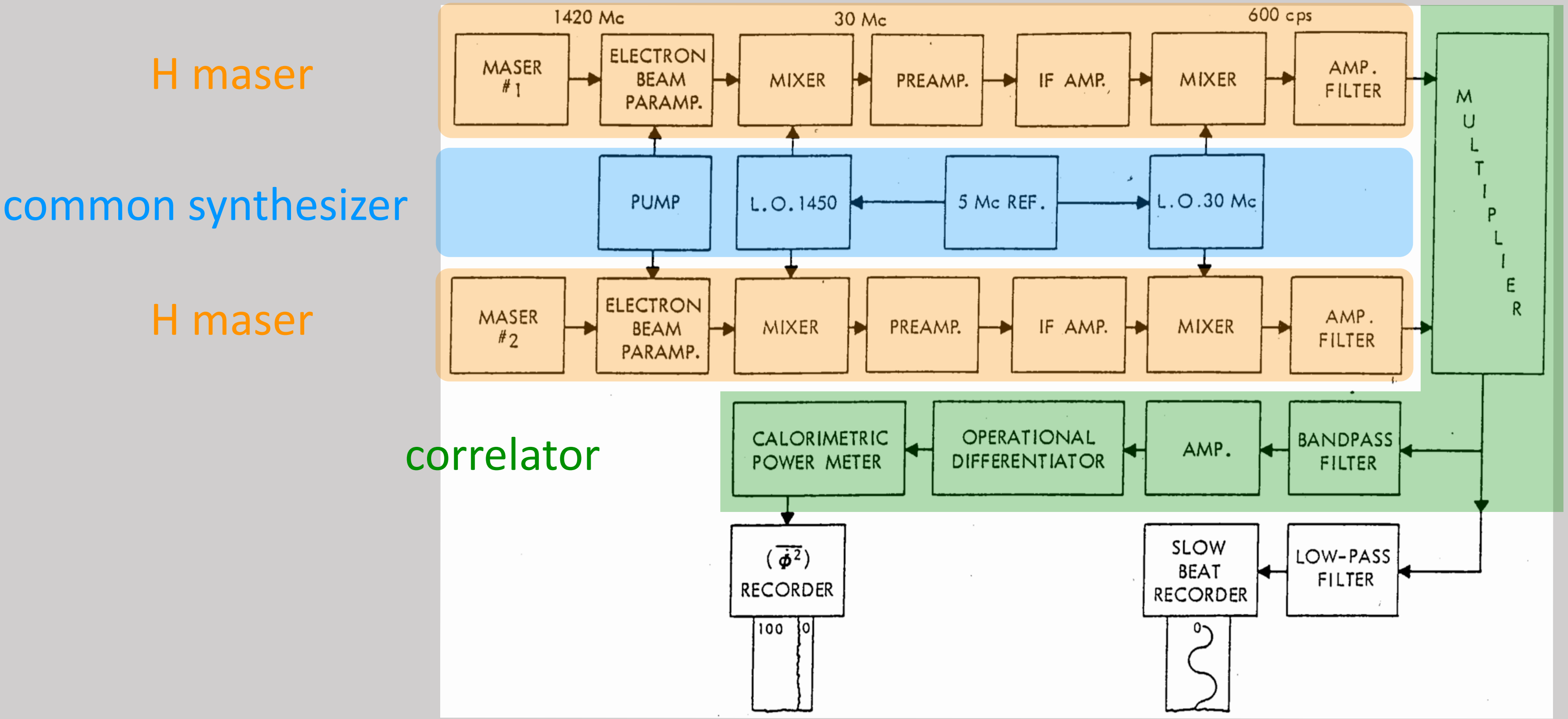
Spectral analysis at the single frequency f_0, in the bandwidth B

Need a filter pair for each Fourier frequency

Frequency noise of a H-maser

For the lectures on oscillators

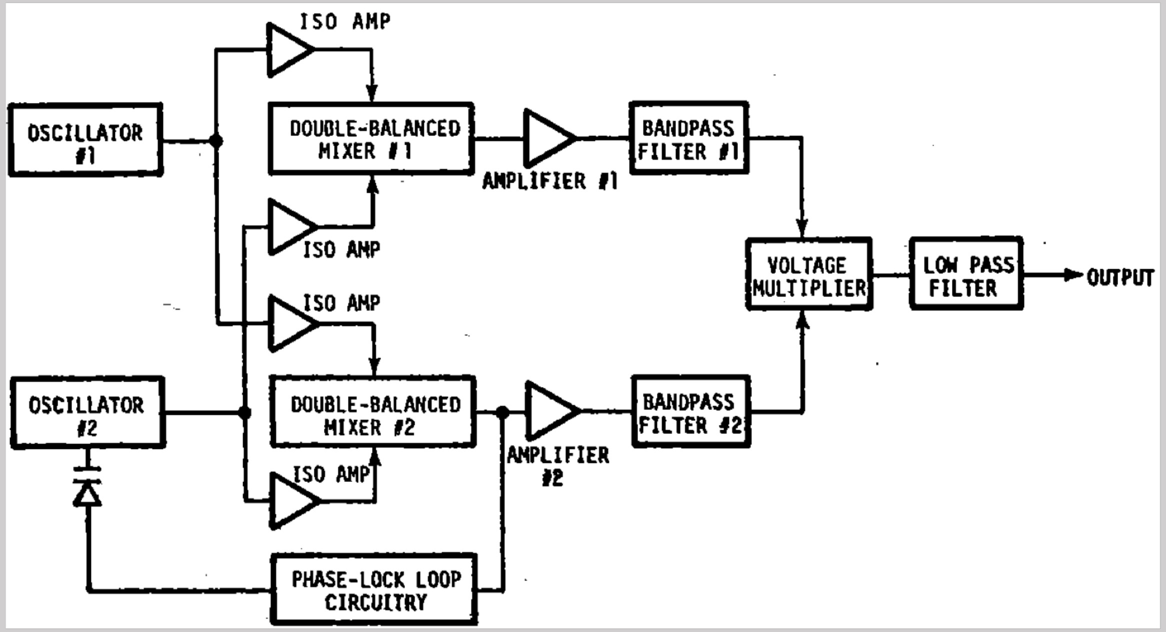
R. F. C. Vessot & al., Fig.10-4



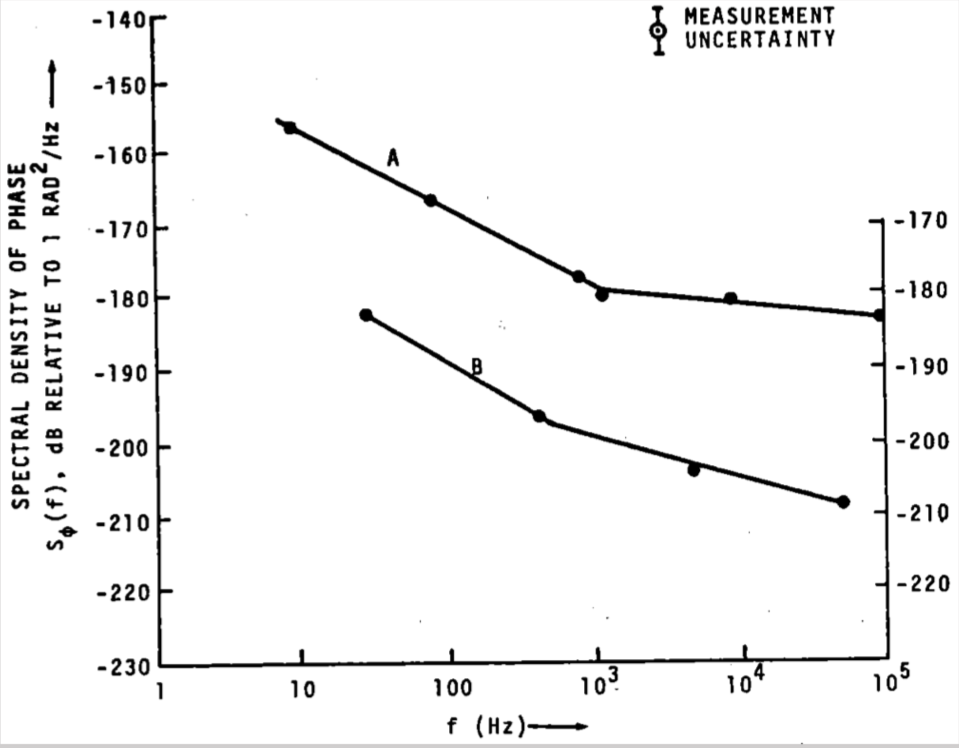
R. F. C. Vessot, L. F. Mueller, J. Vanier, Proc. NASA Symp. on Short Term Frequency Stability p.111-118, Greenbelt, MD, 23-24 Nov 1964

Article made publicly available by NASA https://ntrs.nasa.gov/api/citations/19660001092/downloads/19660001092.pdf

Phase noise measurement



F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976, Fig.7



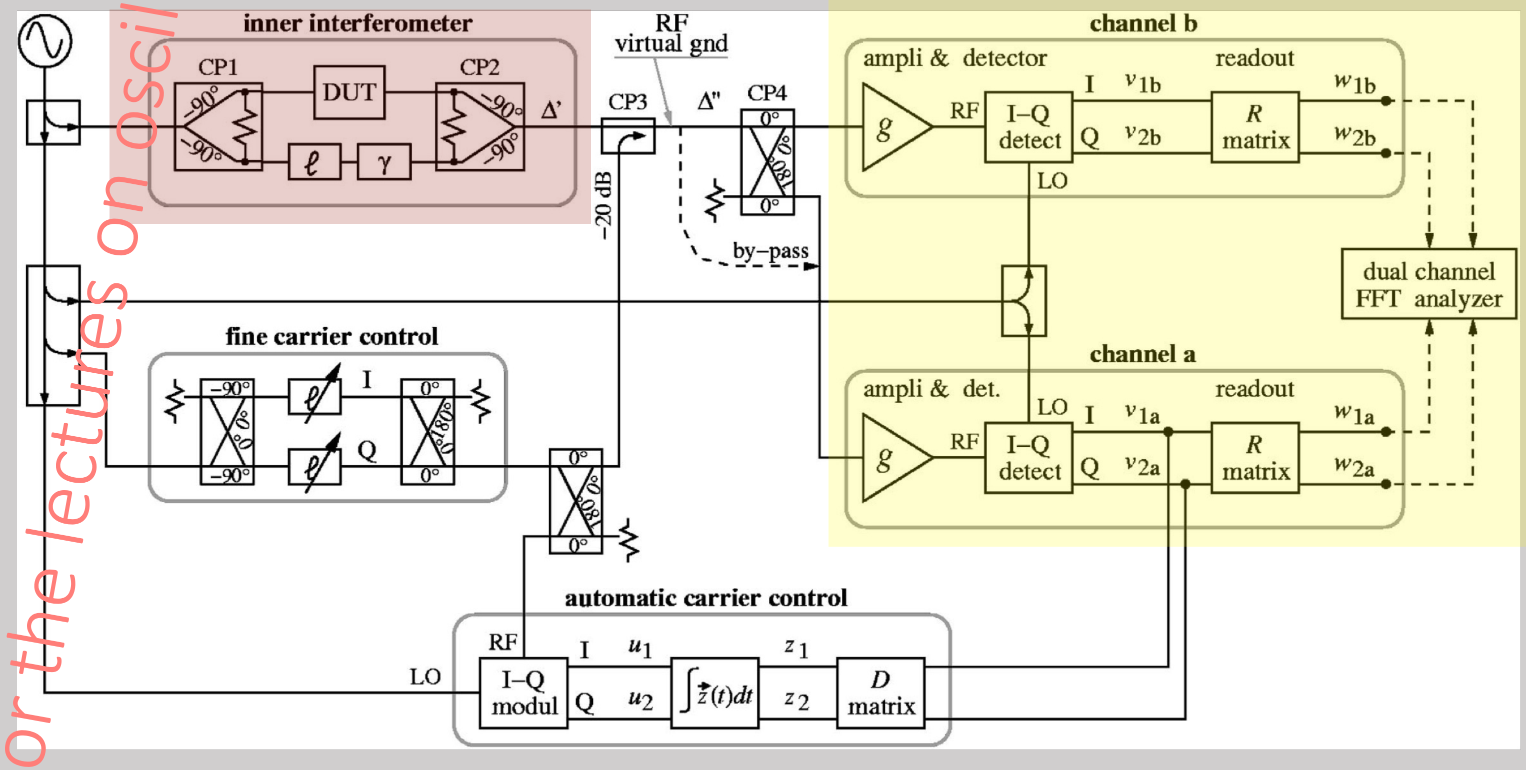
F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976, Fig.8

(relatively) large correlation bandwidth
provides low noise floor in a reasonable time

F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976
More popular after W. Walls, Proc. 46th FCS pp.257-261, 1992

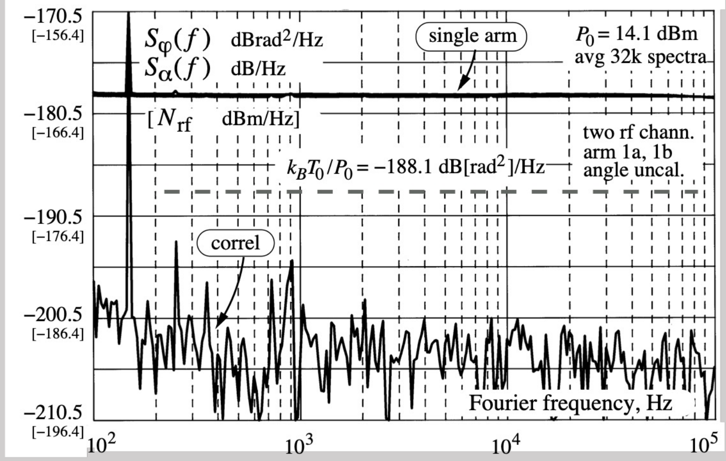
Phase Noise Measurement

E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.2



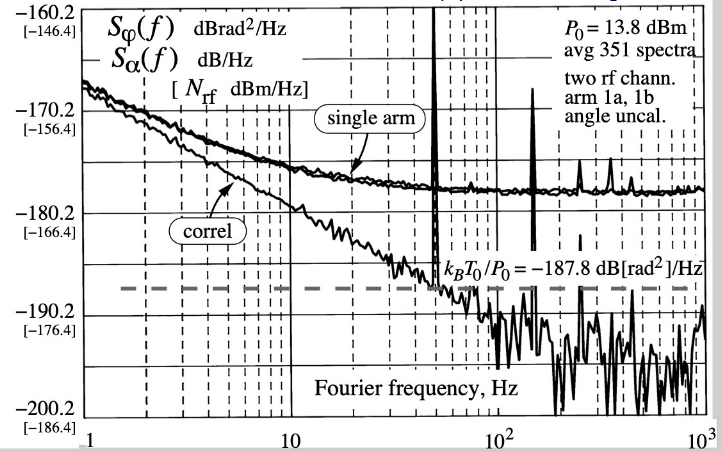
background noise

E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.10



by-step attenuator

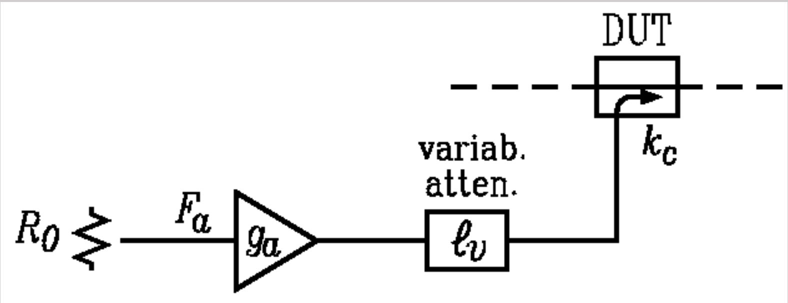
E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.11



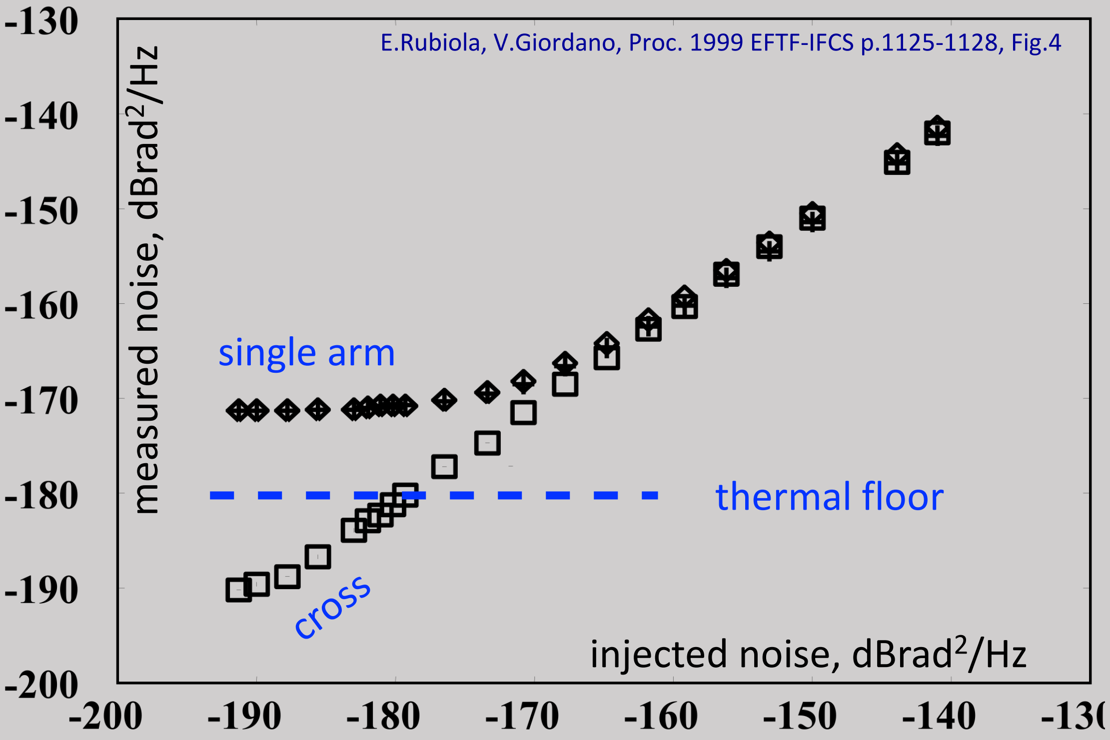
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 71(8) p.3085-3091, aug 2000
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 73(6) pp.2445-2457, jun 2002

Below the standard thermal floor

100 MHz prototype,
carrier power $P_o = 8$ dBm



E.Rubiola, V.Giordano, Proc. 1999 EFTF-IFCS p.1125-1128, Fig.3

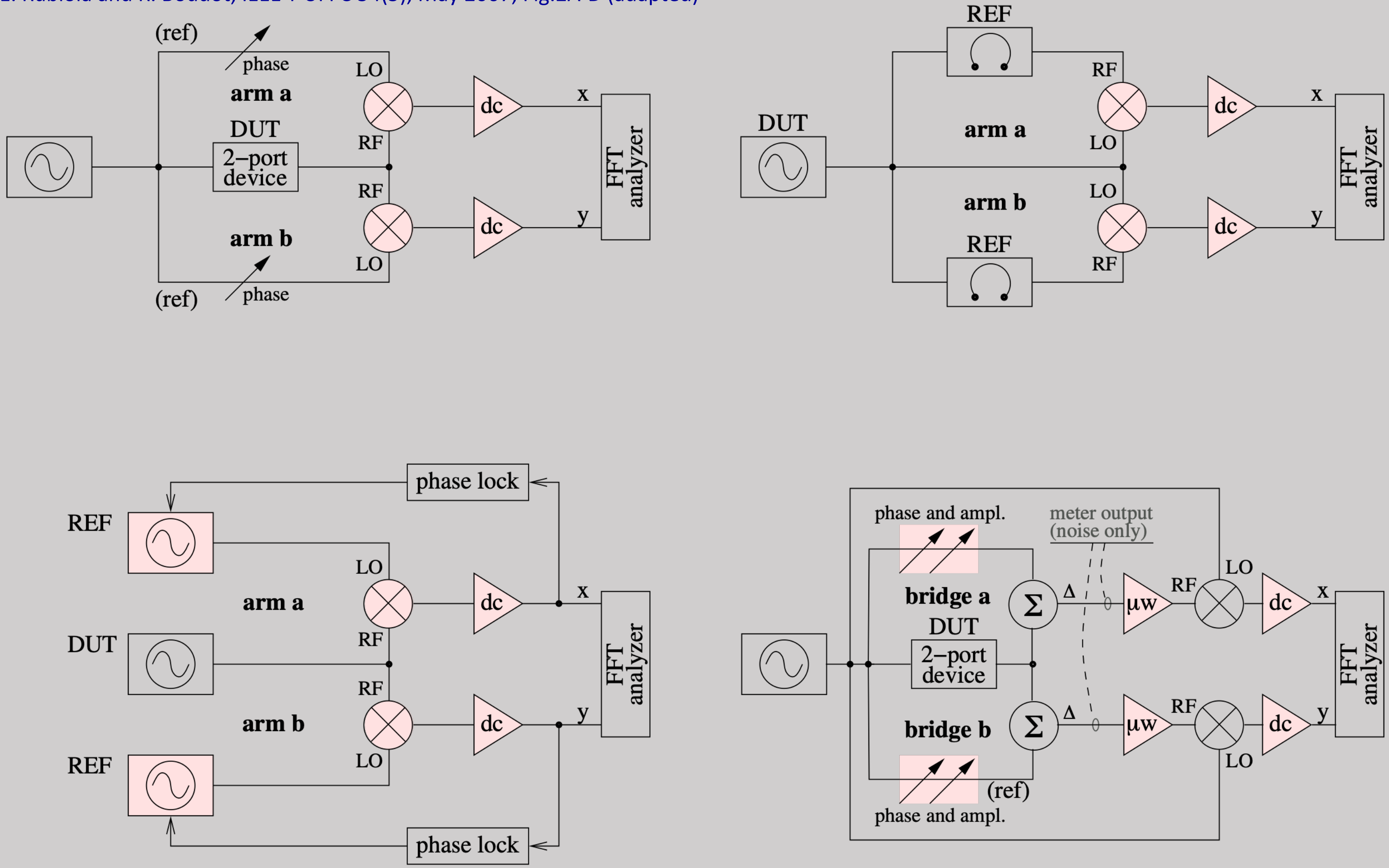

E.Rubiola, V.Giordano, Proc. 1999 EFTF-IFCS p.1125-1128, Fig.4

E.Rubiola, V.Giordano, Correlation-Based Noise Measurements
Below the Thermal Noise Floor, Proc. 1999 EFTF-IFCS p.1125-1128

Phase noise

For the lectures on oscillators

E. Rubiola and R. Boudot, IEEE T UFFC 54(5), May 2007, Fig.2A-D (adapted)



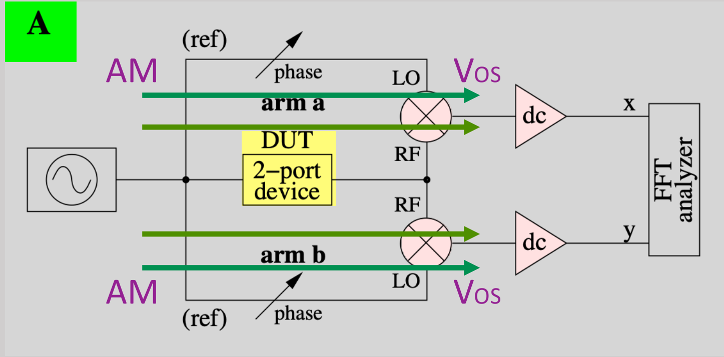
E. Rubiola and R. Boudot, The effect of AM Noise on Correlation Phase-Noise Measurements, IEEE Transact. UFFC 54(5) p.926-932, May 2007

Effect of amplitude noise

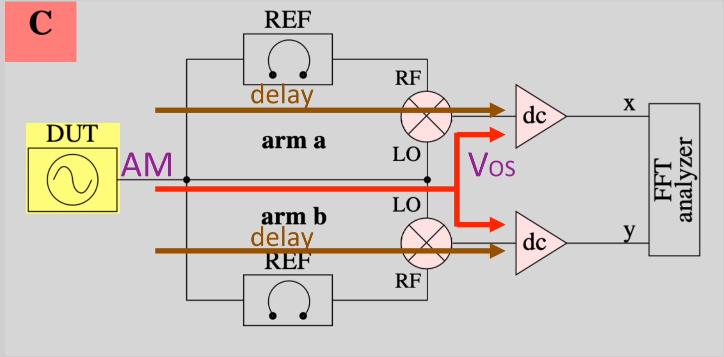
For the lectures on oscillators

E. Rubiola and R. Boudot, IEEE T UFFC 54(5), May 2007, Fig.2A-D (adapted)

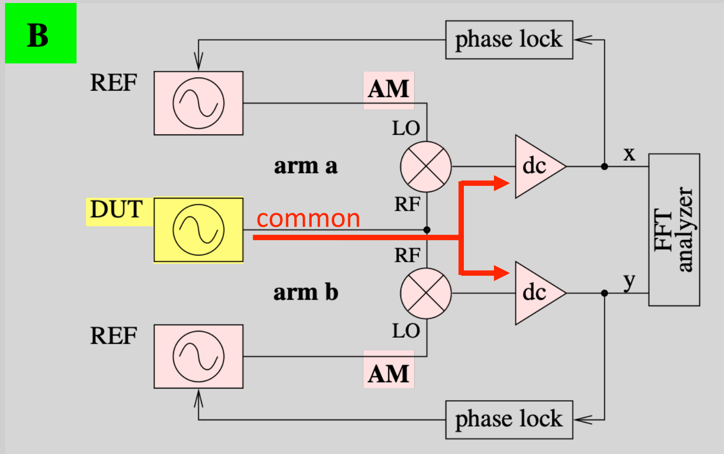
Should set both channels at the sweet point, if exists



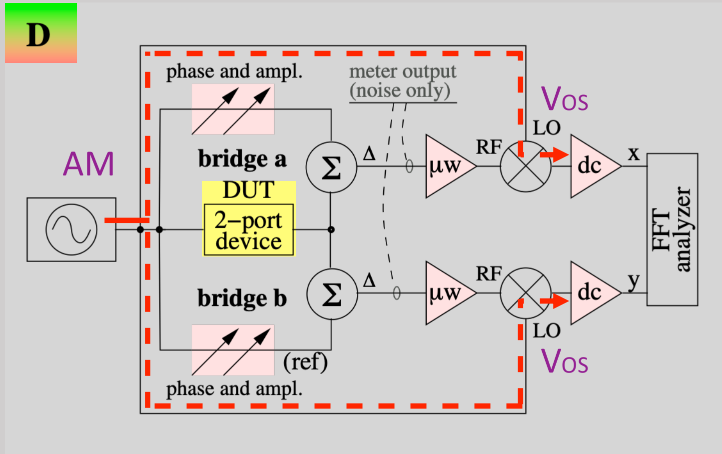
The delay de-correlates the two inputs, so there is no sweet point



pink: noise rejected by correlation and averaging



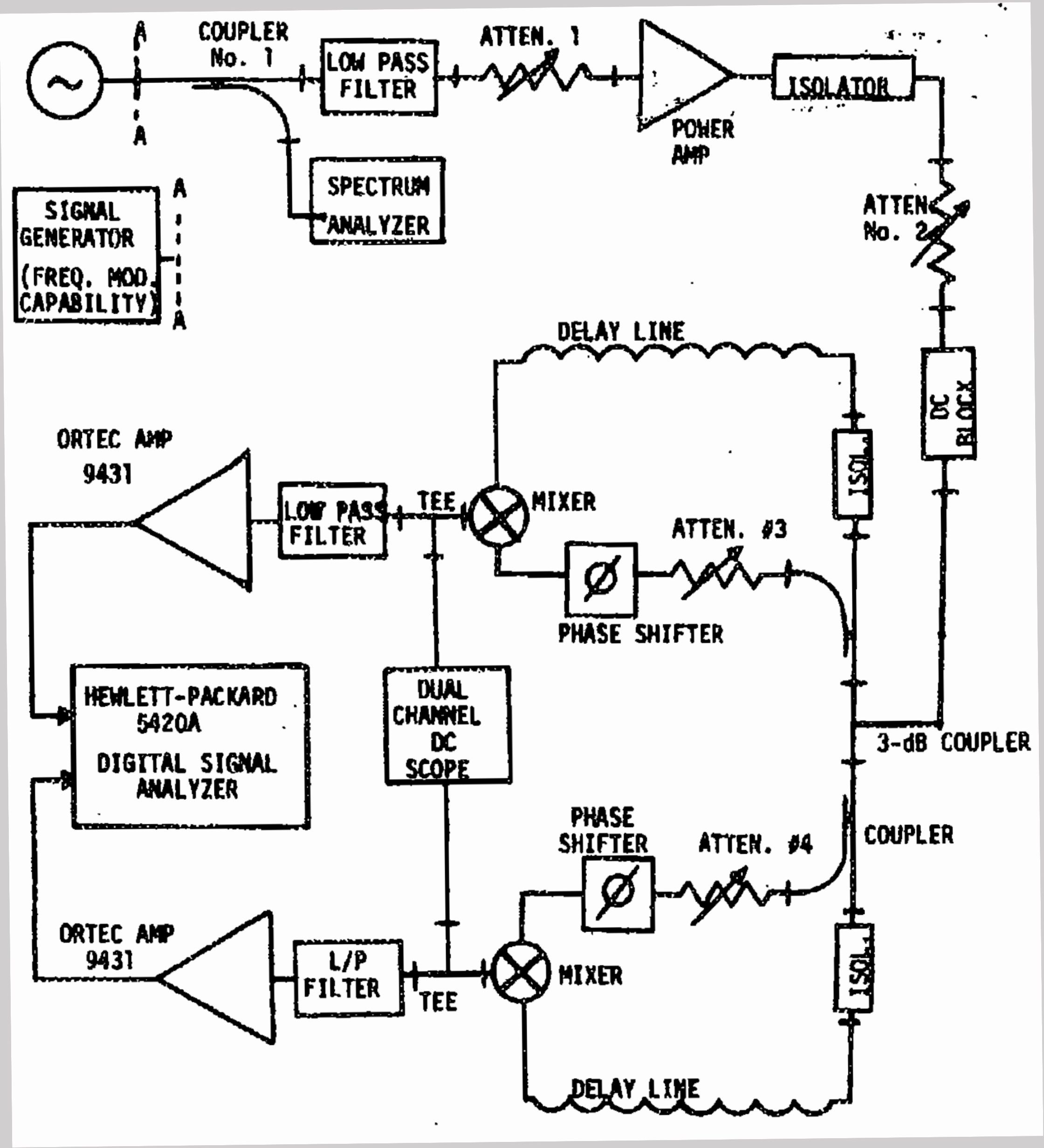
Should set both channels at the sweet point of the RF input, if exists, by offsetting the PLL or by biasing the IF



The effect of the AM noise is strongly reduced by the RF amplification

E. Rubiola and R. Boudot, The effect of AM Noise on Correlation Phase-Noise Measurements, IEEE Transact. UFFC 54(5) p.926-932, May 2007

Dual-delay-line method



A.L. Lance et al., ISA Transact. 21(4), Apr 1982, Fig.6

(arguably) Original idea by
D. Halford's NBS notebook F10 p.19-38, apr 1975

First published: A. L. Lance & al, CPEM Digest, 1978

The delay line converts the frequency noise into phase noise

The high loss of the coaxial cable limits the maximum delay

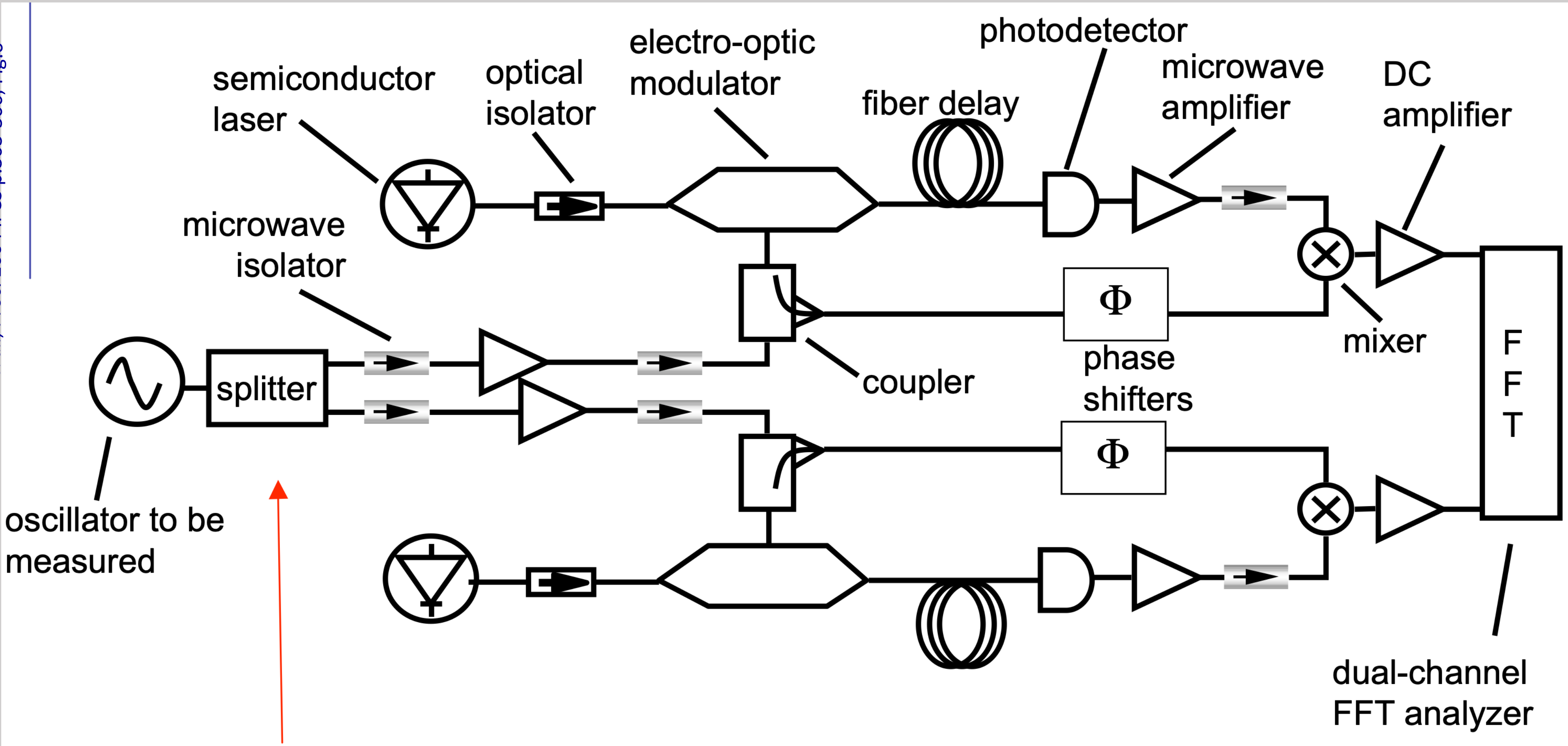
Updated version:
The optical fiber provides long delay with low attenuation
(0.2 dB/km or 0.04 dB/μs)

A.L. Lance, W.D. Seal, F. Labaar, Phase Noise Measurement Systems, ISA Transact. 21(4) p.37-84, Apr 1982

Optical dual-delay-line

Two completely separate systems measure the same oscillator under test

E. Salik & al., Proc. 2004 IFCS p.303-306, Fig.6

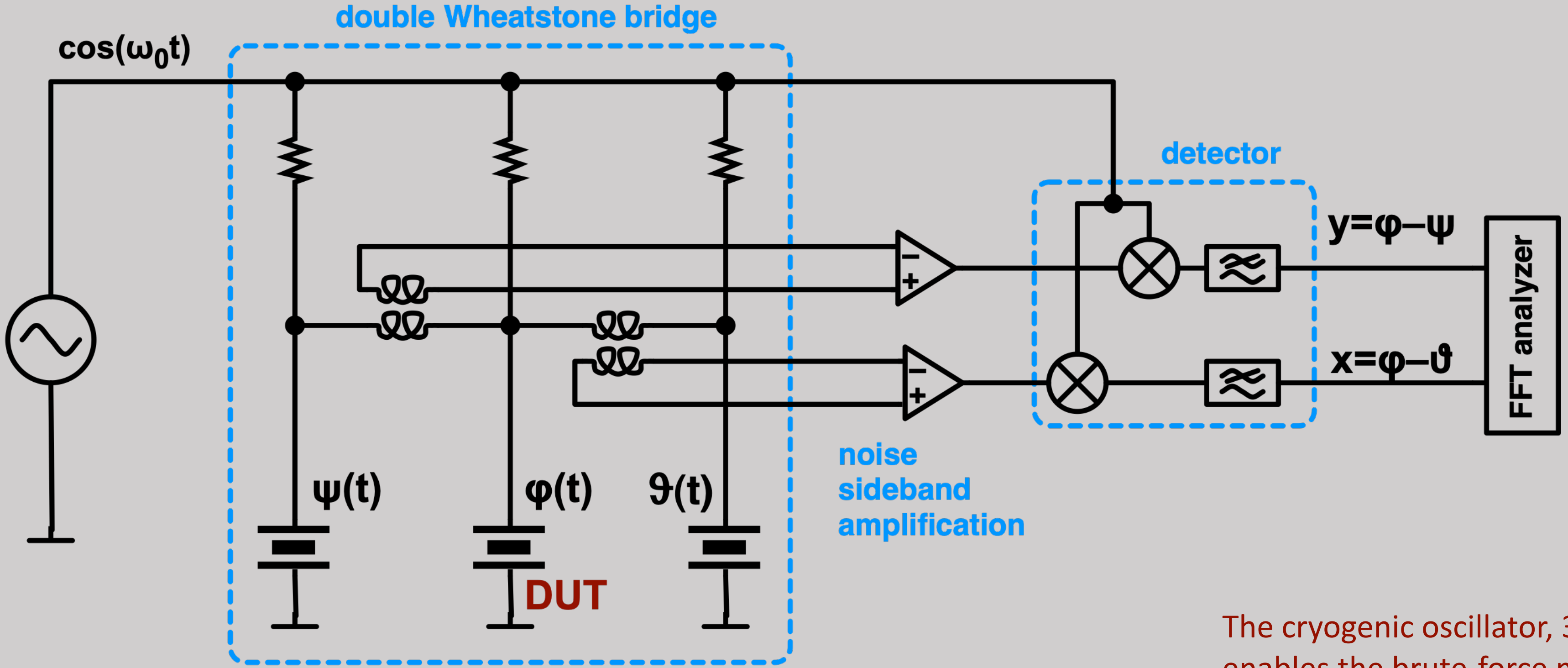


The only common part of the setup is the power splitter.

E. Salik, N. Yu, L. Maleki, E. Rubiola, Proc. IFCS, Montreal, Aug 2004 p.303-306

Volyanskiy & al., JOSAB 25(12) 2140-2150, Dec.2008. Also arXiv:0807.3494v1 [physics.optics] July 2008

Frequency stability of a resonator



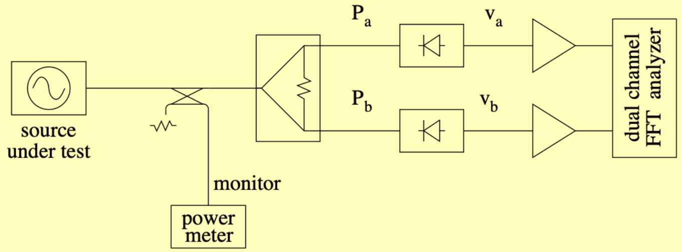
The cryogenic oscillator, 3×10^{-16} stability, enables the brute-force measurement of a single resonator

- Bridge in equilibrium
  - The amplifier cannot flicker around ω_0, which it does not know
- The fluctuation of the resonator natural frequency is estimated from phase noise
- Q matching prevents the master-oscillator noise from being taken in
- Correlation removes the noise of the instruments and the reference resonators

Amplitude noise & laser RIN

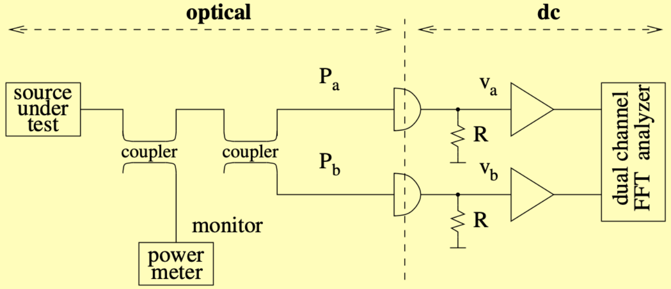
E. Rubiola, Proc. 2006 IFCS p.750-758, Fig.1

AM noise of RF/microwave sources



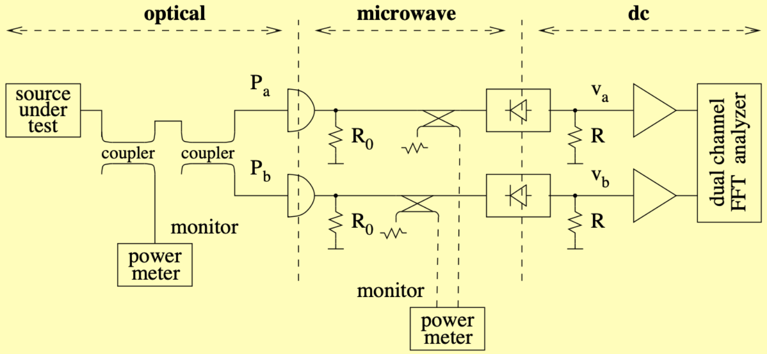
E. Rubiola, Proc. 2006 IFCS p.750-758, Fig.7

Laser RIN



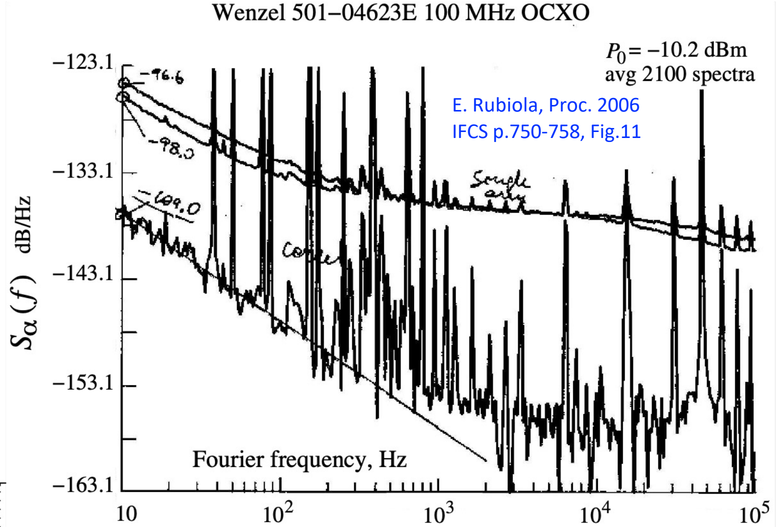
E. Rubiola, Proc. 2006 IFCS p.750-758, Fig.8

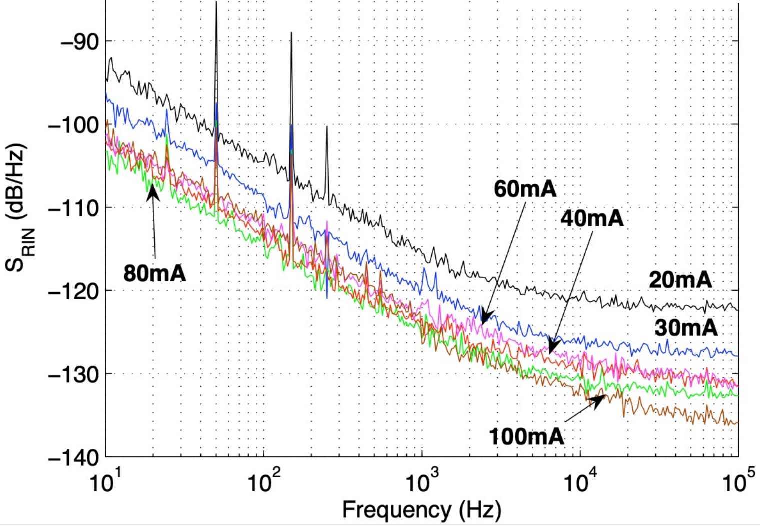
AM noise of photonic RF/microwave sources



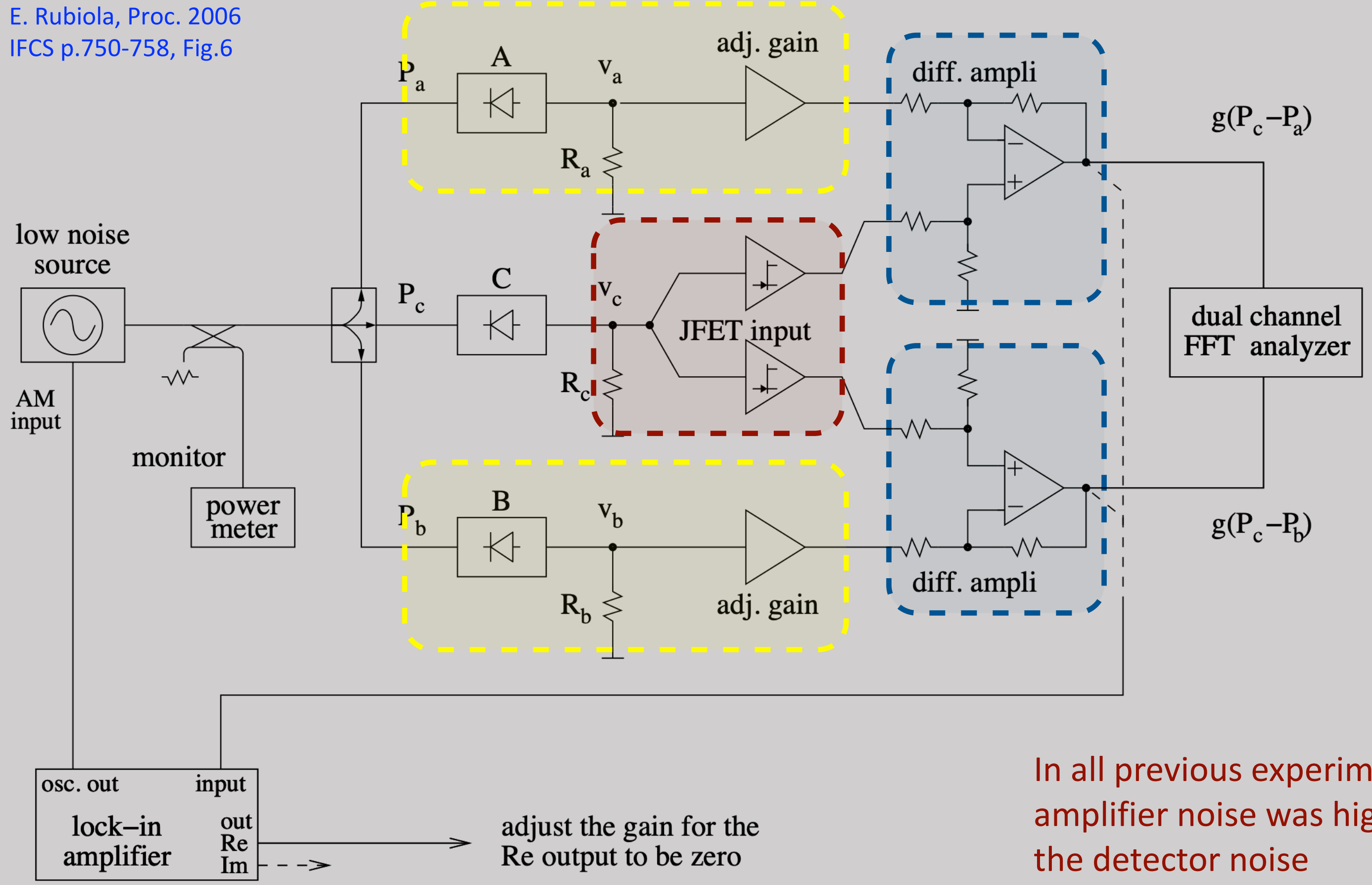
E. Rubiola, The measurement of AM noise, Proc. IFCS p.750-758, June 2006. Also arXiv:physics/0512082v1 [physics.ins-det], Dec 2005

- Cannot measure the background removing the DUT
- Correlation enables to validate the instrument





Detector noise



E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.6

Basic ideas

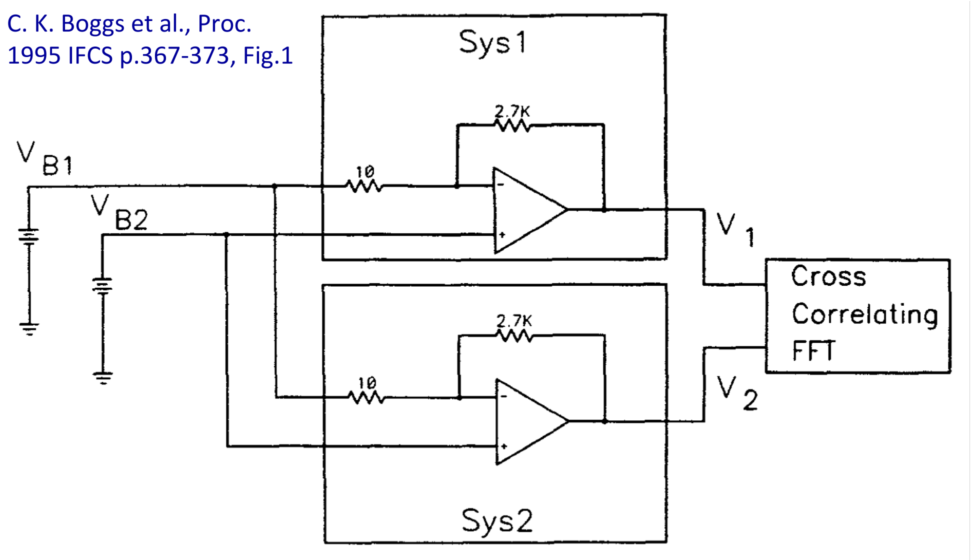
- Remove the noise of the source by balancing C–A and C–B
  - Use a lock-in amplifier to get a sharp null measurement
- Channels A and B are independent –> noise is averaged out
- Two separate JFET amplifiers are needed in the C channel
  - JFETs have virtually no bias-current noise
- Only the noise of the detector C remains

In all previous experiments, the amplifier noise was higher than the detector noise

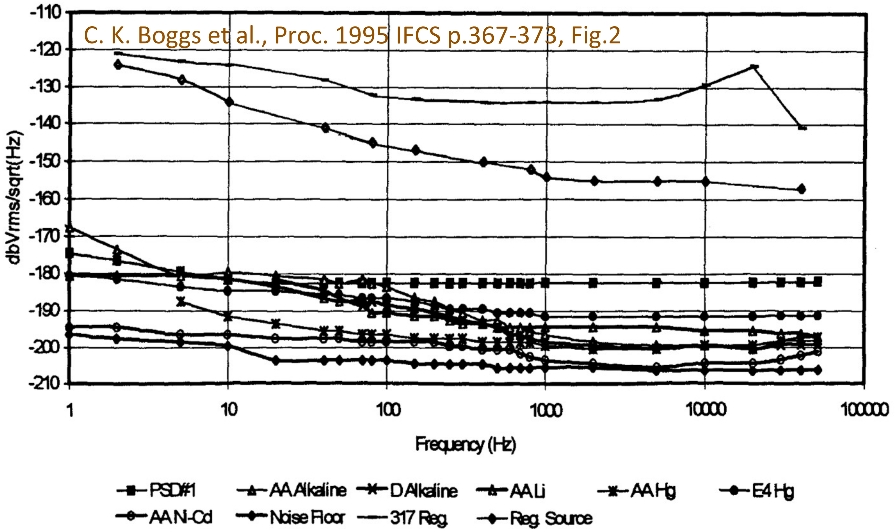
E. Rubiola, The measurement of AM noise, Proc. IFCS p.750-758, June 2006. Also arXiv:physics/0512082v1 [physics.ins-det], Dec 2005

S. Grop, E. Rubiola, Flicker Noise of Microwave Power Detectors, Proc. IFCS p.40-43, April 2009

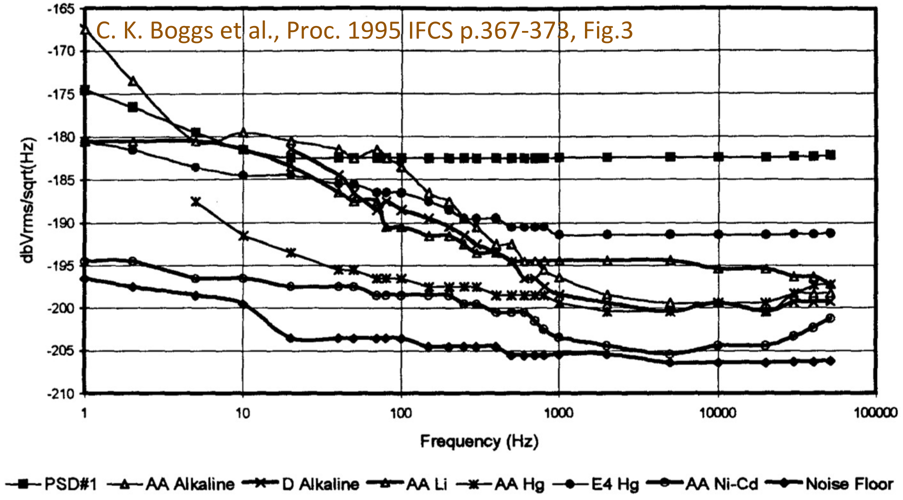
Noise in chemical batteries



C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.1



C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.2



C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.3

- Do not waste DAC bits for a constant DC, $V = V_{B2}–V_{B1}$ has (almost) zero mean
- Two separate amplifiers measure the same quantity V
- Correlation rejects the amplifier nose, and the FFT noise as well

C. K. Boggs, A. D. Doak, F. L. Walls, Measurement of Voltage Noise in Chemical Batteries, Proc. IFCS p.367-373, 1995

Noise in semiconductors

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.2

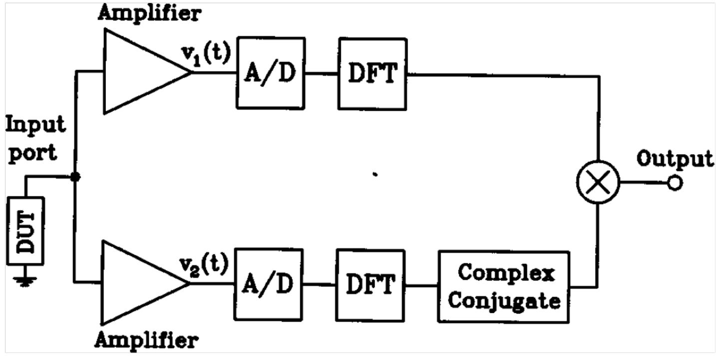


FIG. 2. Schematics of the building blocks of our correlation spectrum analyzer performing the suppression of the uncorrelated input noises by a digital processing of sampled data.

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.3

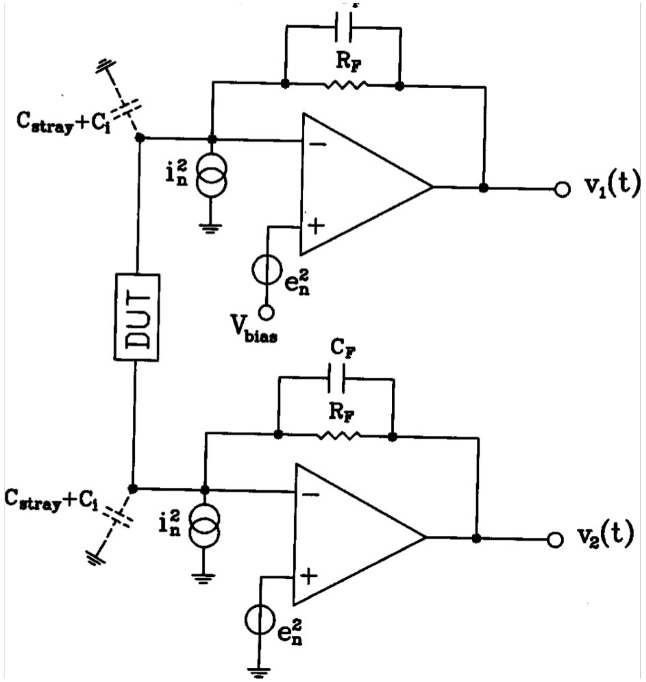


FIG. 3. Schematics of the active test fixture for current noise measurements.

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.9

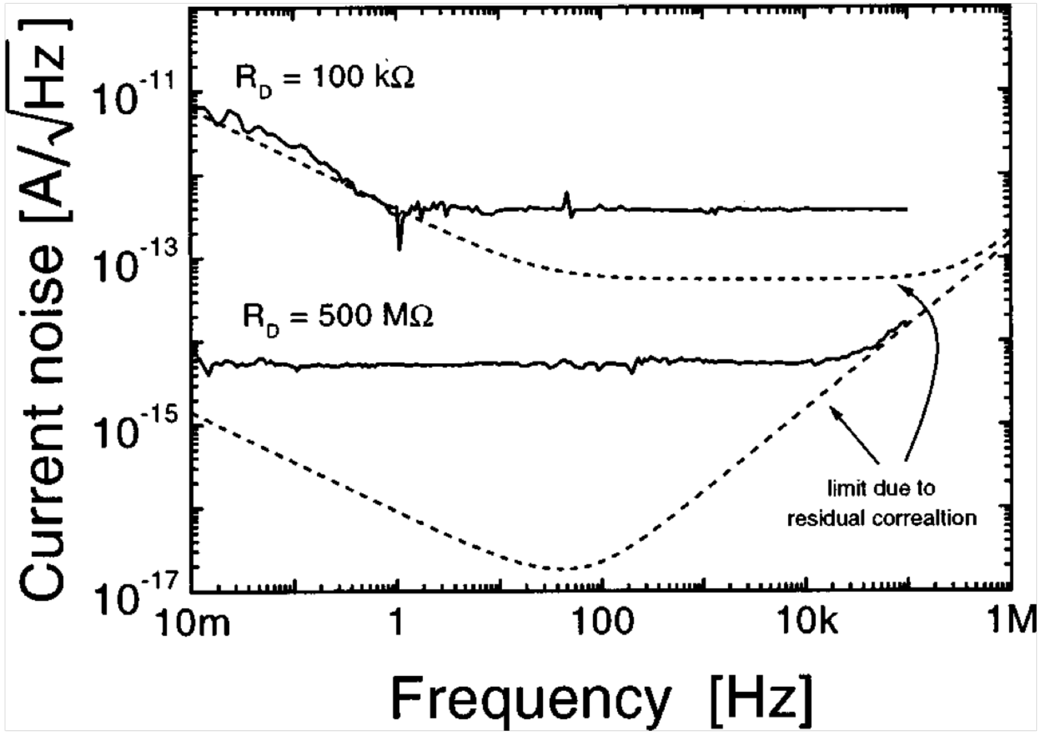
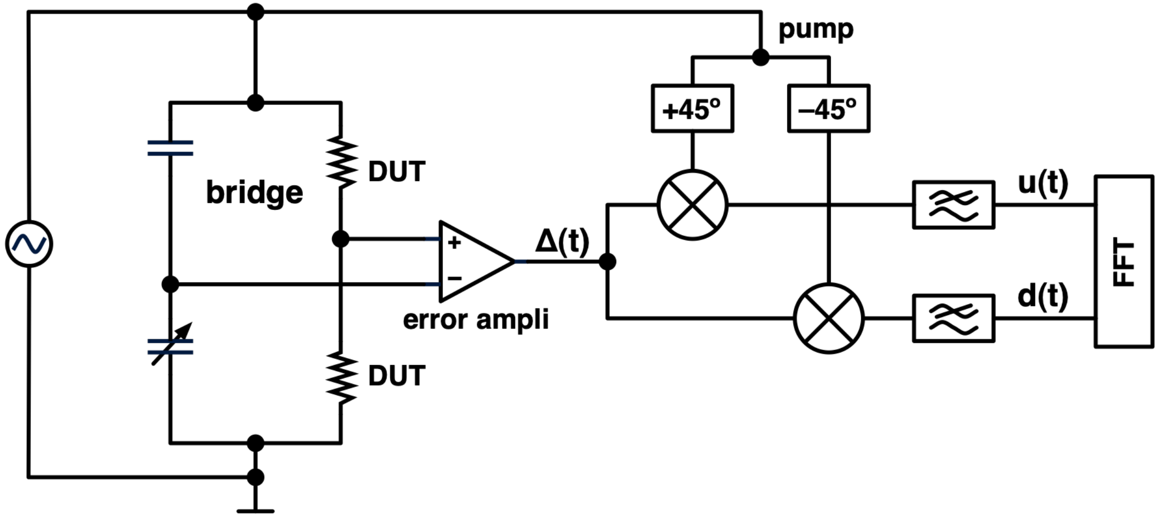


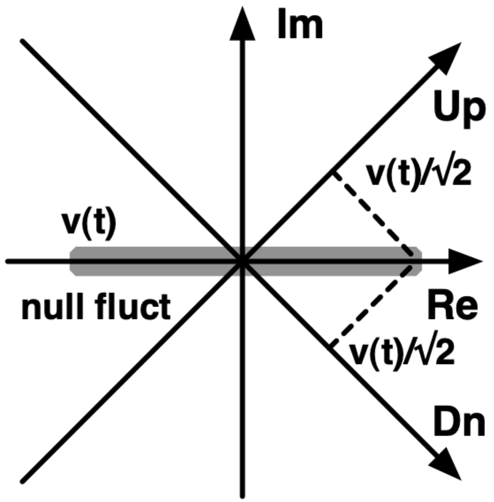
FIG. 9. Experimental frequency spectrum of the current noise from DUT resistances of 100 kΩ and 500 MΩ (continuous line) compared with the limits (dashed line) given by the instrument and set by residual correlated noise components.

Sampietro M, Fasoli L, Ferrari G - Spectrum analyzer with noise reduction by cross-correlation technique on two channels - RSI 70(5) p.2520-2525, May 1999

Electro-migration in thin films



- Random noise: X' and X'' (real and imag part) of a signal are statistically independent
- The detection on two orthogonal axes eliminates the amplifier noise.
  This work with a single amplifier!
- The DUT noise is detected



$$S_{ud}(f) = \frac{1}{2}\left[S_\alpha(f) - S_\varphi(f)\right]$$

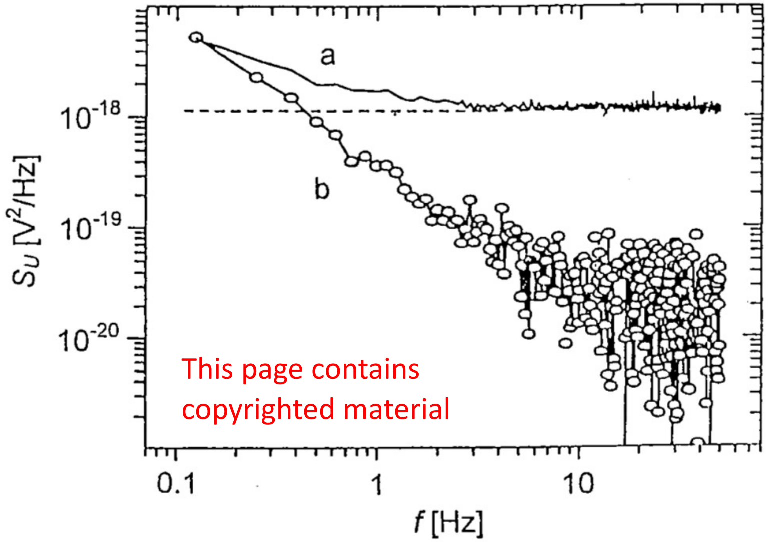


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Fig. 1 1/f noise of an $\mathrm{AlSi_{0.01}Cu_{0.002}}$ thin film measured at room temperature (a) without and (b) with the phase-sensitive ac correlation technique. The Johnson noise level is indicated by the dashed line.

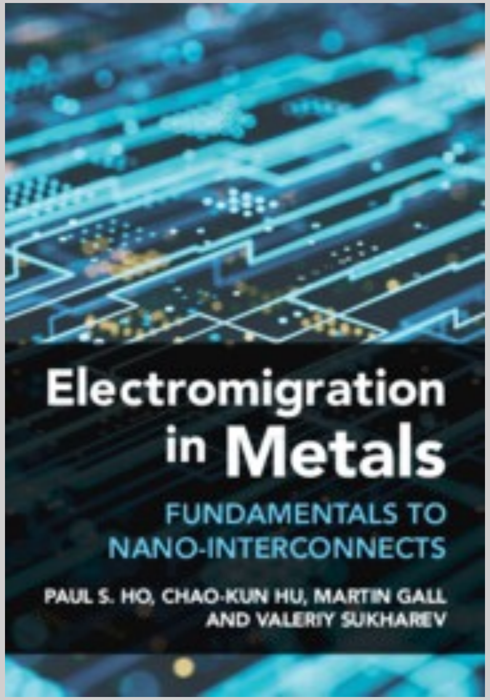
A. Seeger, H. Stoll, Proc. 1999 ICNF p.162-167, Fig.1

A. Seeger, H. Stoll, 1/f noise and defects in thin metal films, Proc. ICNF p.162-167, Hong Kong 23-26 aug 1999
RF/microwave version: E. Rubiola, V. Giordano, H. Stoll, IEEE Transact. IM 52(1) pp.182-188, feb 2003

Electromigration in metals is still a hot topic

Paul S. Ho, Chao-Kun Hu, Martin Gall, Valeriy Sukharev, Siemens, *Electromigration in Metals*, Cambridge, May 2022

ISBN: 9781107032385

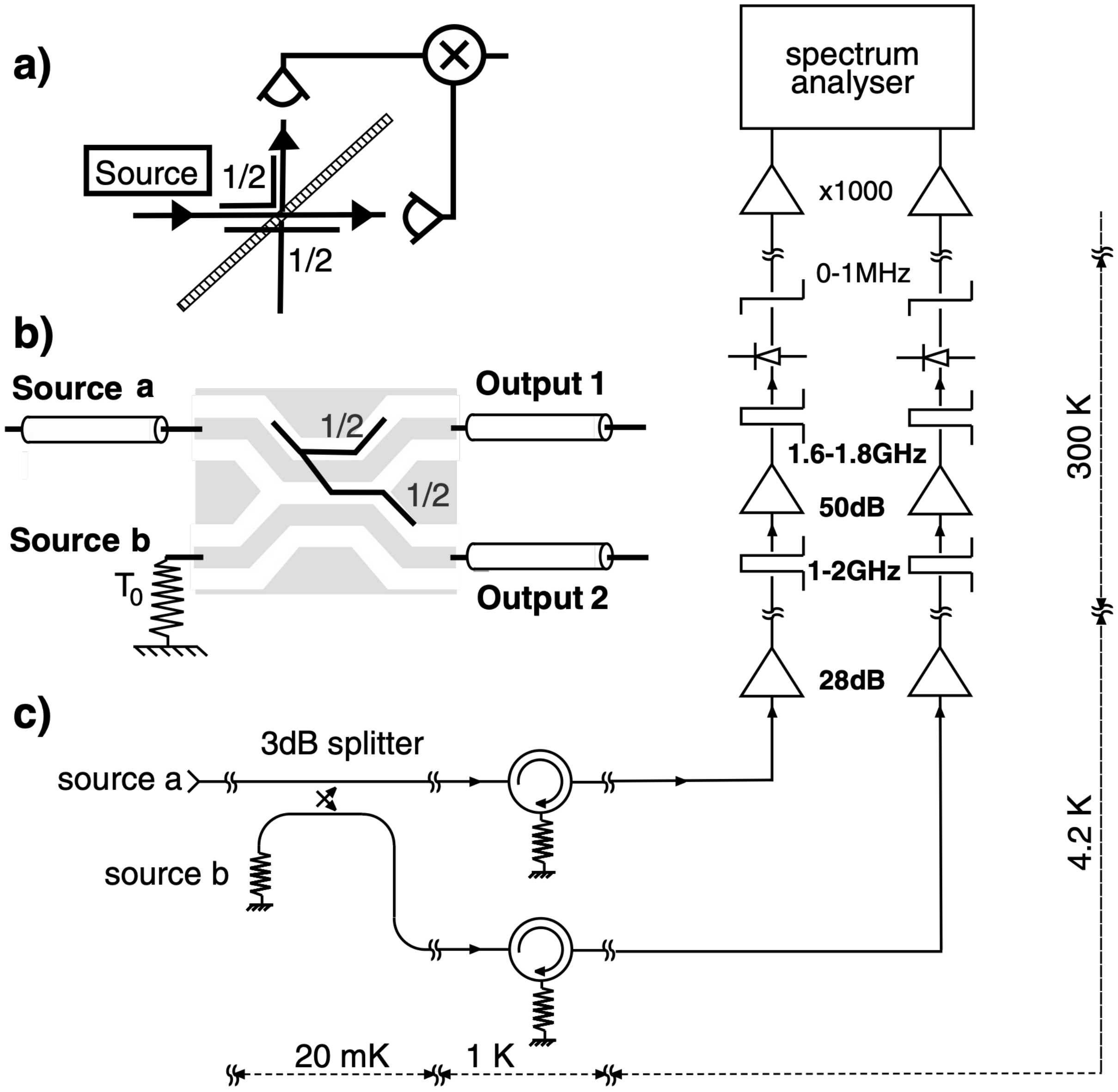

Electromigration in Metals
FUNDAMENTALS TO NANO-INTERCONNECTS
PAUL S. HO, CHAO-KUN HU, MARTIN GALL AND VALERIY SUKHAREV

Hanbury Brown – Twiss Effect

Anti-correlation shows up in single-photon regime

Also observed in microwaves Gabelli…Glattli, PRL 93(5) 056801, Jul 2004

20 mK and 1.7 GHz
$kT = 2.7\times10^{-25}$ J
$h\nu = 1.12\times10^{-24}$ J
$kT/h\nu = -6.1$ dB



J.Gabelli et al, PRL 93(5) 056801, 27 July 2004, Fig.1

Featured reading (optics)
Hanbury Brown R, Twiss RQ - Correlation Between Photons in Two Coherent Beams of Light - Nature 4497 p.27-29, 7 January 1956

Featured reading (microwave port)
Gabelli J, Reydellet LH, Feve G, Berroir JM, Placais B, Roche P, Glattli DC, Hanbury-Brown Twiss Correlation to Probe the Population Statistics of GHz Photons Emitted by Conductors, PRL 93(5) 056801, 27 July 2004

Lecture 3 ends here

Lecture 4

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Fourier statistics
- The cross spectrum method (theory)
- Applications of the cross spectrum method

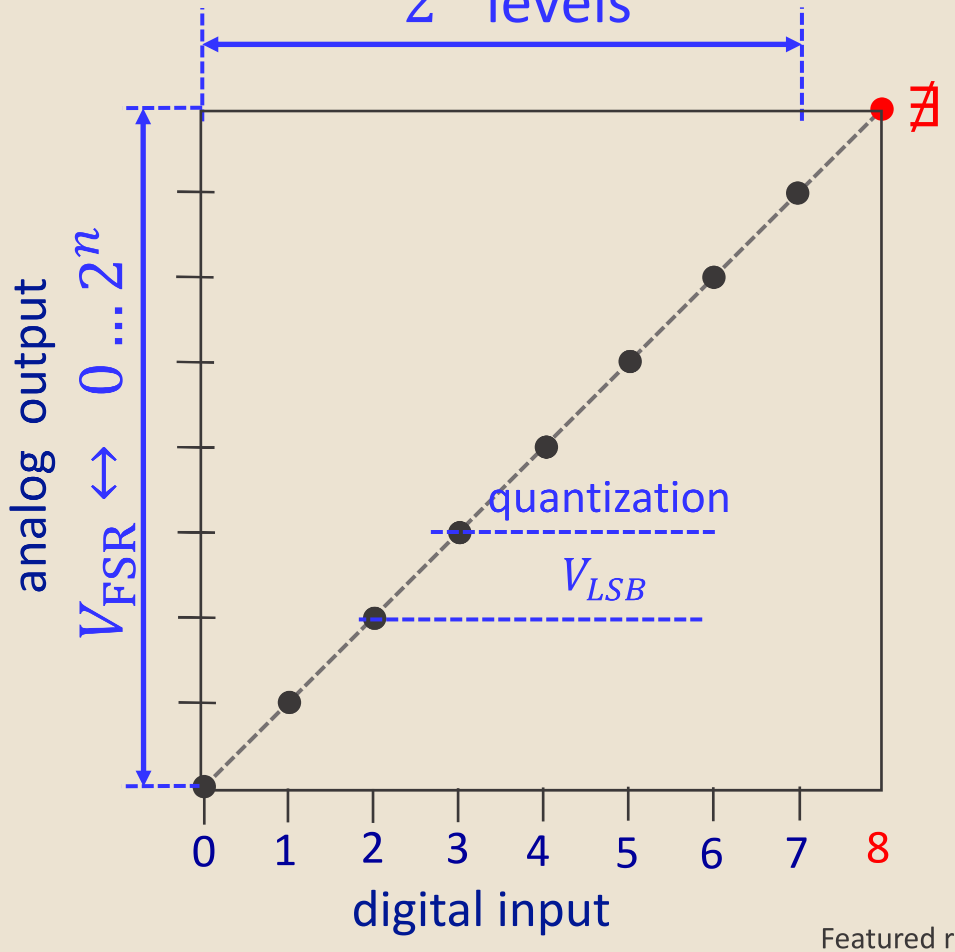
ORCID 0000-0002-5364-1835

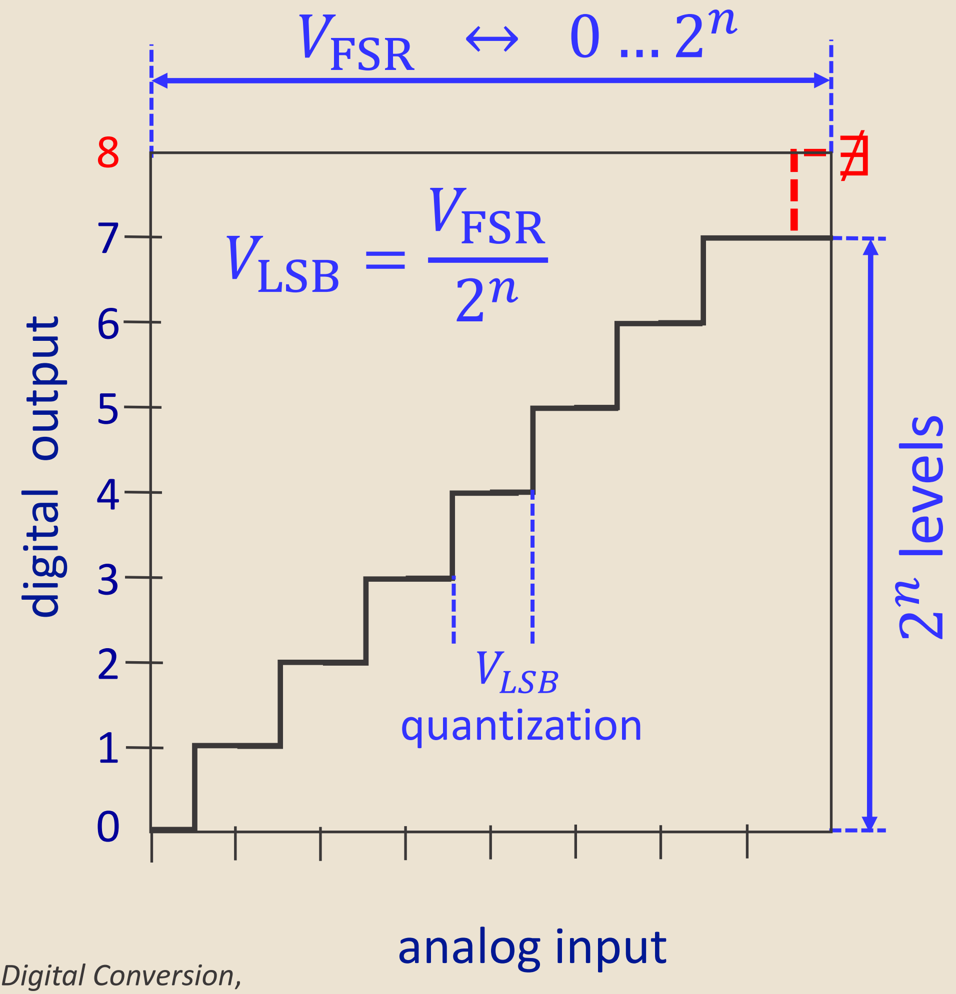
home page http://rubiola.org

Analog-to-Digital Conversion

Transfer function and quantization

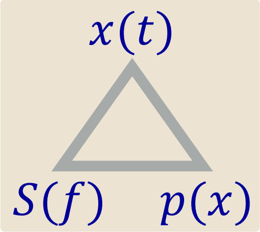
n bits $\qquad V_{\mathrm{LSB}} = \frac{V_{\mathrm{FSR}}}{2^n}$





Featured reading: Kester W (ed), *Analog-Digital Conversion*,
Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Variance (signal power)



Parseval Theorem

Wiener Khinchin Theorem

Power spectral density

$$\sigma^2 = \int_0^\infty S(f)df$$

The frequency interval

$f, f + \delta f$

parries power

$S(f)\,\delta f$

Time domain

$$\sigma^2 = \frac{1}{T}\int_0^T |x(t) - \mu|^2 dt$$

The fluctuation

$x(t) - \mu$

carries instantaneous power

$P(t) = |x(t) - \mu|^2$

Probability density function

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx$$

The interval

$x, x + \delta x$

occurs with probability

$p(x)\delta x$

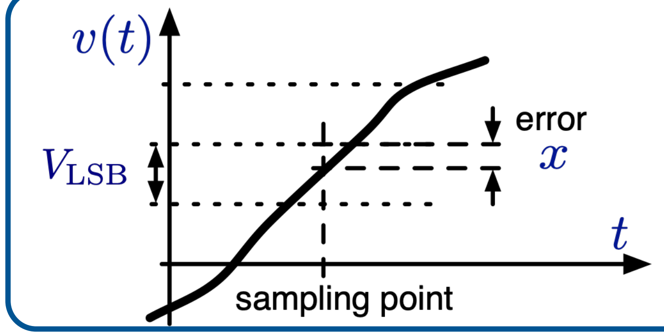
and contains a power

$P(x)\delta x = (x - \mu)^2\delta x$

Do you remember Monte Carlo averaging?

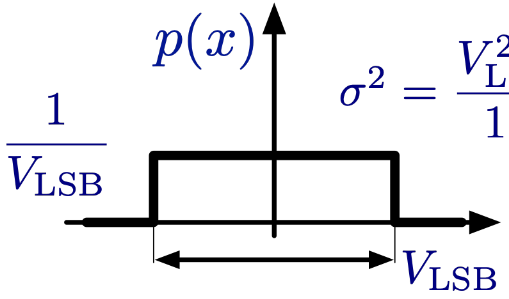
Quantization noise

W. R. Bennett, Spectra of quantized signals, Bell System Tech J. 27(4), July 1948



Analog-to-digital conversion introduces a quantization error x $[-V_{\mathrm{LSB}}/2 \leq x \leq +V_{\mathrm{LSB}}/2]$

n-bit conversion $\quad V_{\mathrm{LSB}} = \dfrac{V_{\mathrm{FSR}}}{2^n}$



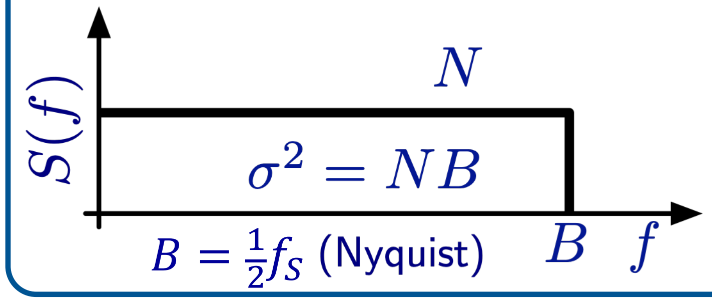
Wiener–Khintchine theorem: in ergodic systems, interchange time & ensemble

The noise can be calculated with statistics

$$\sigma^2 = \frac{V_{\mathrm{FSR}}^2}{12\times 2^n} \quad \mathrm{V}^2$$

$1/12 \rightarrow -10.8$ dB

$2^{2n} \rightarrow 6$ dB/bit



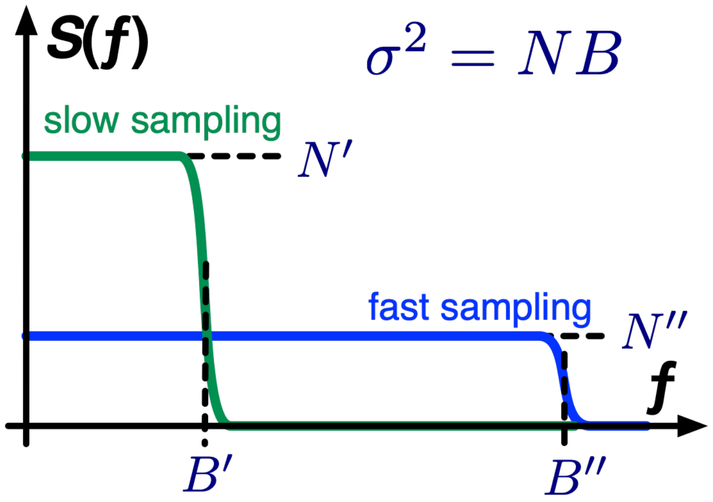
Parseval theorem: Energy (power) calculated in time and in frequency is the same

$$N = \frac{V_{\mathrm{FSR}}^2}{6\times 2^{2n} f_S} \quad \mathrm{V}^2/\mathrm{Hz}$$

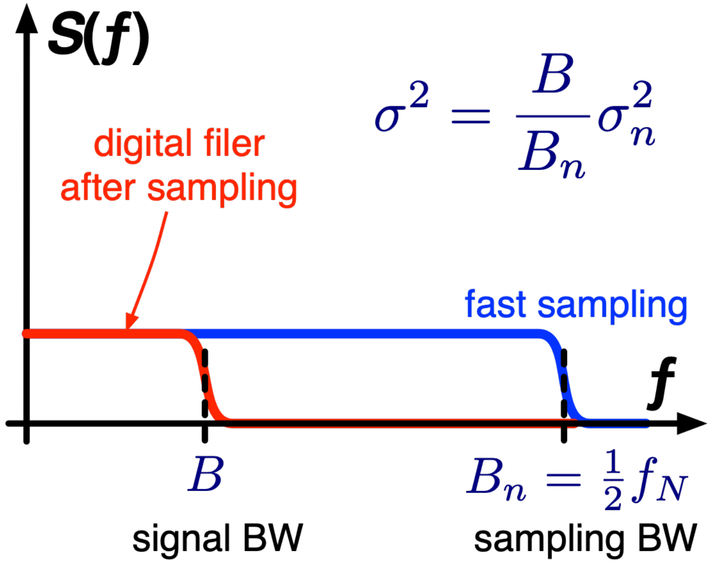
Digital filter and decimation

Noise, Sampling, and the Parseval theorem

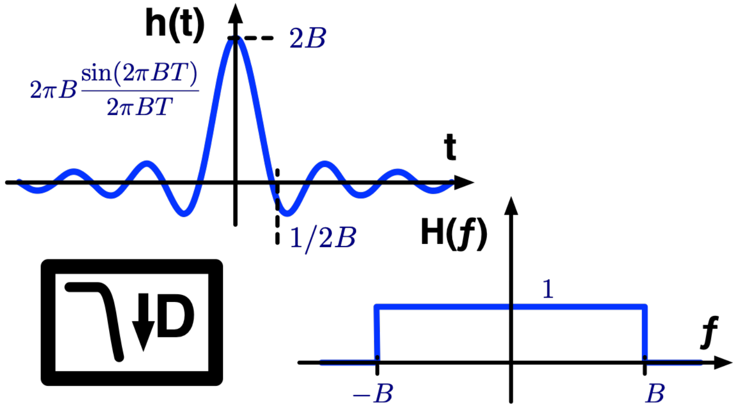
$$\sigma^2 = \frac{V_{\mathrm{FSR}}^2}{12\times 2^{2m}}$$



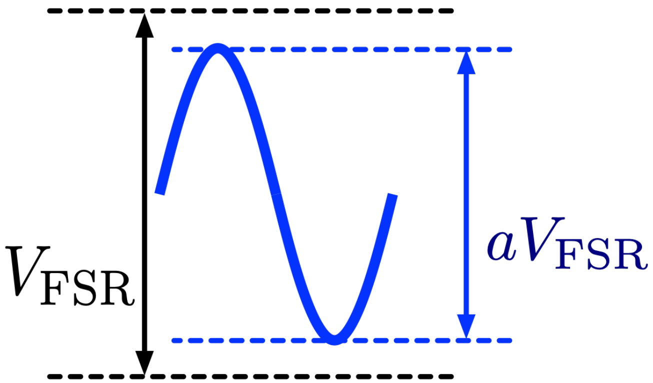
Solution:
Fast sampling, and filter



- Convolution with low-pass h(t)
- 127 coeff. Blackman-Harris kernel provides 70 dB stop-band attenuation
- Before reducing σ², additional bits are required to keep quantization low



Quantization and sinusoidal signals



Assume that the noise power is equally distributed between 0 and $B = f_S/2$

This is not true when signal and clock are highly coherent (Widrow-Kollar, Appendix G)

Provisionally, take uniform distribution

Max SNR (a = 1)

$$\mathrm{SNR}_{\mathrm{dB}} = 6.02n + 1.76\ \mathrm{dB}$$

Signal power

$$P_0 = \frac{V_{pp}^2}{8} = \frac{a^2 V_{\mathrm{FSR}}^2}{8}$$

Remind: noise power and PSD

$$\sigma^2 = \frac{V_{\mathrm{FSR}}^2}{12 \times 2^{2n}} \qquad N = \frac{V_{\mathrm{FSR}}^2}{6 \times 2^{2n} f_S}$$

Max Signal-to-Noise Ratio (SNR), $a = 1$

$$\mathrm{SNR} = \frac{P_0}{\sigma^2} = \frac{V_{\mathrm{FSR}}^2}{8} \, \frac{12 \times 2^{2n}}{V_{\mathrm{FSR}}^2} = \frac{3}{2} 2^{2n}$$

$$\mathrm{SNR}_{\mathrm{dB}} = 6.02n + 1.76\ \mathrm{dB}$$

SNR in 1 Hz bandwidth

$$\frac{P_0}{N} = \frac{a^2 V_{\mathrm{FSR}}^2}{8} \, \frac{6 \times 2^{2n} f_S}{V_{\mathrm{FSR}}^2} = \frac{3}{4} a^2 \, 2^{2n} \, f_S$$

1/SNR in 1 Hz BW = PM noise = AM noise

$$S_\varphi = S_\alpha = \frac{4}{3} \, \frac{1}{a^2 \, 2^{2n} \, f_S}$$

Feature reading: B. Widrow B, I. Kollar, *Quantization Noise*, Cambridge 2008

Resolution and entropy

Entropy (information theory)

$$H = -\sum_{i=1}^{N} p_i \log_2(p_i) \quad [\text{bit}]$$

Example: 1024 equally probable values,
i.e. $p_1 = 1/1024$, $\log_2(p_i) = -10$, $N = 1024$

$$H = -1024\left[\frac{1}{1024}\times(-10)\right] = 10\text{bit}$$

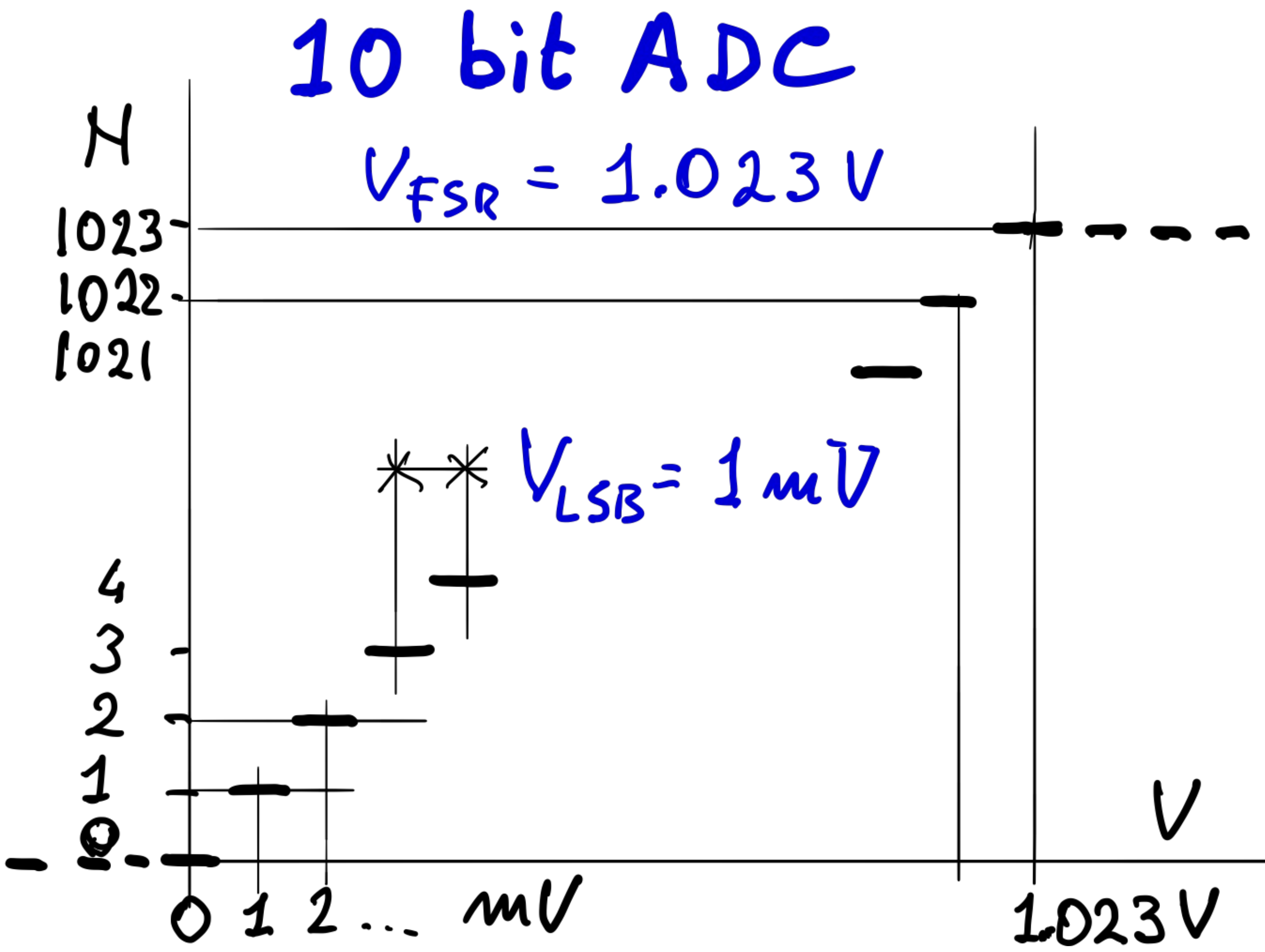
Non-uniform probability —> $H < H_{\max}$

Entropy in ADC

$$n = \log_2\left(1 + \frac{V_{\text{FSR}}}{V_{\text{LSB}}}\right)$$

The number *n* of bits is the same thing as *H* (assumes uniform quantization)

Unit	bit	nat	Hartley
Log base	2	e	10



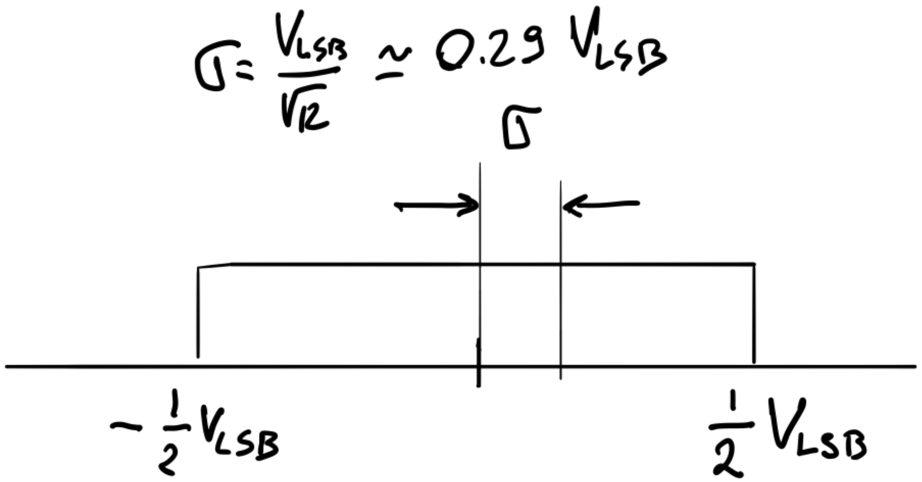
Entropy and transition noise

This is an approximation – Reality is way more complex, read Widrow & Kolar

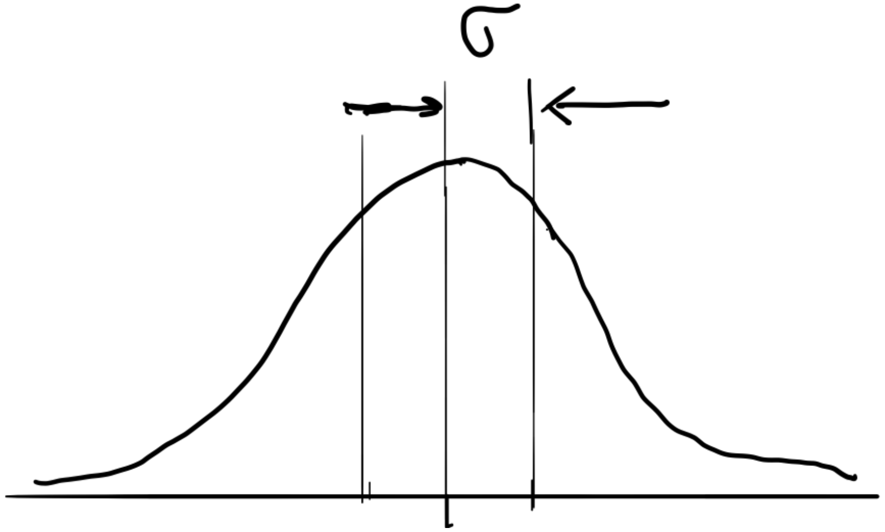
$$H = \log_2\left(1 + \frac{V_{FSR}}{V_{LSB}}\right)$$

Replace $V_{LSB} \rightarrow \sqrt{12}\,\sigma$

$$H = \log_2\left(1 + \frac{V_{FSR}}{\sqrt{12}\sigma}\right)$$



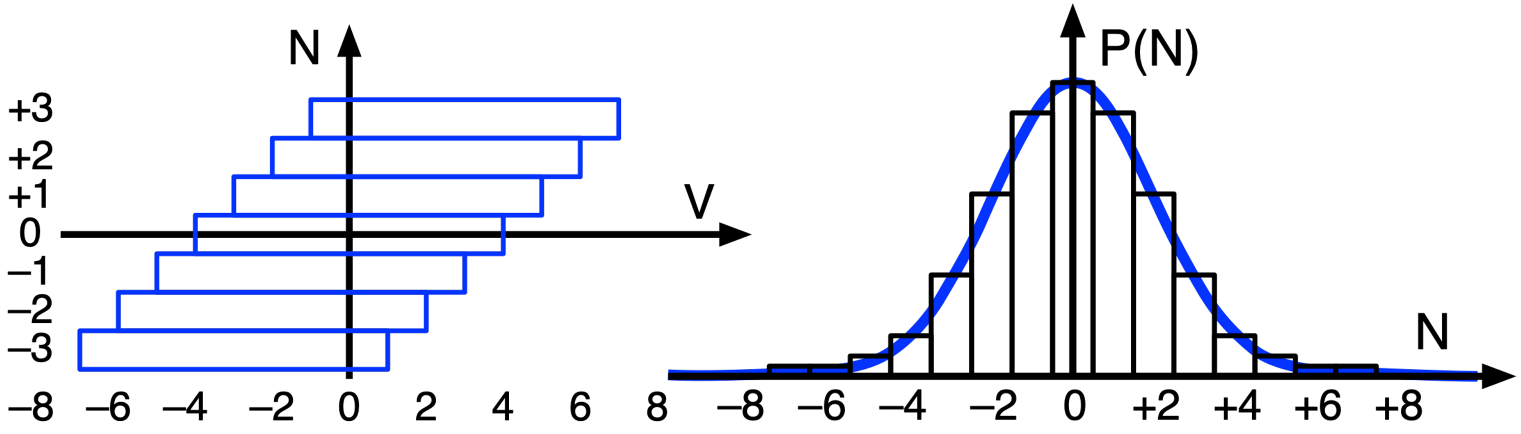
$$\sigma = \frac{V_{LSB}}{\sqrt{12}} \simeq 0.29\, V_{LSB}$$



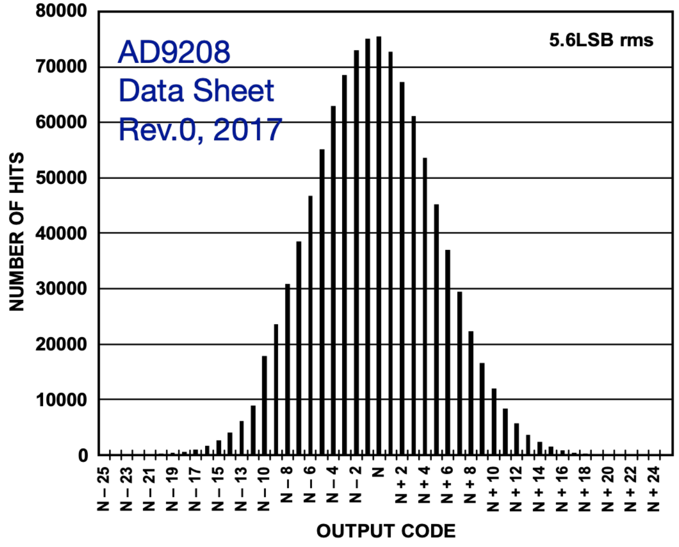
Take this as a heuristic explanation.
This approximation is reasonably close to the exact result.

Widrow B, Kollar I, *Quantization Noise*, Cambridge 2008, ISBN 978-9-521-88671-0

Transition noise



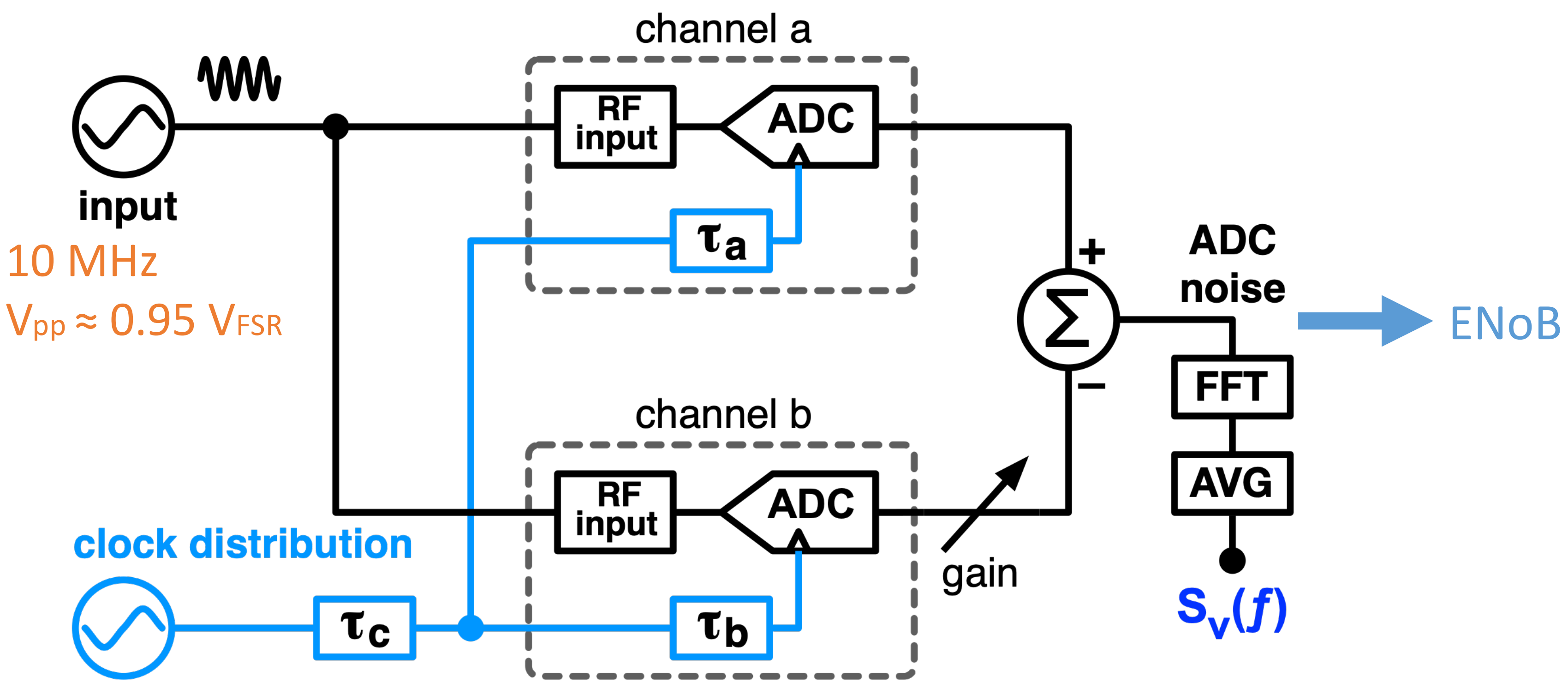
Example



AD9208
Data Sheet
Rev.0, 2017

- Actual noise includes quantization, analog noise, and distortion
- Total noise $\sigma_v^2 = \sigma_q^2 + \sigma_a^2 + \sigma_d^2$
- Random distribution of output N
- Metrology suggests to make σ^2_q negligible because BUS bits are cheap

Transition Noise Measurement

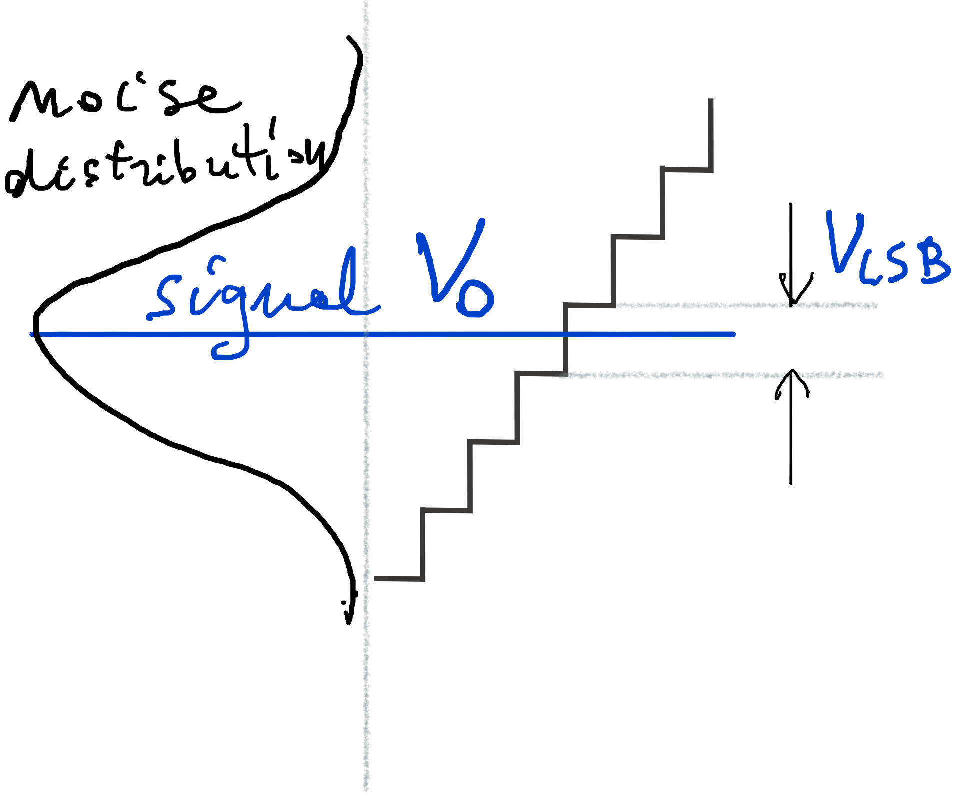


The differential clock jitter introduces additional noise due to the asymmetry between AM and PM

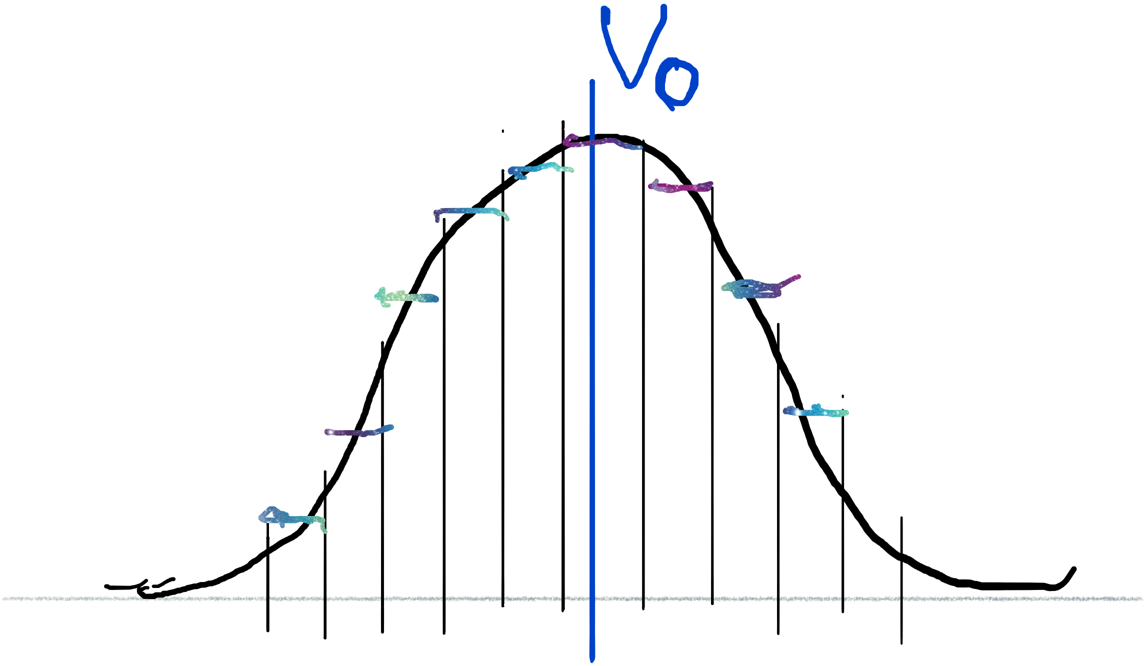
At 10 MHz input, the effect of ≈100 fs jitter does not show up

Dithering

Historical challenge: resolution of a fraction of V_{LSB}



- Add white noise and average
- Estimate the center of the distribution



Signal-To-Noise And Distortion ratio (SINAD)

Actual noise includes quantization, analog noise, and distortion

Total noise

$$\sigma_v^2 = \sigma_q^2 + \sigma_a^2 + \sigma_d^2$$

Start from

$$\text{SNR} = \frac{3}{2} 2^{2n}$$

Holds for full-scale-range sinusoidal signal

Replace

$$n \rightarrow \text{ENOB}$$

$$\text{SNR} \rightarrow \text{SINAD}$$

Signal-To-Noise And Distortion ratio

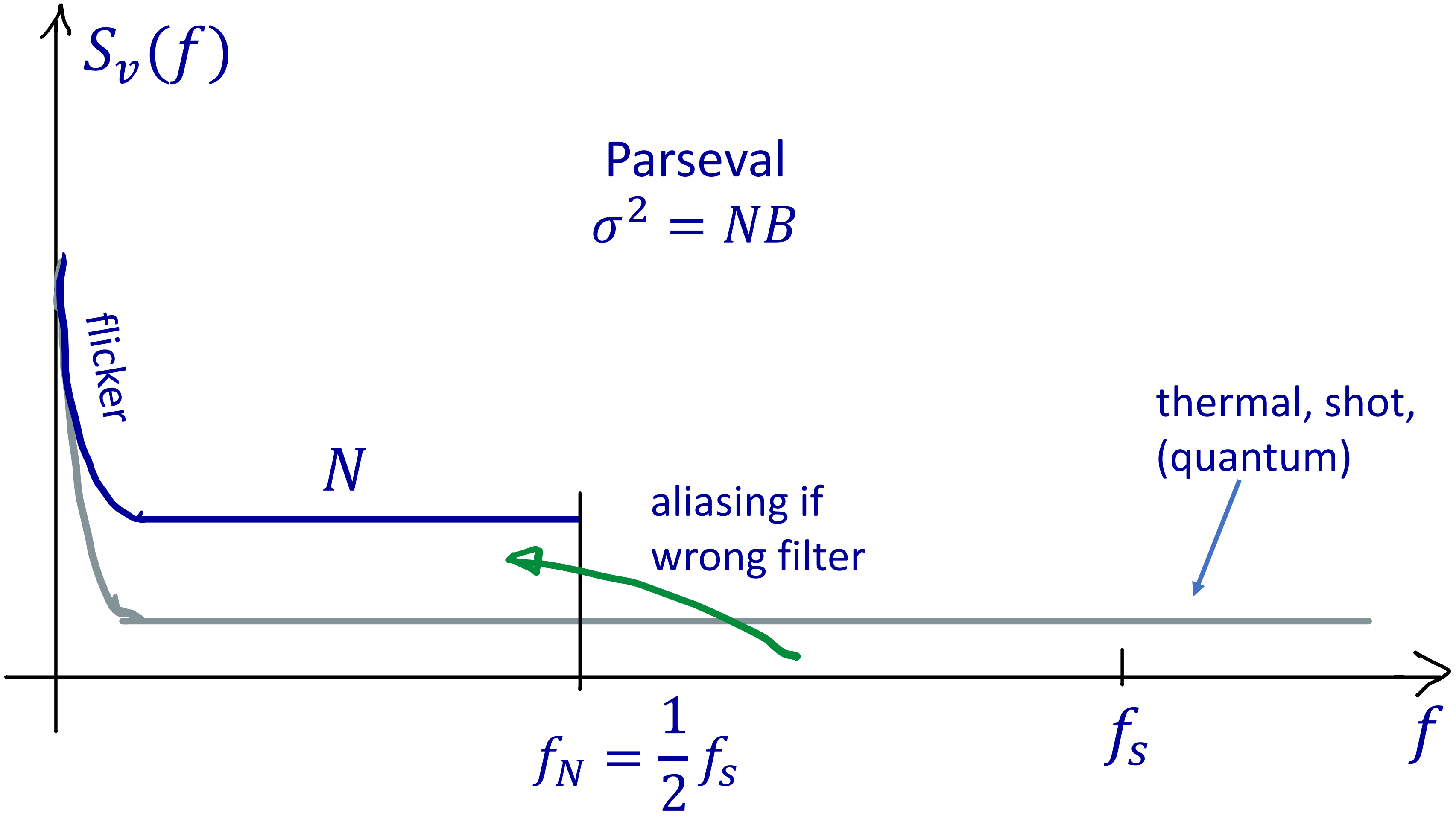
$$\text{SINAD} = \frac{P_0}{\sigma_q^2 + \sigma_a^2 + \sigma_d^2}$$

Get

$$\text{SINAD} = \frac{3}{2} 2^{2\times\text{ENOB}}$$

Often written as $\text{ENOB} = \frac{\text{SINAD}_{\text{dB}} - 1.76}{6.02}$ $\quad \text{dB}$

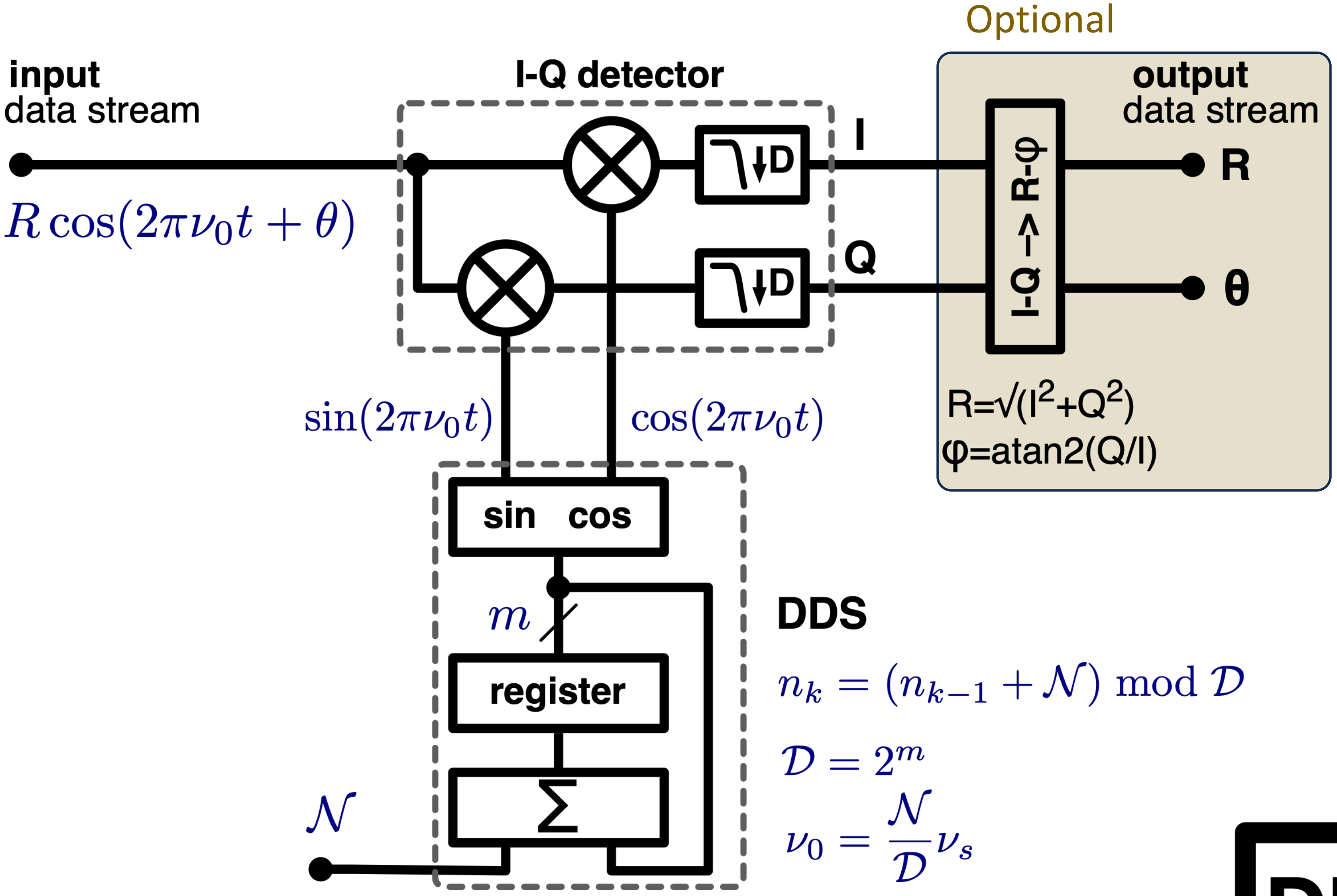
Ultimate limits



Parseval

$$\sigma^2 = NB$$

Digital Down Conversion



DDS

$$n_k = (n_{k-1} + \mathcal{N}) \bmod \mathcal{D}$$

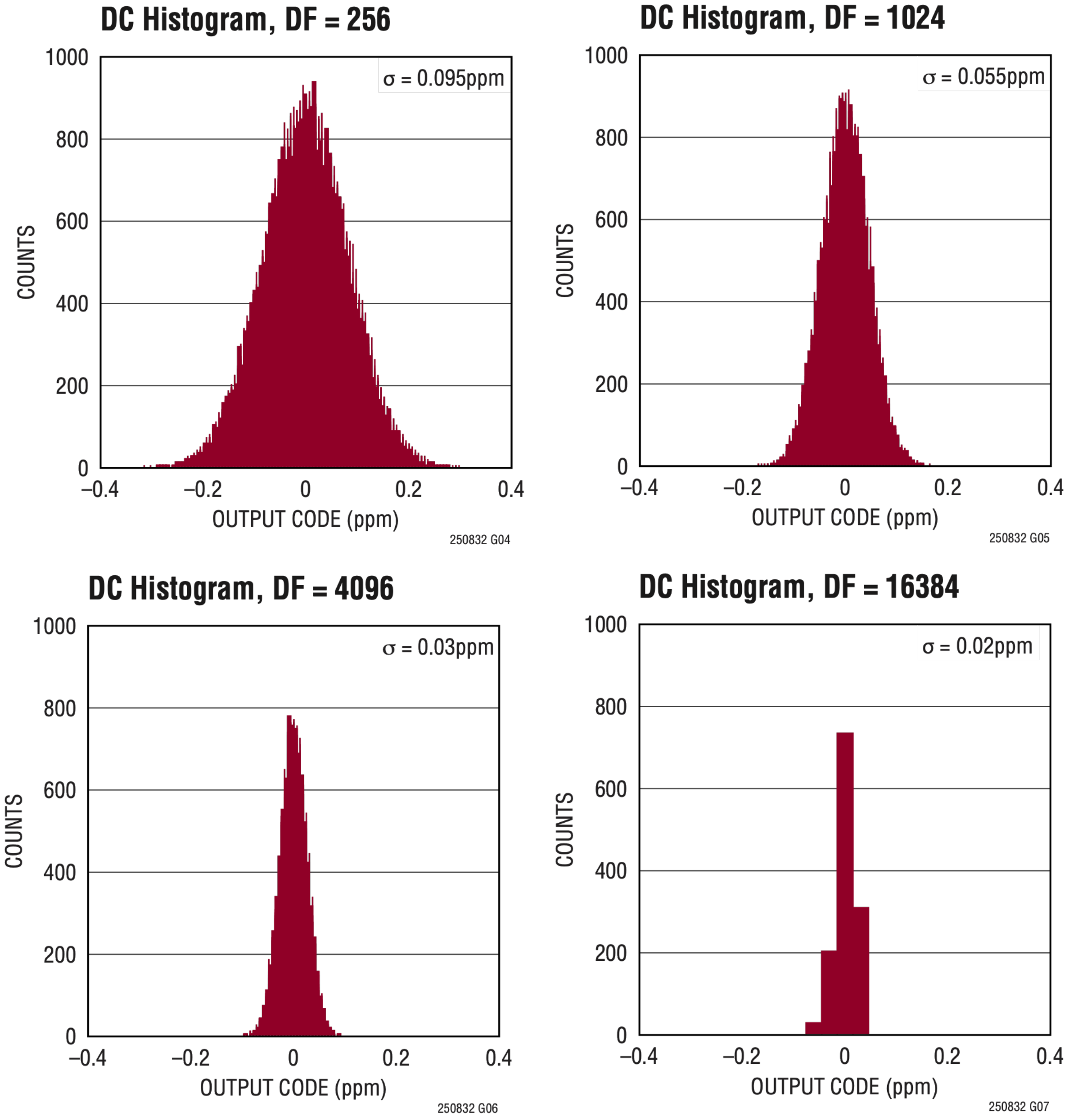
$$\mathcal{D} = 2^m$$

$$\nu_0 = \frac{\mathcal{N}}{\mathcal{D}}\nu_s$$

DDC

Down sampling (example)

Figures: LTC2508-32 data sheet p.7, ©2016 Linear Technology / Analog Devices



- DF is the Decimation Factor
  $$\mathrm{DF} = B_{\mathrm{max}}/B\mathrm{Bandwidthratio}$$
- A factor 4 in B_{max}/B results in 1 bit resolution increase

ADS1262, Texas Instrument
LTC2508-32, Linear Technology / Analog Devices

Selected High-Speed ADCs

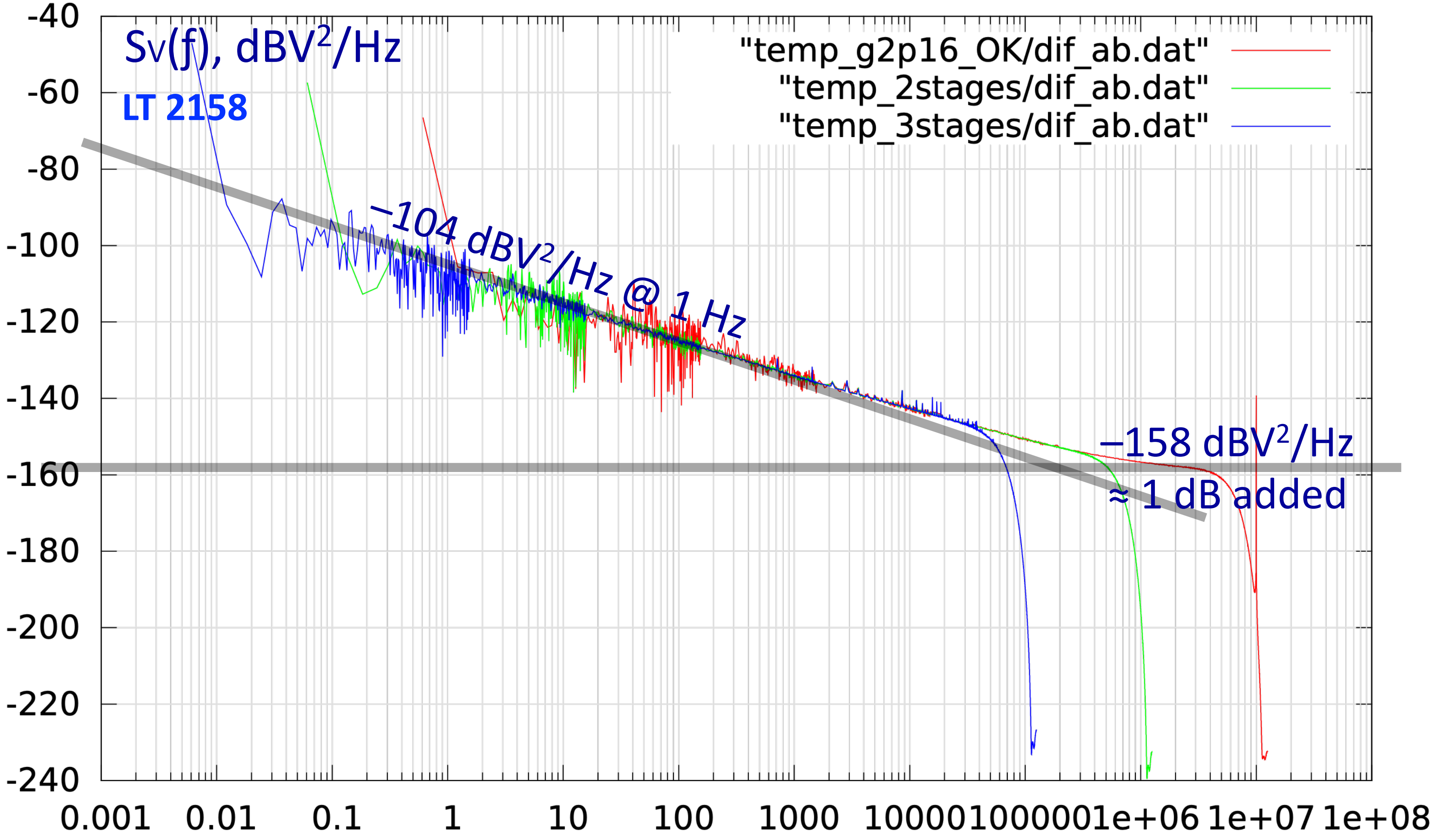
ADC type	AD9467 / Single Alazartech board)	LTC2145 / Dual Red Pitaya board	LTC2158 / Dual Eval board
Platform	Computer	Zynq (onboard)	Zynq (separated)
Sampling f Input BW	250 MHz 900 MHz	125 MHz 750 MHz	310 MHz 1250 MHz
Bits / ENoB	16 / 12	14 / 12	14 / 12
Expected noise (2 V_{fsr})	–158 dBV^2/Hz	–155 dBV^2/Hz	–159 dBV^2/Hz
Delay & Jitter	1.2 ns & 60 fs	0? & 100 fs diff 0? & 80 fs single	1 ns & 150 fs
Power supply	1.8 V & 3.3 V 1.33 W	1.8 V 190 mW	1.8 V 725 mW

Dissipation is relevant to thermal stability

For reference, 100 fs jitter is equivalent to

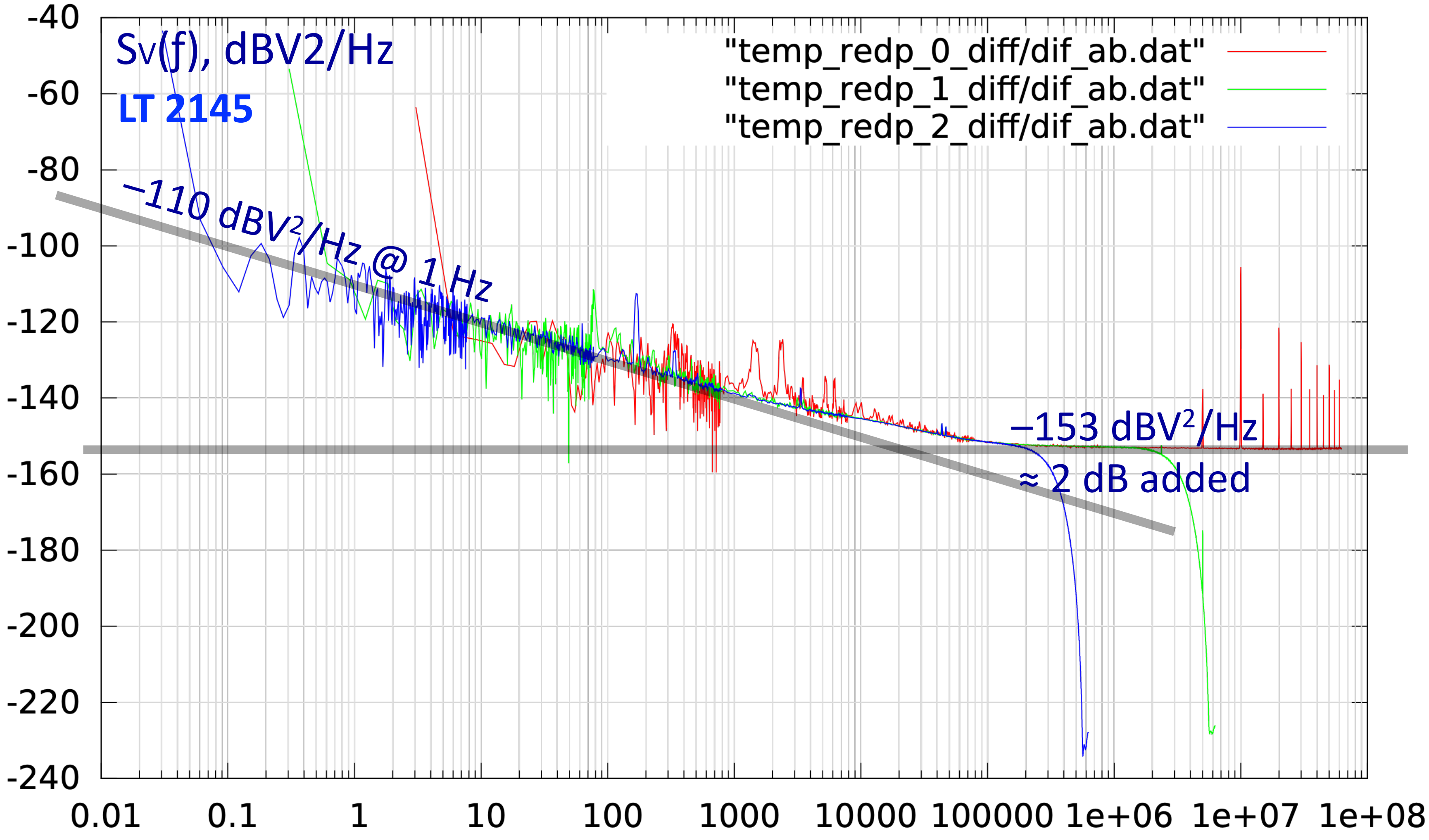
carrier f	φ rms	$S_\varphi(f) = b_0$	10 Log$_{10}$[L(f)]
10 MHz	6.3 µrad	4x10^{-18} rad^2/Hz	–177 dBc/Hz
100 MHz	63 µrad	4x10–17 rad2/Hz	–167 dBc/Hz

LT 2158 noise



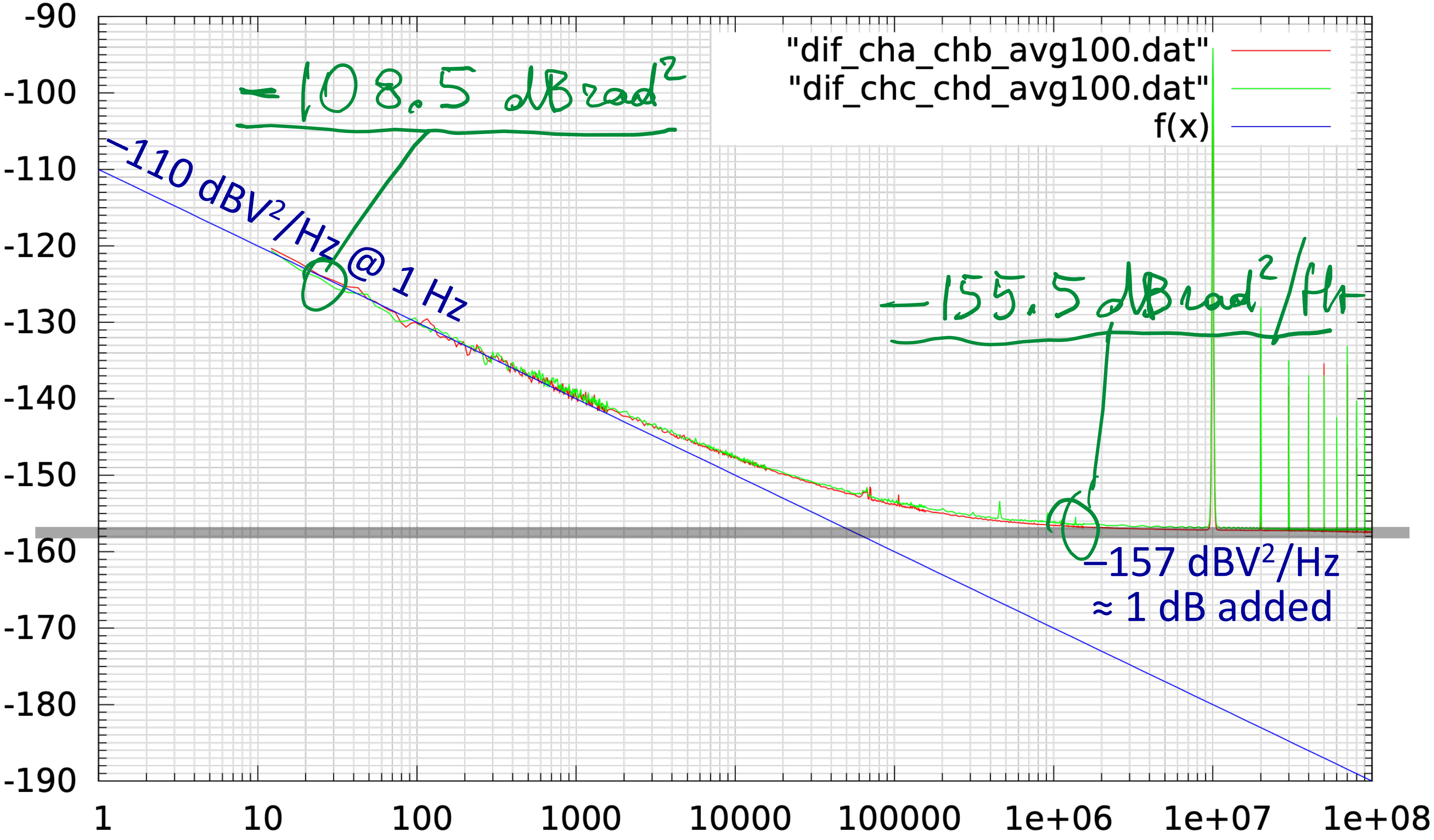
10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

LT2145 noise (Red Pitaya)



10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

Background noise – Example AD9467 (Alazartech)



−110 dBV²/Hz @ 1 Hz

−108.5 dBrad²

−155.5 dBrad²/Hz

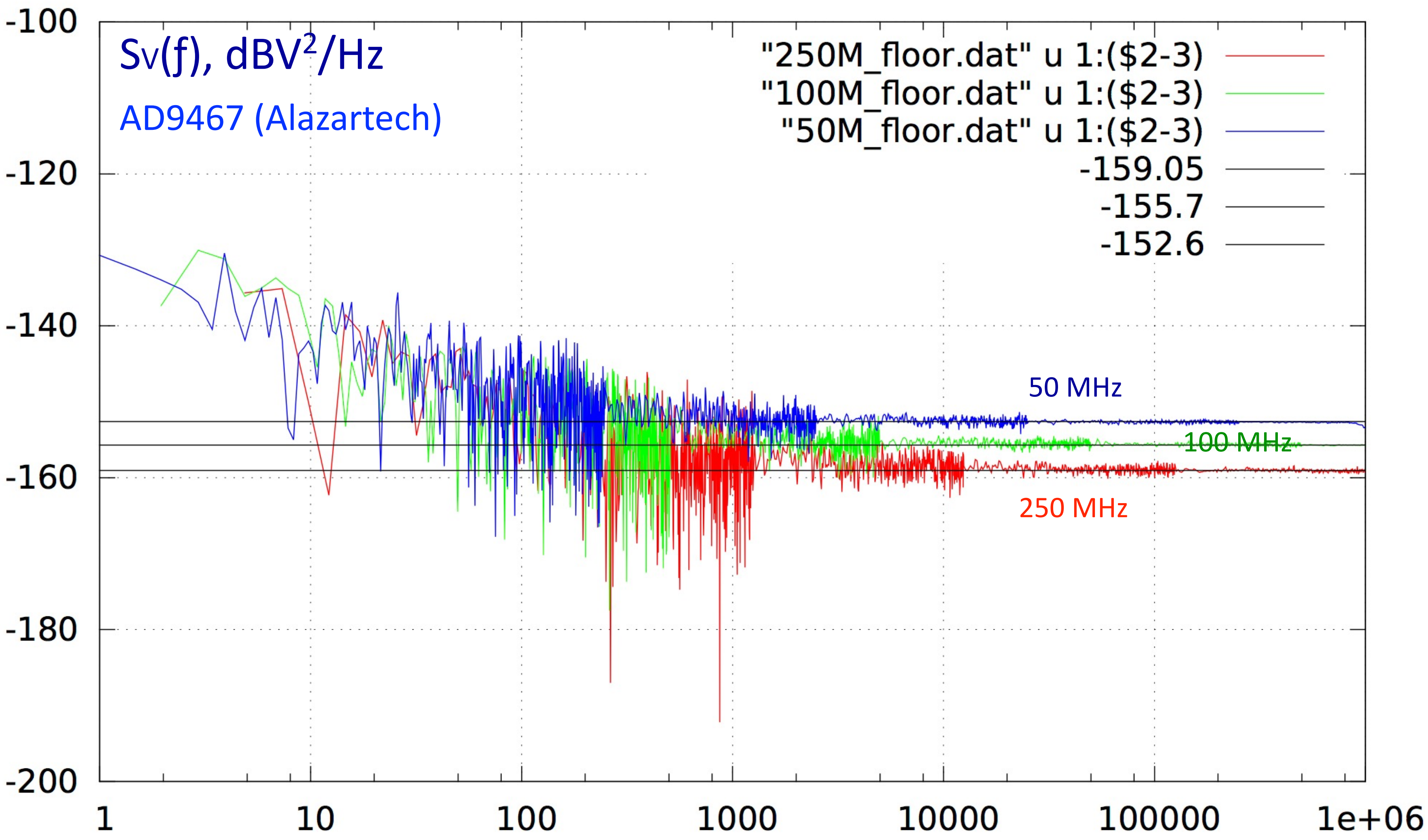
−157 dBV²/Hz
≈ 1 dB added

$V_{FSR} = 2.5\ V_{pp}$
$= 0.88\ V_{rms}$

$0.95 V_{FSR} \rightarrow 0.84\ V_{rms}$
-1.5 dBV

10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

AD9467 (Alazartech) sampling frequency



The observed floor fits the theory

We always use the highest sampling frequency

Low-resolution ADCs (1-4 bits)

Currently used where the SNR is extremely poor
Typically, radio astronomy and in consumer GNSS ("GPS") receivers

- van der Wal PW, van Willigen D - Hard limiting and sequential detection applied to Loran-C - IEEE T AES 14(4), July 1978
- van der Wal JC, van Willigen D - Hard limiter performance as a polarity detector for extremely polluted signals - IEEE AES 18(5), September 1982
- Josse C, van Willigen D - Analysis and design considerations of hard limiters for LF and VLF navaid receivers - IEEE T AES 20(3), May 1984
- Takasu T - Pocket SDR, A Seminar for GNSS Software Defined Receivers - Chubu-Univ, 2024-11-19
- Hegarty CJ - Analytical Model for GNSS Receiver Implementation Losses - Navigation 58(1), March 2011

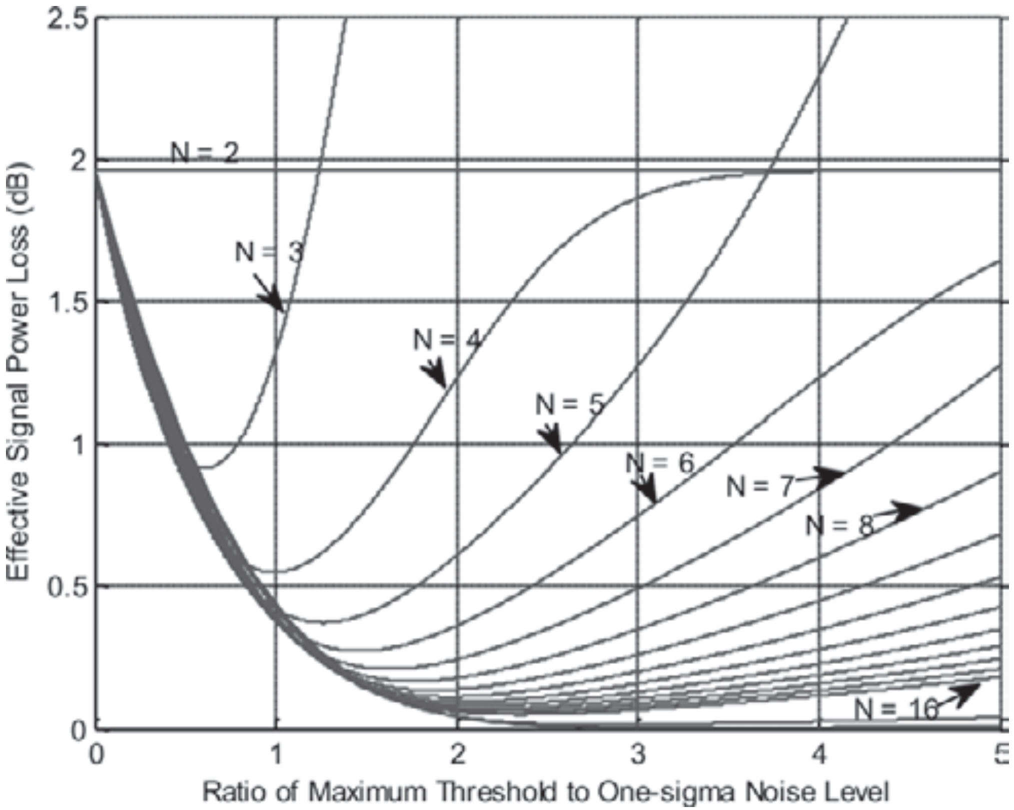
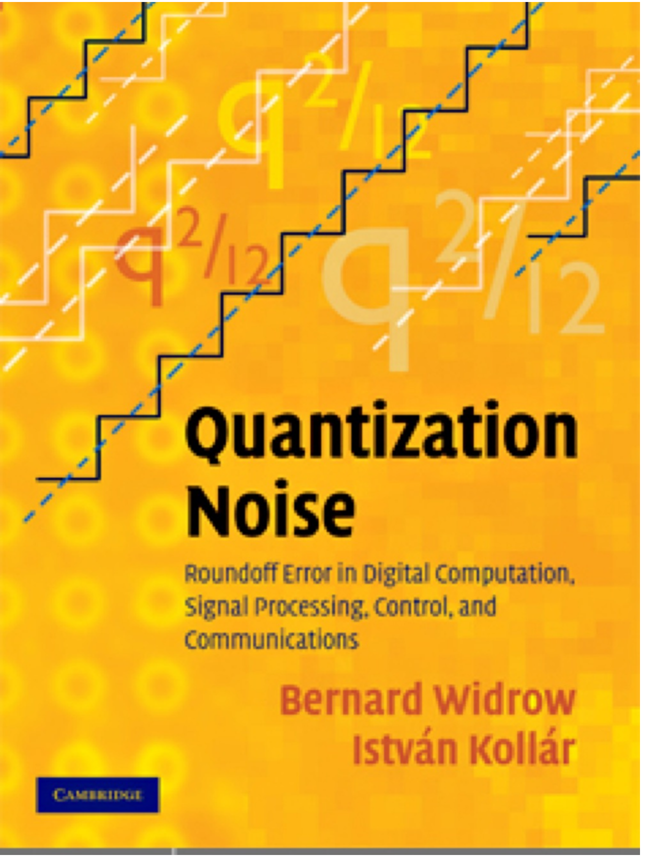


Fig. 5–Signal losses for quantizers with N levels vs. maximum input threshold

Fig.5 from Hegarty CJ - Analytical Model for GNSS Receiver Implementation Losses - Navigation 58(1), March 2011



B. Widrow, I. Kollar, *Quantization Noise*, Cambridge 2008, ISBN 978-0-511-40990-5

Chapter 15: Roundoff noise in FIR digital filters and in FFT calculations

Appendix G: Quantization of a sinusoidal input

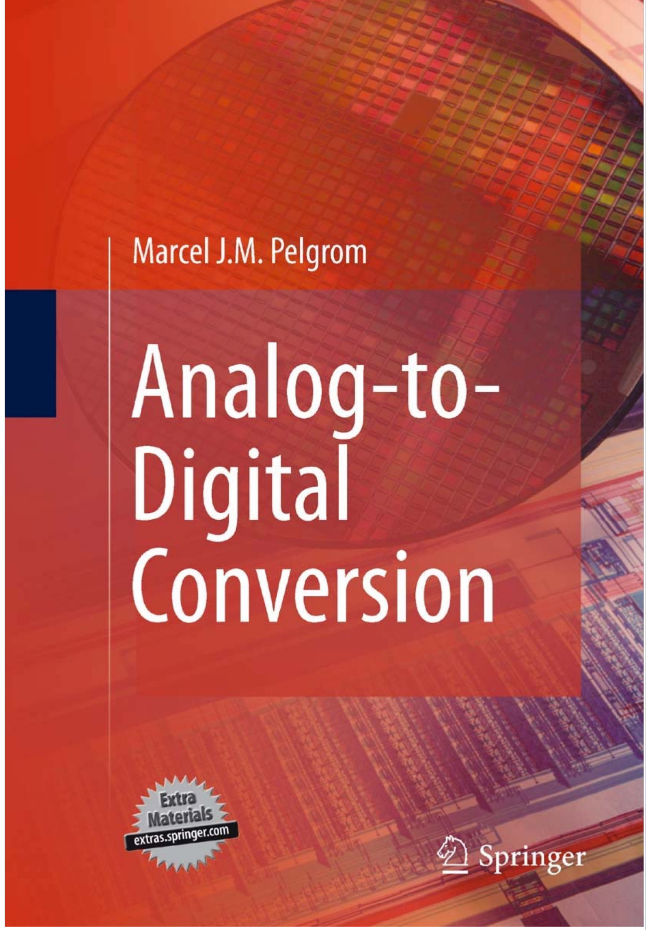
ANALOG-DIGITAL CONVERSION

Walt Kester

Editor

ANALOG DEVICES

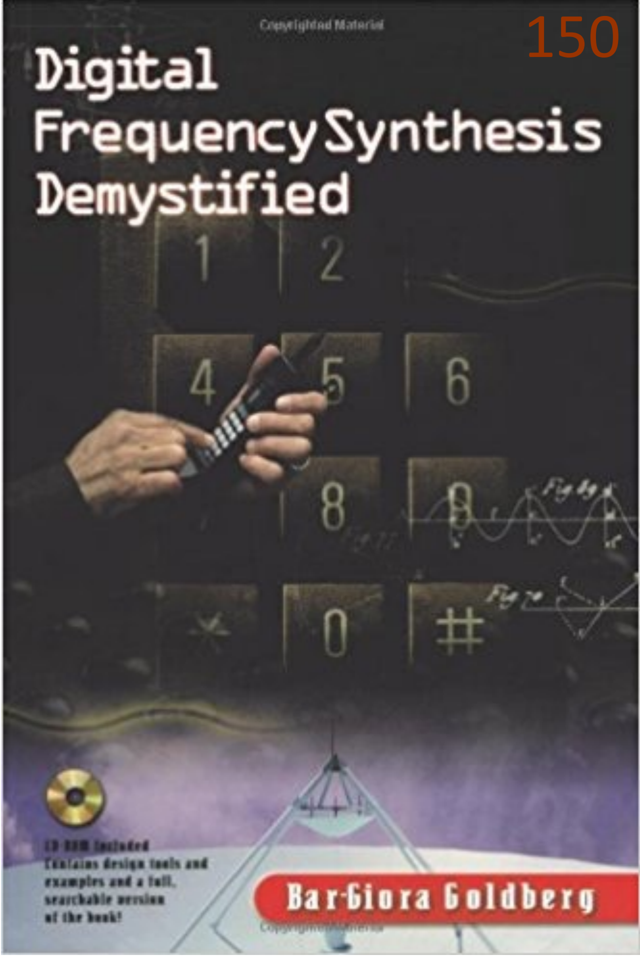
Walt Kester (editor), *Analog-Digital Conversion*, Analog Devices 2004.
ISBN 0-916550-27-3.
Free of charge



Marcel J. M. Pelgrom

Analog-to-Digital Conversion

Springer 2010
ISBN 978-90-481-8888-8



Bar-Giora Goldberg

Digital Frequency Synthesis Demystified

Newnes 1999
ISBN 978-1-878707-47-5

Our Articles

- C. E. Calosso, A. C. Cárdenas Olaya E. Rubiola, Phase-Noise and Amplitude-Noise Measurement of DACs and DDSs, IEEE Transact UFFC vol.67 no.2 p.431-439 February 2020
- A. C. Cárdenas Olaya, C. E. Calosso, J.-M. Friedt, S. Micalizio, E. Rubiola, "Phase Noise and Frequency Stability of the Red-Pitaya Internal PLL," IEEE Transact. UFFC vol.66 no.2 p.412-416, Feb 2019
- C. E. Calosso, F. Vernotte, V. Giordano, C. Fluhr, B. Dubois, E. Rubiola Frequency Stability Measurement of Cryogenic Sapphire Oscillators with a Multichannel Tracking DDS and the Two-Sample Covariance, IEEE Transact. UFFC vol.66 no.3 p.616-623, March 2019.
- A. C. Cardenas-Olaya, E. Rubiola, J.-M. Friedt, P.-Y. Bourgeois, M. Ortolano, S. Micalizio, and C. E. Calosso Noise characterization of analog to digital converters for amplitude and phase noise measurements, Rev. Scientific Instruments 88, 065108, June 2017.
- C. E. Calosso, Y. Gruson, E. Rubiola, "Phase noise and amplitude noise in DDS," Proc IFCS p.777-782, May 2012
- C. E. Calosso, E. Rubiola, "The Sampling Theorem in Pi and Lambda Digital Frequency Dividers," Proc IEEE IFCS p.960-962, 2013
- A. C. Cardenas Olaya, E. Rubiola, J.-M. Freidt, P.-Y. Bourgeois, M. Ortolano, S. Micalizio, C. E. Calosso, "Noise characterization of analog to digital converters for amplitude and phase noise measurements," Rev Sci Instrum 88, 065108, June 2017

Supplemental Material

Something Funny: The Maxwell's Demon

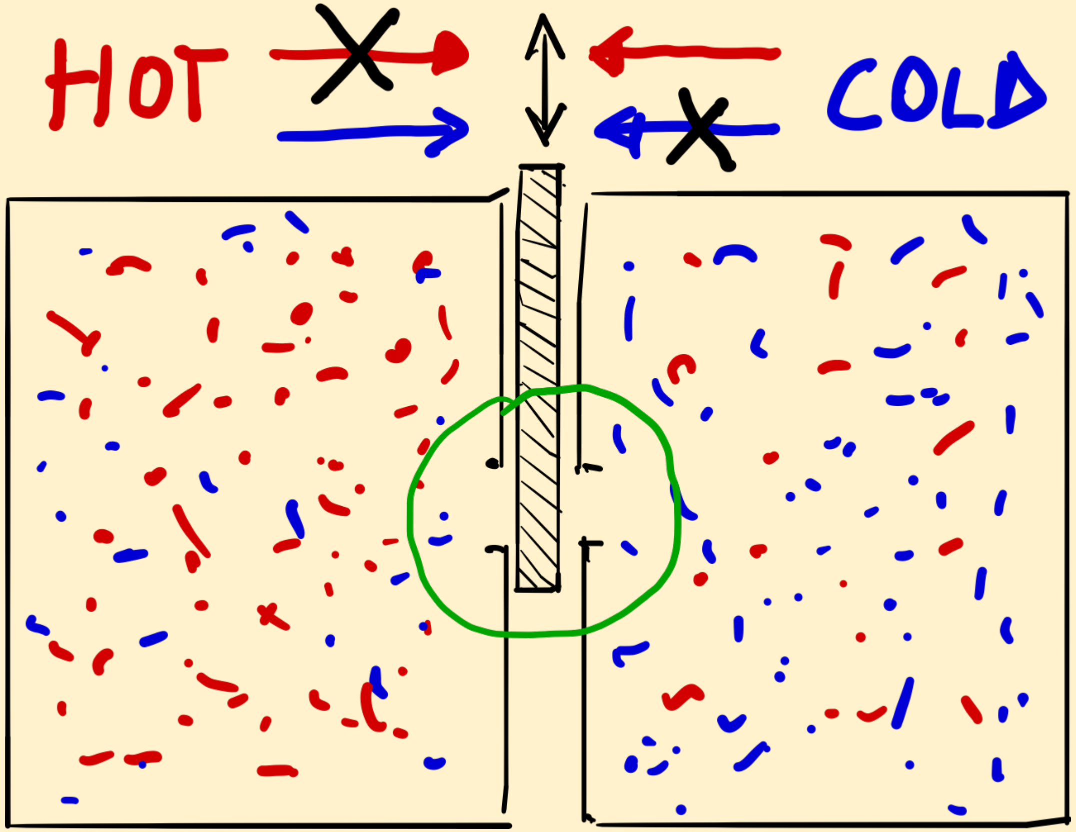
- Intriguing paradox
- Many scientists spent time and brainpower
- Theories: photon energy needed to probe the particles, etc.
- Ultimately, the ND shows the equivalence between thermodynamic entropy and information entropy
- W micro states with probability $1/W$

$$H = -\sum_{i=1}^{W} p\log(p) = \log(W)$$

Units k per nat (nat is like bit, but in natural base)

$$H = k\log(W)$$

The demon
checks on the speed and allows
cold particles —> <— hot particles
The thermodynamical equilibrium is broken



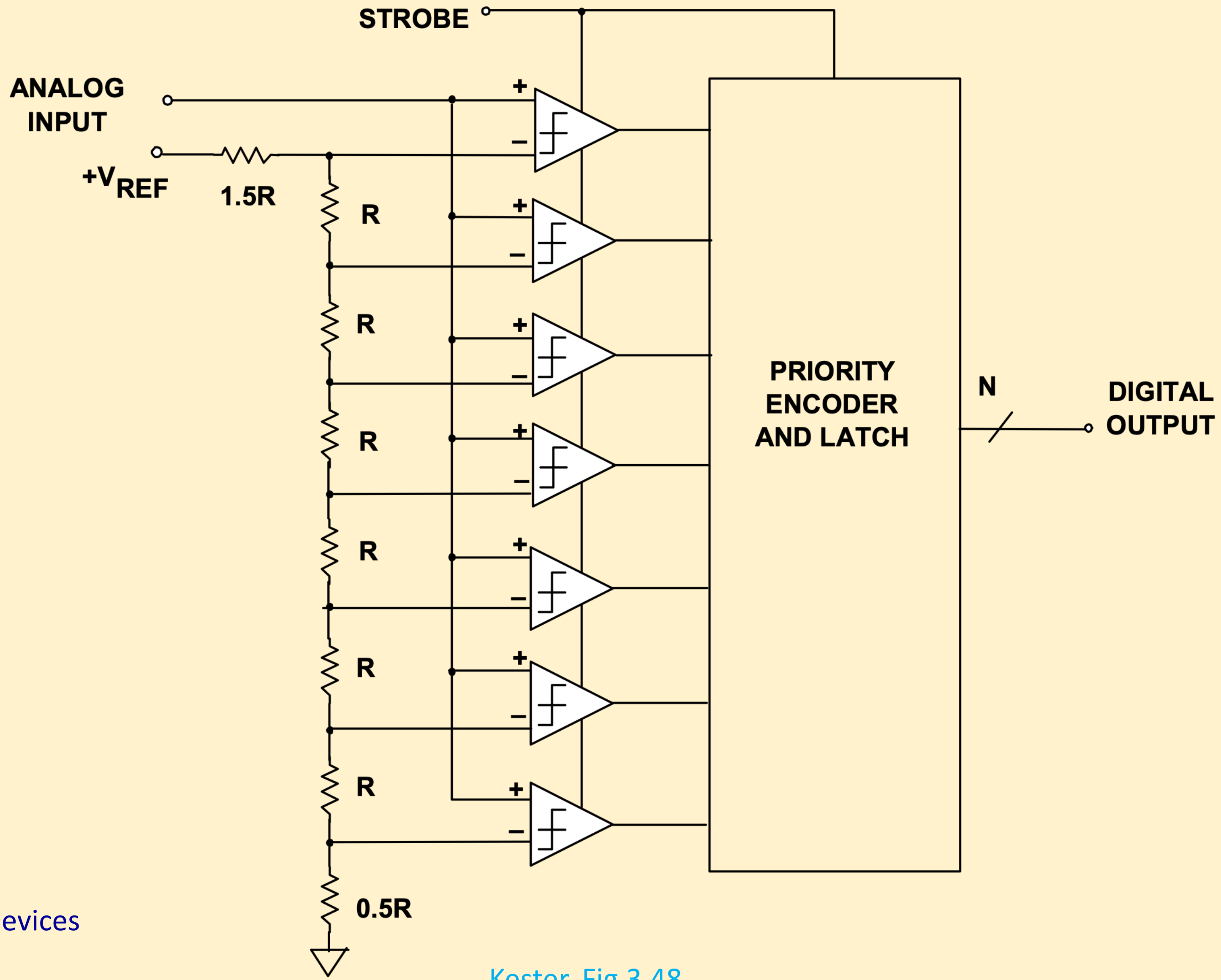
ADC Architectures

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Read it again, again and again

Flash

- Fastest, sub-nanosecond

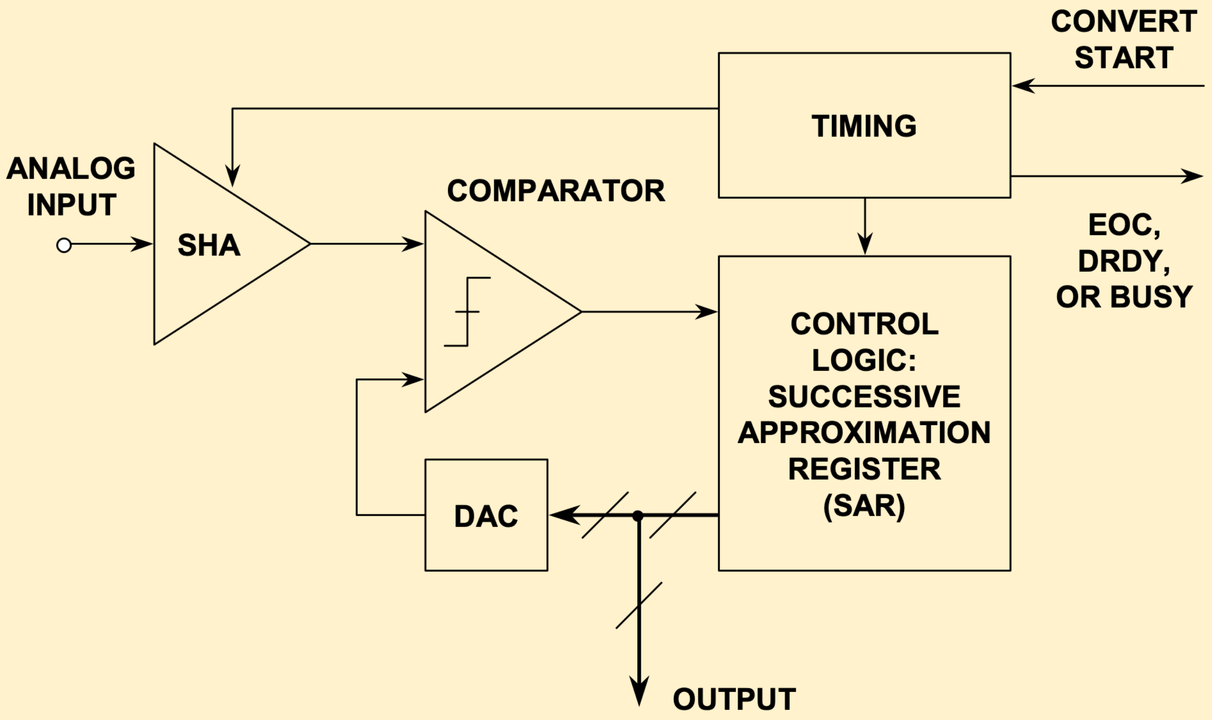


Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Kester, Fig.3.48

Successive Approximation (SAR)

- High accuracy
- High resolution, up to 32 bits
- Testing n bits takes n clock cycles
- Latency and downsampling
  - Slow, full accuracy and resolution
  - Moderate, at cost of accuracy
- The internal DAC uses switched capacitors (resistor network was obsoleted long ago)
- Tracking operation possible
  - Faster, but limited slew rate

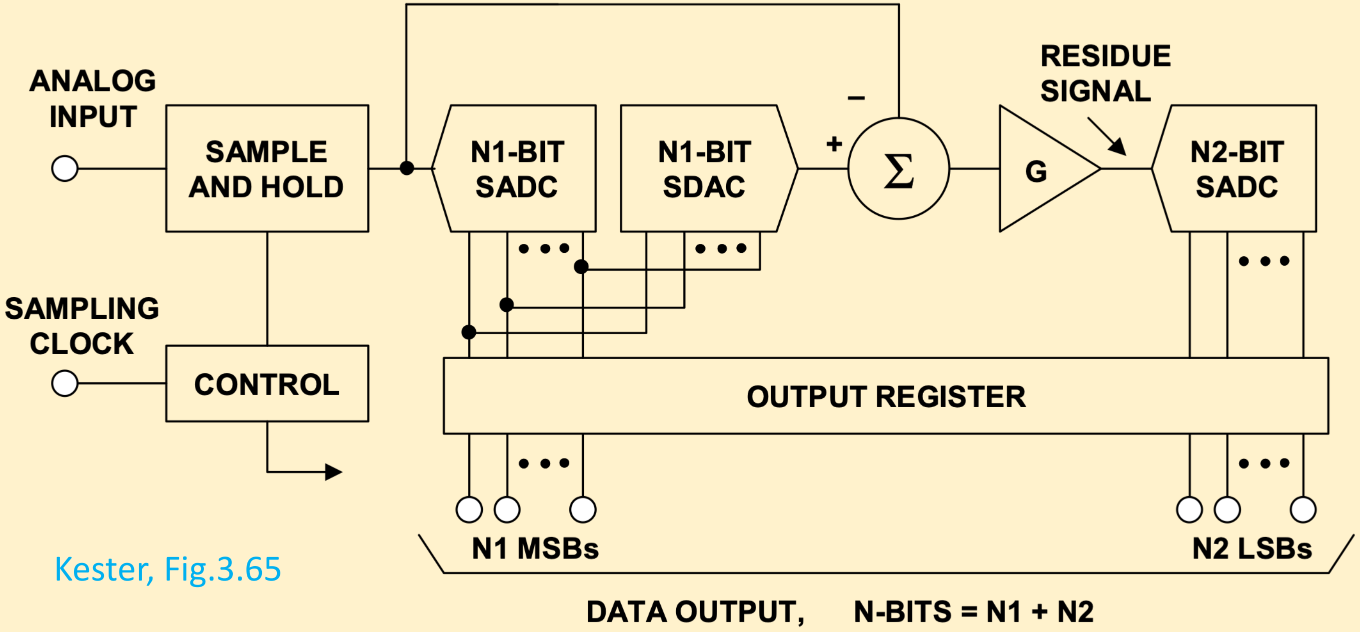


Kester, Fig.3.56

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Subranging

- Pipeline
- Great speed/resolution tradeoff



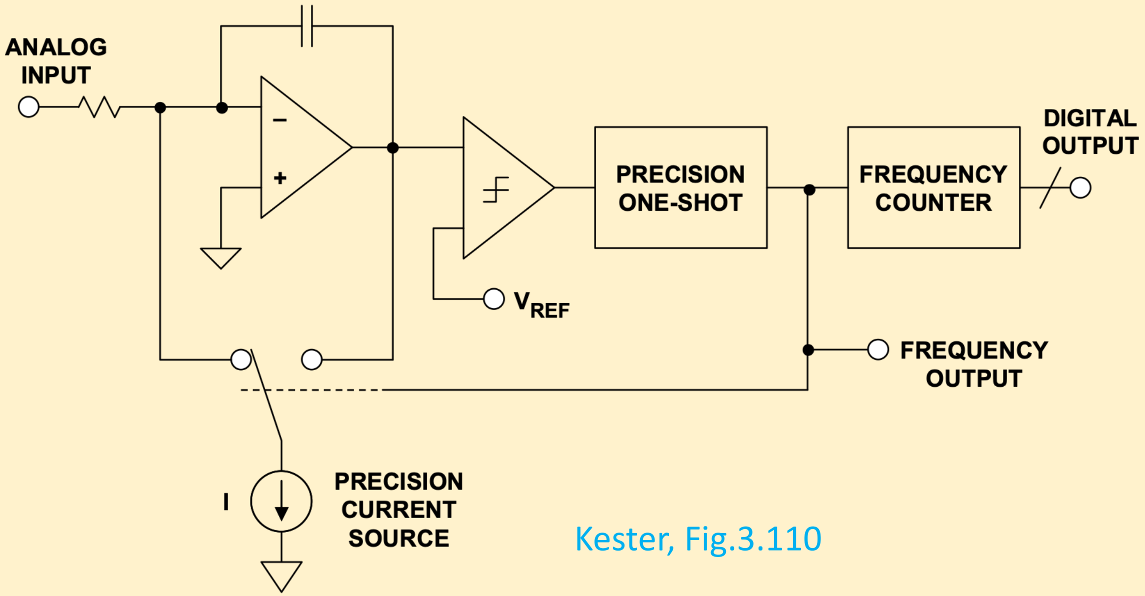
Kester, Fig.3.65

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Counting

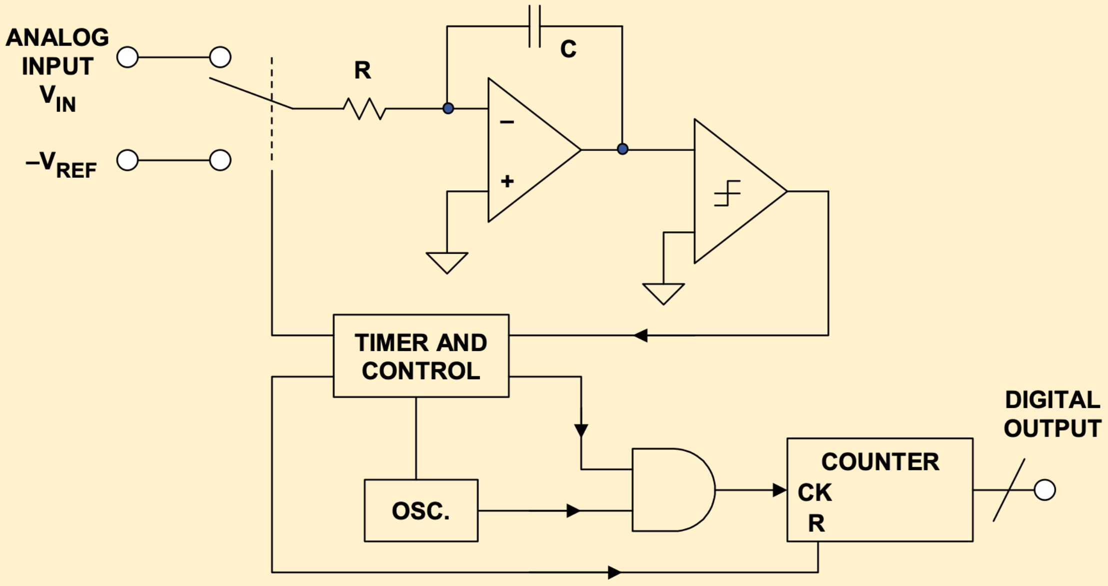
A few techniques – Analog integrator

Voltage-to-frequency converter



Kester, Fig.3.110

Dual slope



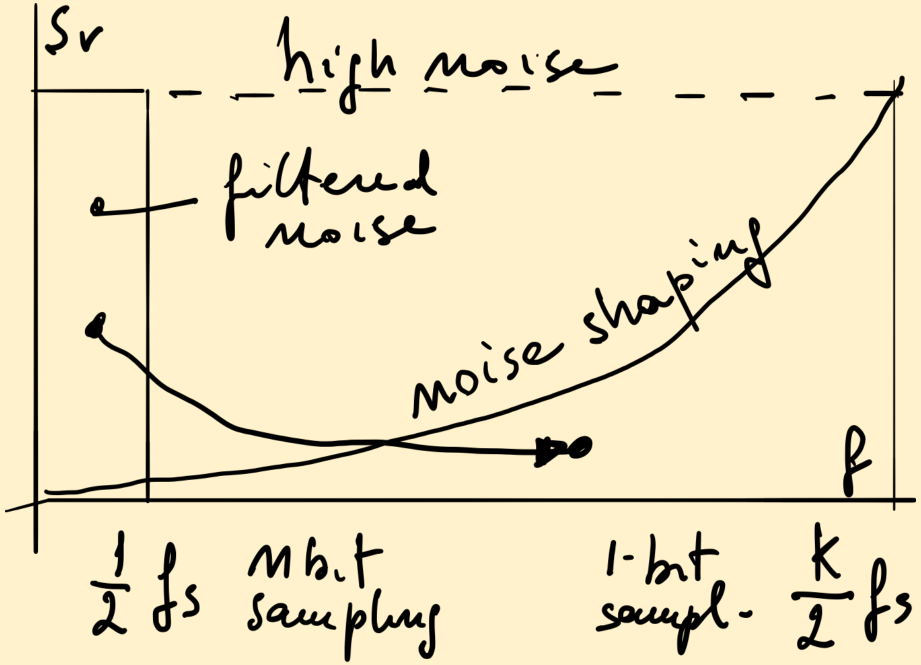
Kester, Fig.3.113

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

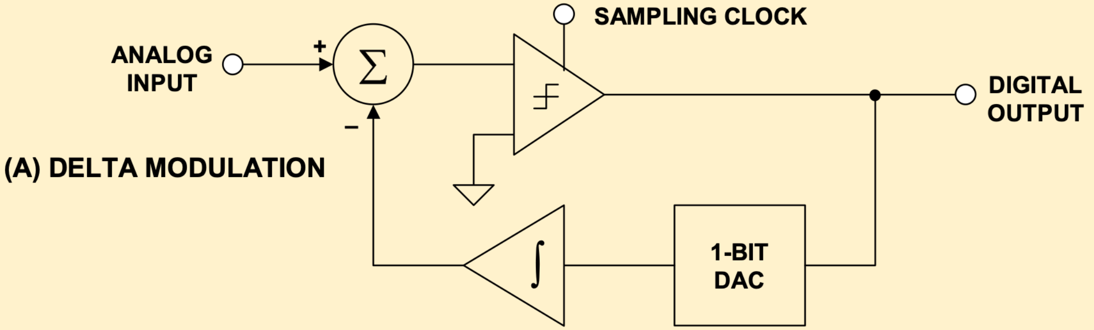
An education version of these converters is in E. Rubiola, *Laboratorio di misure elettroniche* (in Italian), CLUT, Torino, 1993. ISBN 88-7992-081-2

Sigma Delta

- High resolution and low power for cheap
- Simple ideas, but complex mathematics
- Noise shaping



Delta modulation – Kester, Fig.3.119(A)



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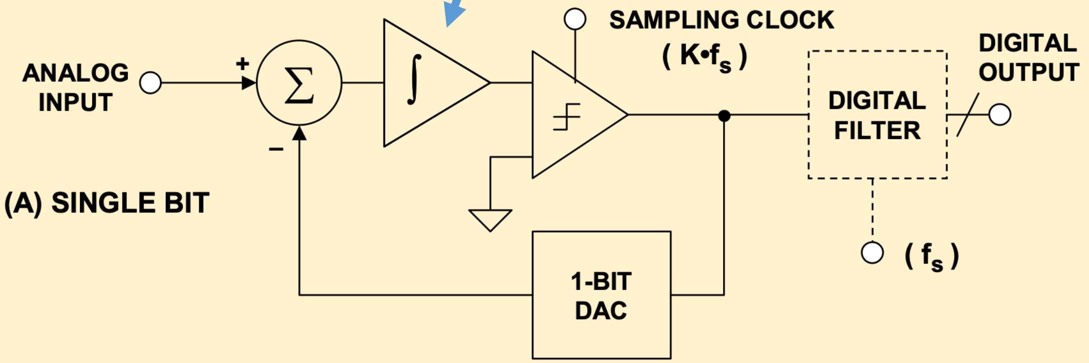


Delta modulation – Kester, Fig.3.121(A)

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Direct Digital Synthesizer (DDS)

Frequency synthesis using digital methods

Belongs to Part 2
(lectures 6–10)

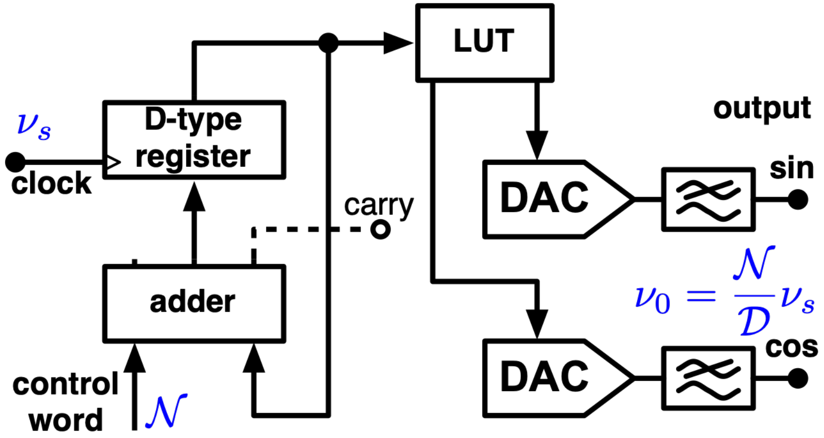
Basic DDS Scheme

replace θ —> ϕ

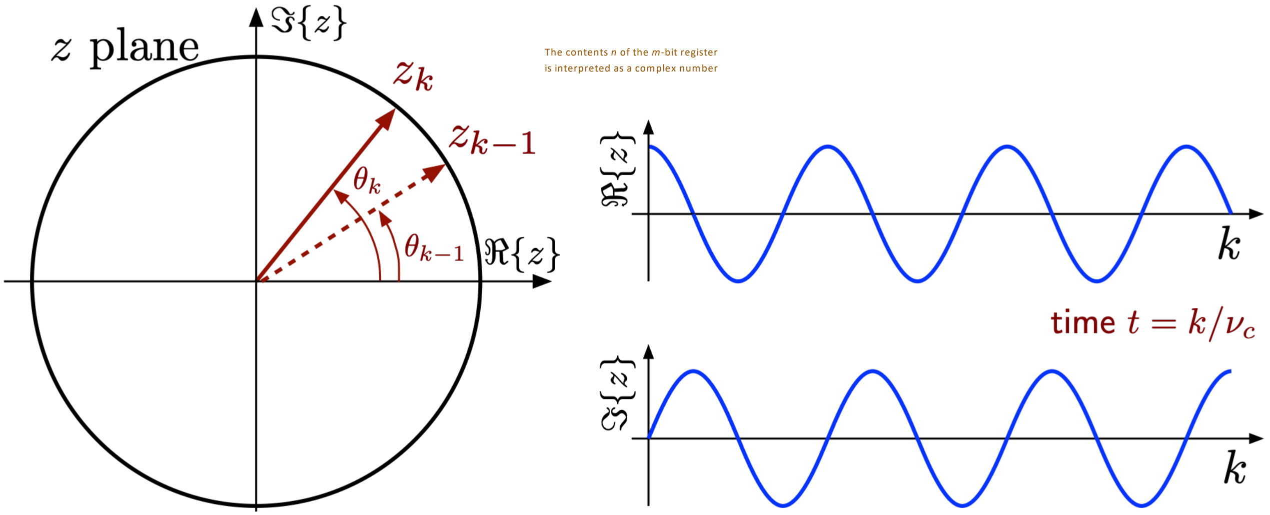
integer: $n_k = (n_{k-1} + \mathcal{N}) \bmod \mathcal{D}$

complex: $z_k = z_{k-1} \exp(j\eta)$

phase: $\theta_k = (\theta_{k-1} + \eta) \bmod 2\pi$



quantity	digital	analog
state variable	n	$\theta = 2\pi \frac{n}{\mathcal{D}}$
assoc. complex		$z = e^{j\theta}$
modulo	$\mathcal{D} = 2^m$	2π
increment	$\mathcal{N}$	$\eta = 2\pi \frac{\mathcal{N}}{\mathcal{D}}$
time	$k, \quad 0, 1, 2, \ldots$	$t = k/\nu_s$
clock freq. ν_s	output freq. $\nu_0 = \frac{\mathcal{N}}{\mathcal{D}}\nu_s$	



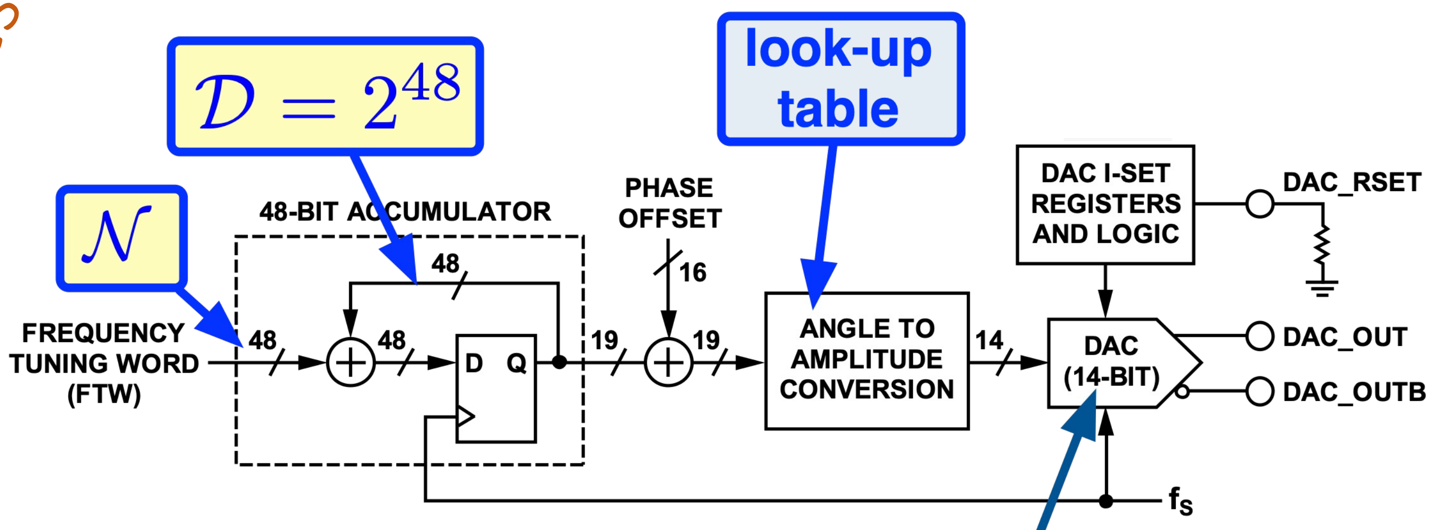
The contents n of the m-bit register is interpreted as a complex number

Belongs to Part 2 (lectures 6–10)

AD9912, a Fast DDS

Belongs to Part 2 (lectures 6–10)

48 bit accumulator, 14 bit DAC, 1 GHz clock



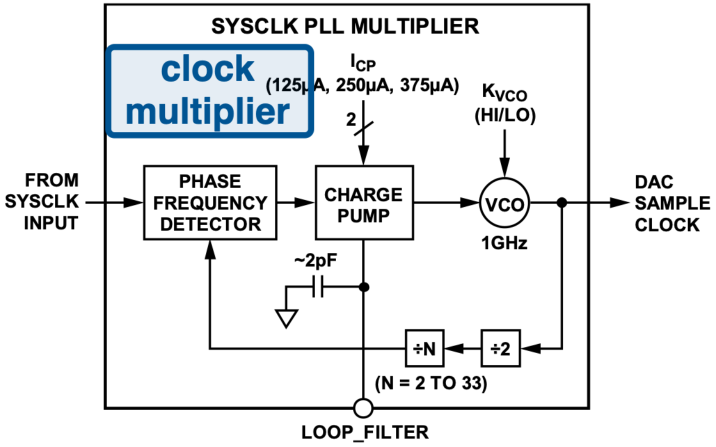
$\mathcal{D} = 2^{48}$

$\mathcal{N}$

look-up table

DAC

AD9912 data sheet, Fig. 40



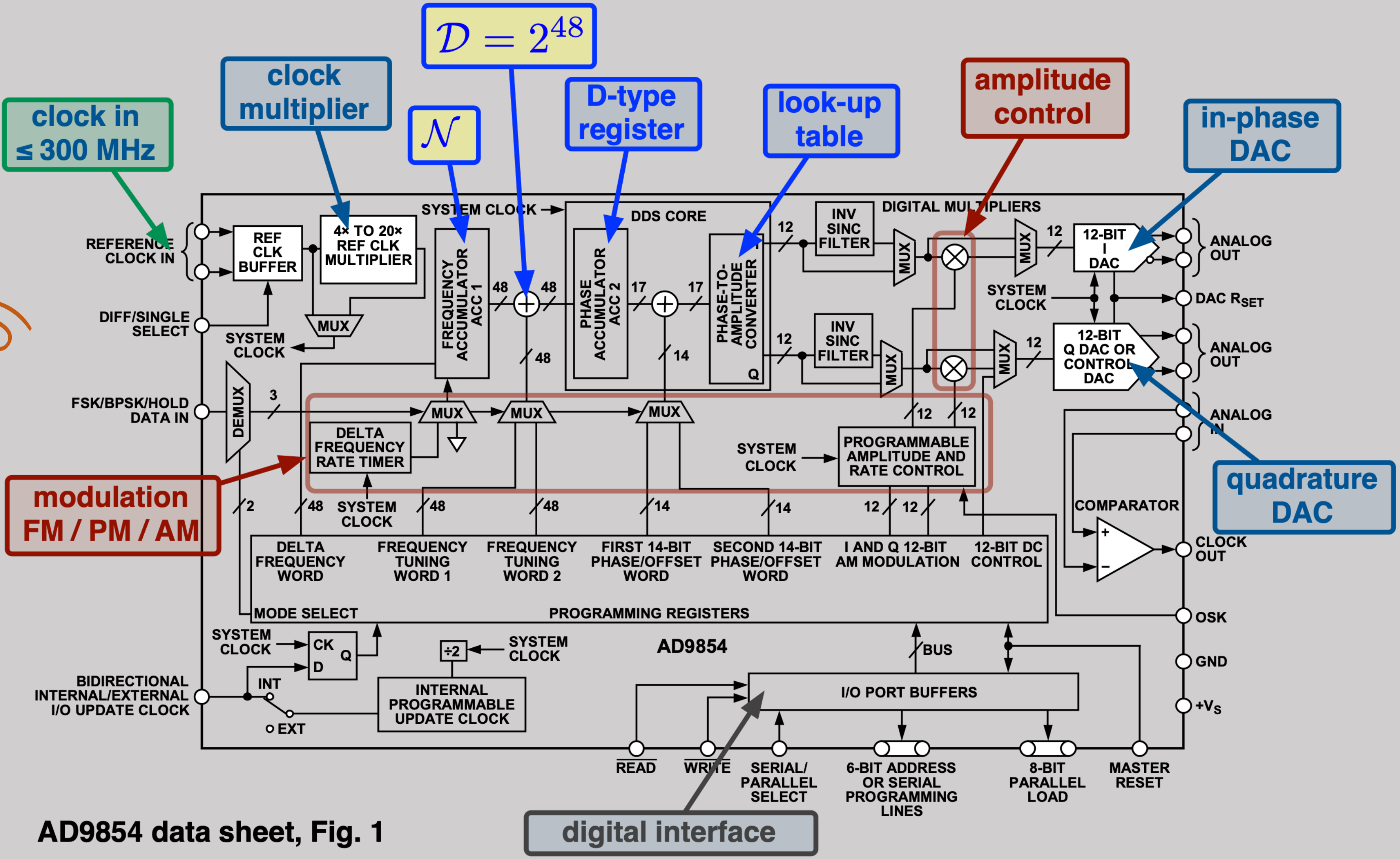
clock multiplier

AD9912 data sheet, Fig. 45

AD9854, a Flexible DDS

48 bit accumulator, 300 MHz clock,
12 bit DAC, I-Q output, AM/PM/FM capability

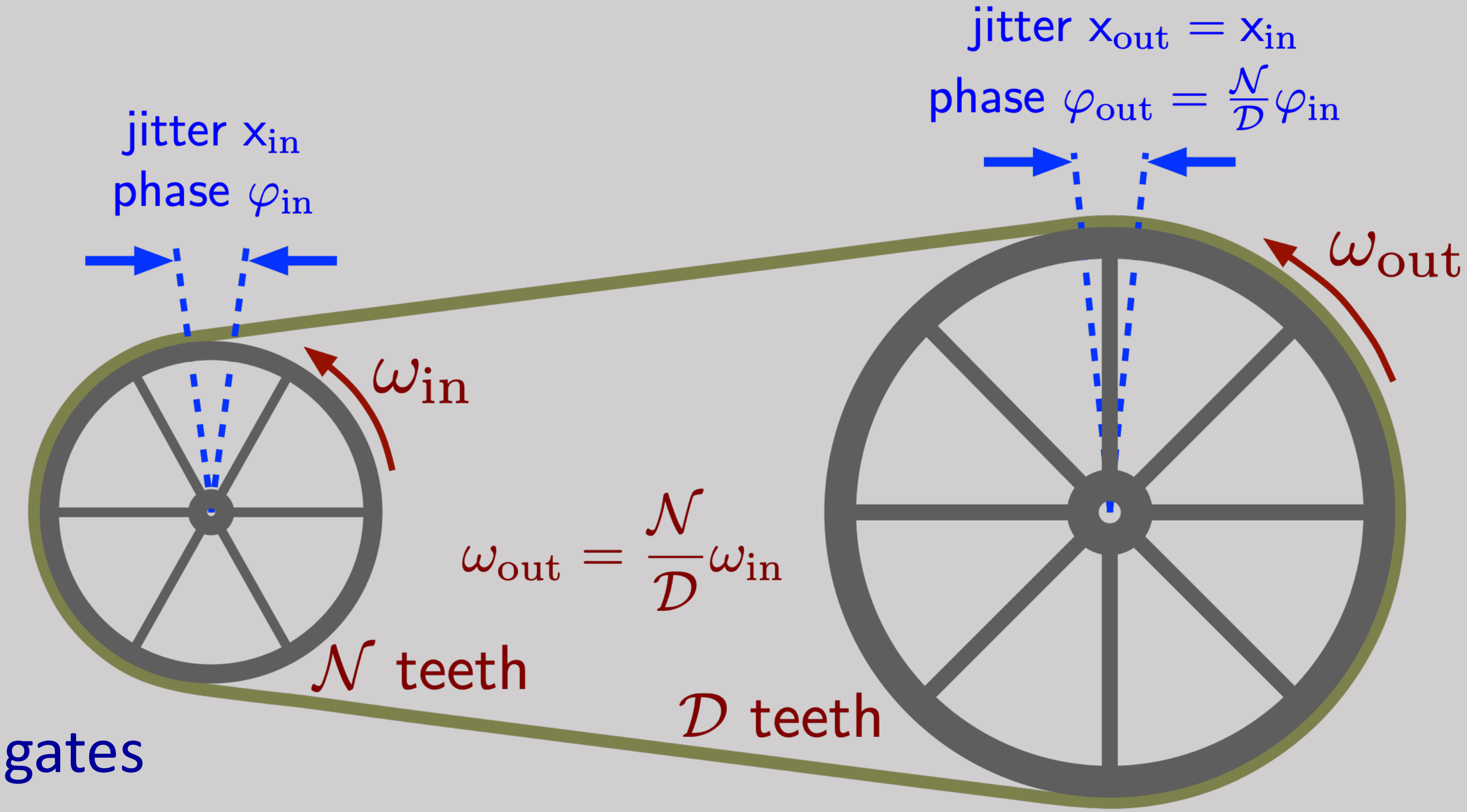
Belongs to Part 2
(lectures 6–10)



AD9854 data sheet, Fig. 1

The Noise-Free Synthesizer

Belongs to Part 2
(lectures 6–10)

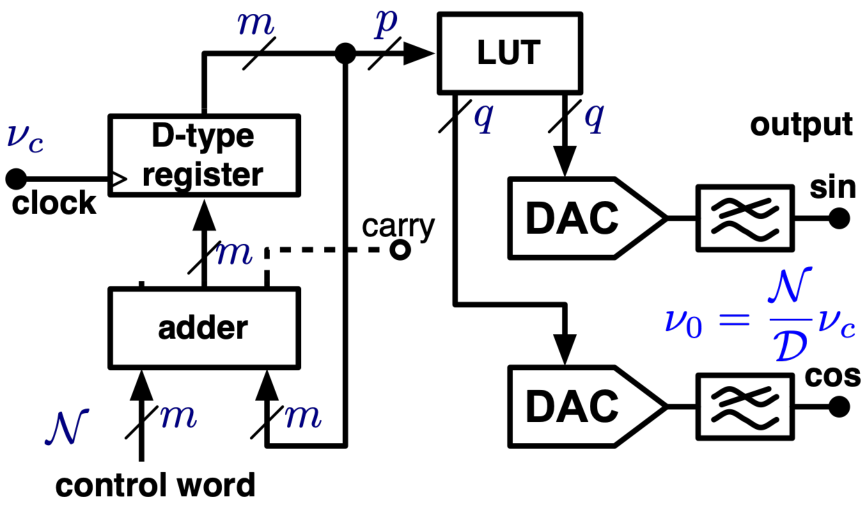


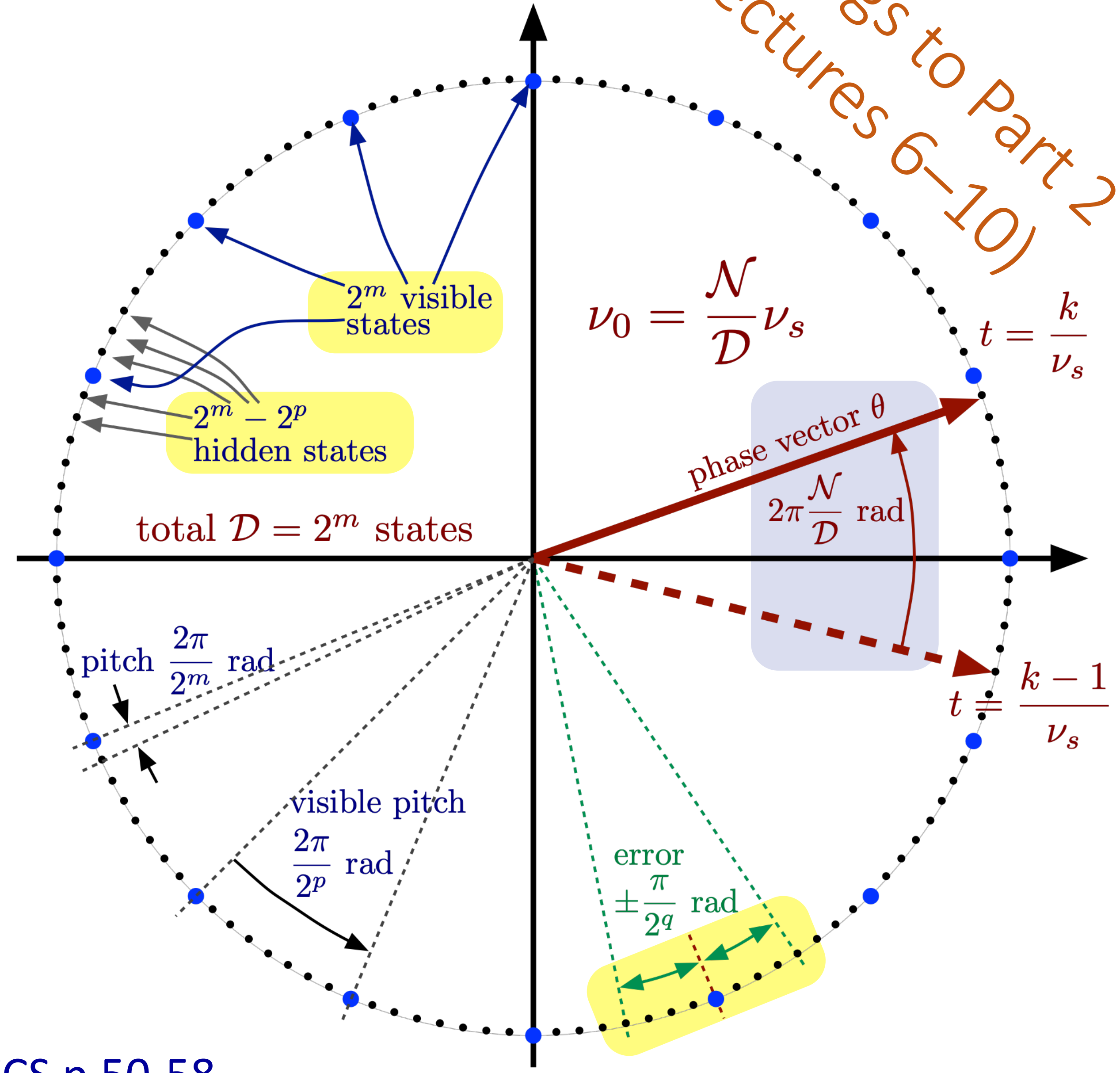
- The noise-free synthesizer propagates the jitter x (phase time)
- So, it scales the phase ϕ as N/D,
- and the phase spectrum Sϕ as (N/D)2
- Notice the absence of sampling

State-Variable Truncation

Belongs to Part 2 (lectures 6–10)

$$n_k = (n_{k-1} + \mathcal{N}) \bmod \mathcal{D}, \quad \mathcal{D} = 2^m$$

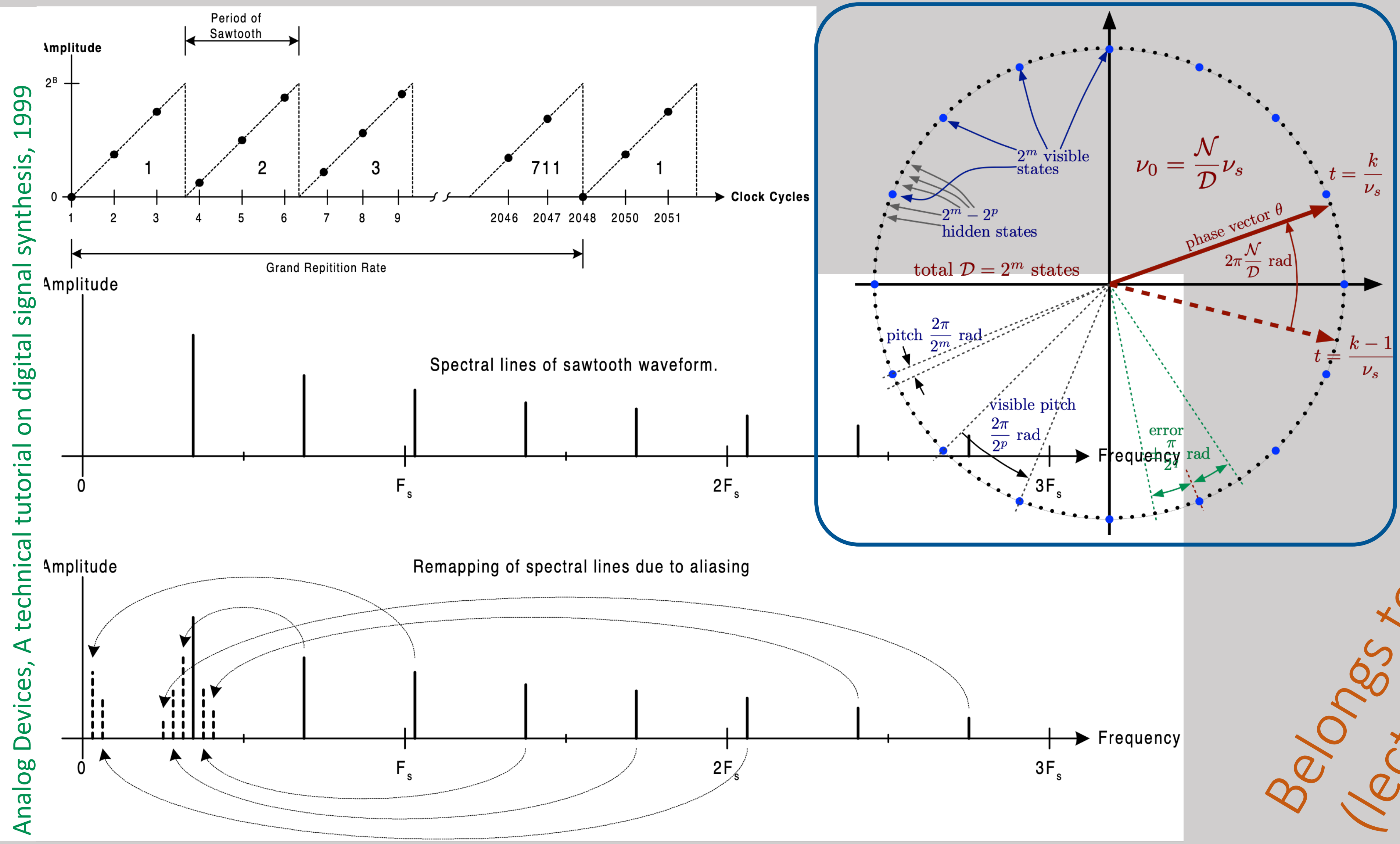




- Only quantization shows up with full m-bit conversion
- Technology –> q max
- Why p > q
- Slow pseudorandom beat
  48 bit –> 3d 6h 11m 15s @ 1 GHz

Spurs: Torosyan A, Wilson AN jr, Proc 2005 IFCS p.50-58

Truncation Generates Spurs

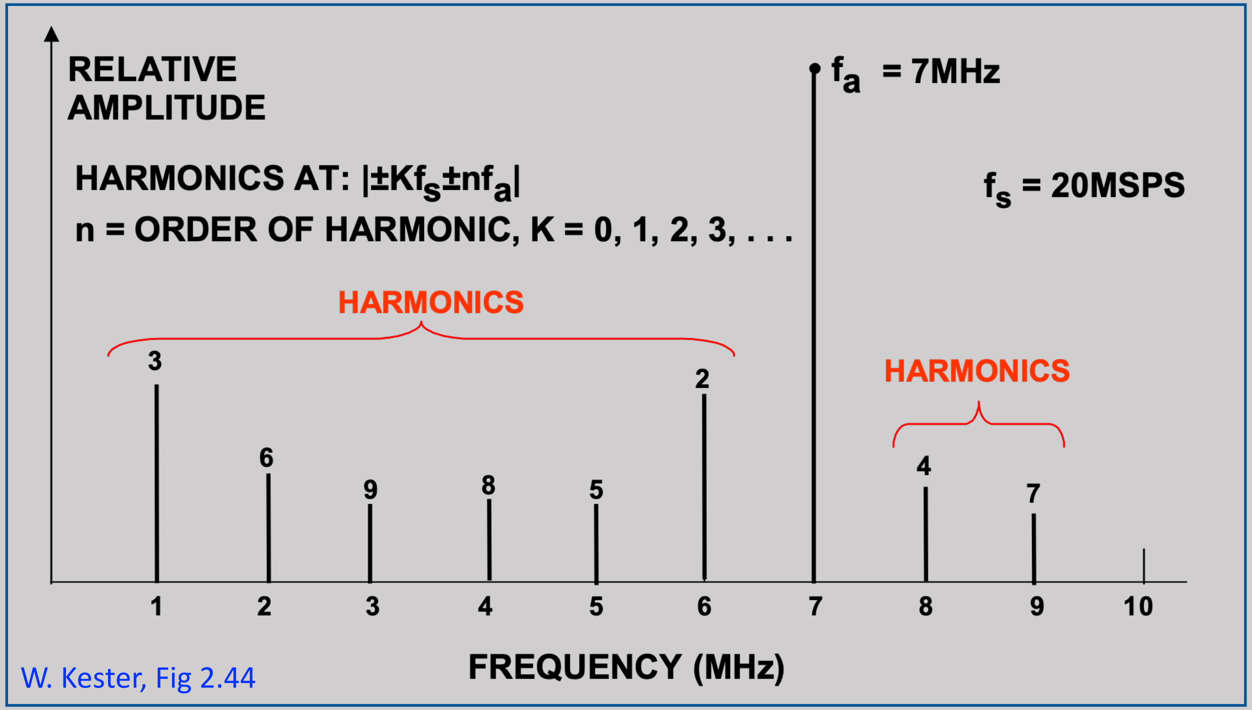


Analog Devices, A technical tutorial on digital signal synthesis, 1999

Belongs to Part 2 (lectures 6–10)

The power of spurs comes at expenses of white noise – yet not as one-to-one

Distortion and Aliasing



W. Kester, Fig 2.44

Sampling f_s = 20 MHz
Nyquist f_N = 10 MHz
Output f_a = 7 MHz

2nd. 2 x 7 = 14, mirror to 10 —> 6
3rd. 3 x 7 = 21, take away 20 —> 1
4th. 4 x 7 = 28, take away 20 —> 8
5th. 5 x 7 = 35, take away 20 —> 15, mirror to 10 —> 5
6th. 6 x 7 = 42, take away 40 —> 2
7th. 7 x 7 = 49, take away 40 —> 9
8th. 8 x 7 = 56, take away 40 —> 16, mirror to 10 —> 4
9th. 9 x 7 = 63, take away 60 —> 3

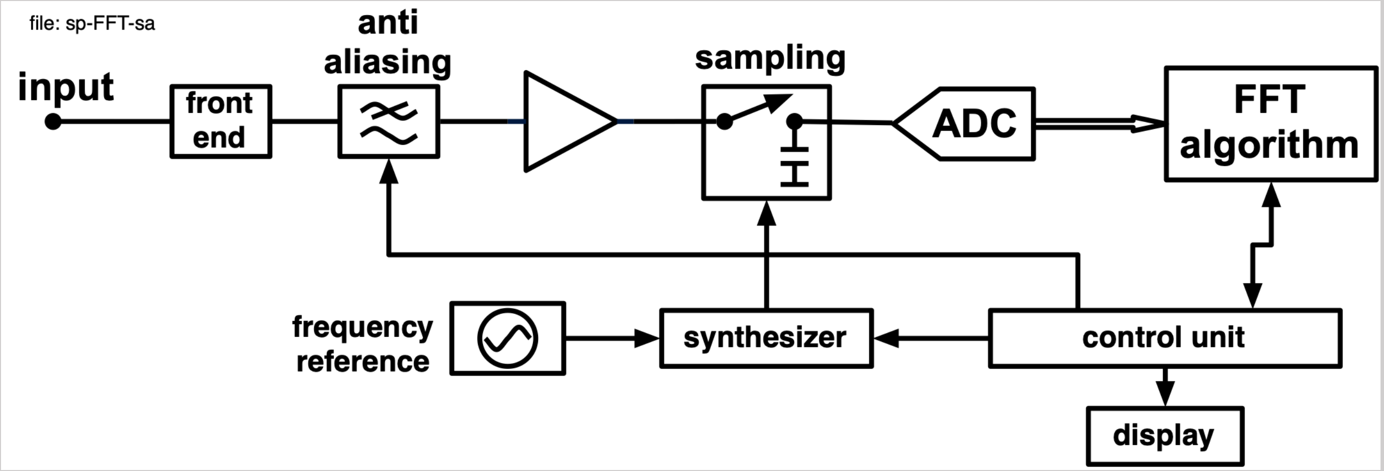
W. Kester ed., Analog-Digital Conversion, Analog Devices 2004, ISBN 0-916550-27-3

Belongs to Part 2
(lectures 6–10)

Spectrum Analyzers

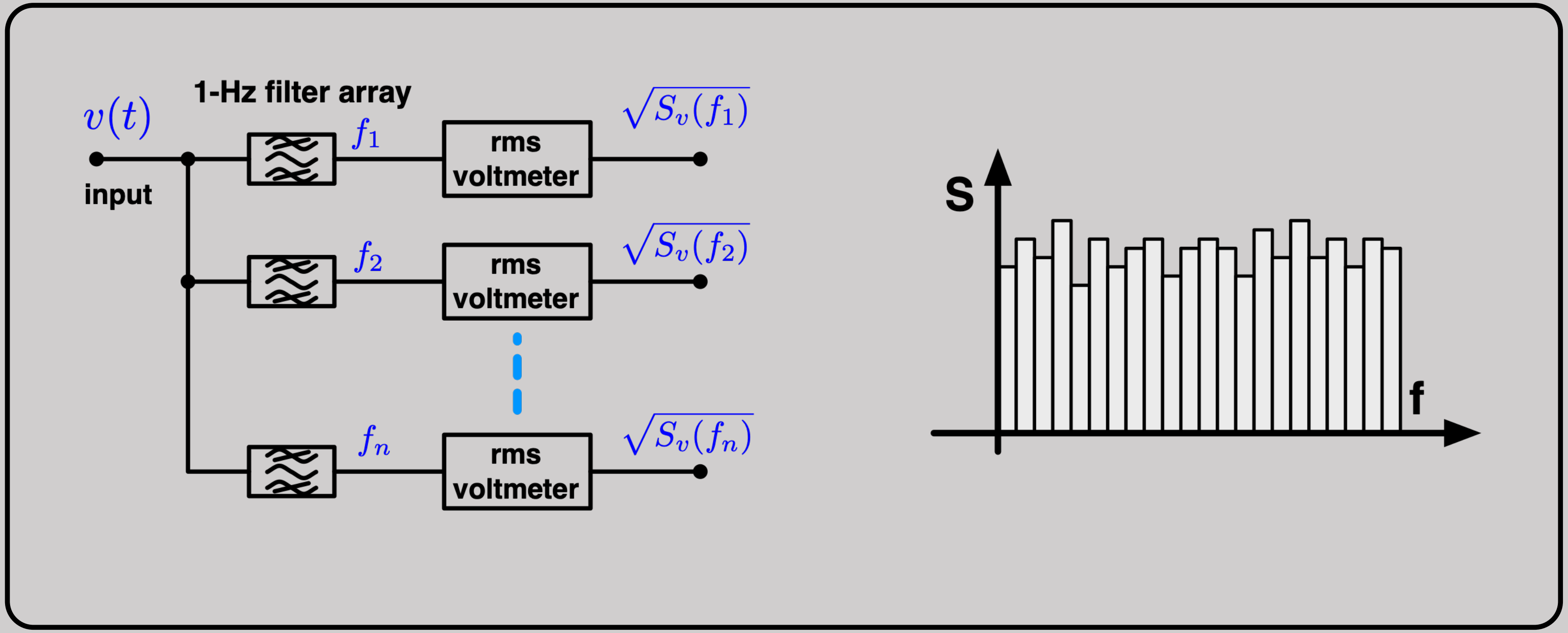
Excerpt from 03 Power Spectra

FFT spectrum analyzer



- Direct digitization of the input signal
- Fully digital process
- Practical limit $f \leq 0.4\ f_s$
- Tough tradeoff between resolution and max frequency

Parallel spectrum analyzer



Rice representation

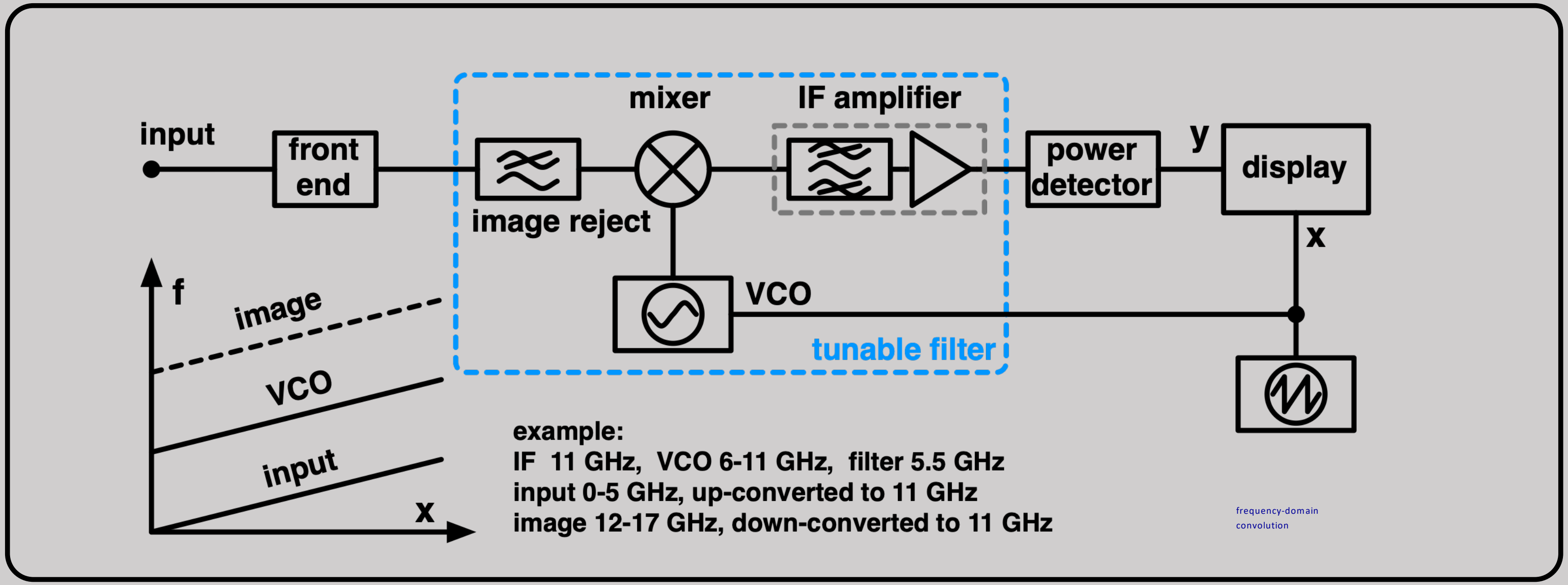
Integration over a finite time

$$x(t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\omega_0 t) - b_n(t)\sin(n\omega_0 t)$$

$$S_x(n\omega_0) = \left[a_n^2 + b_n^2\right]/\omega_0$$

ω_0 is the analysis bandwidth

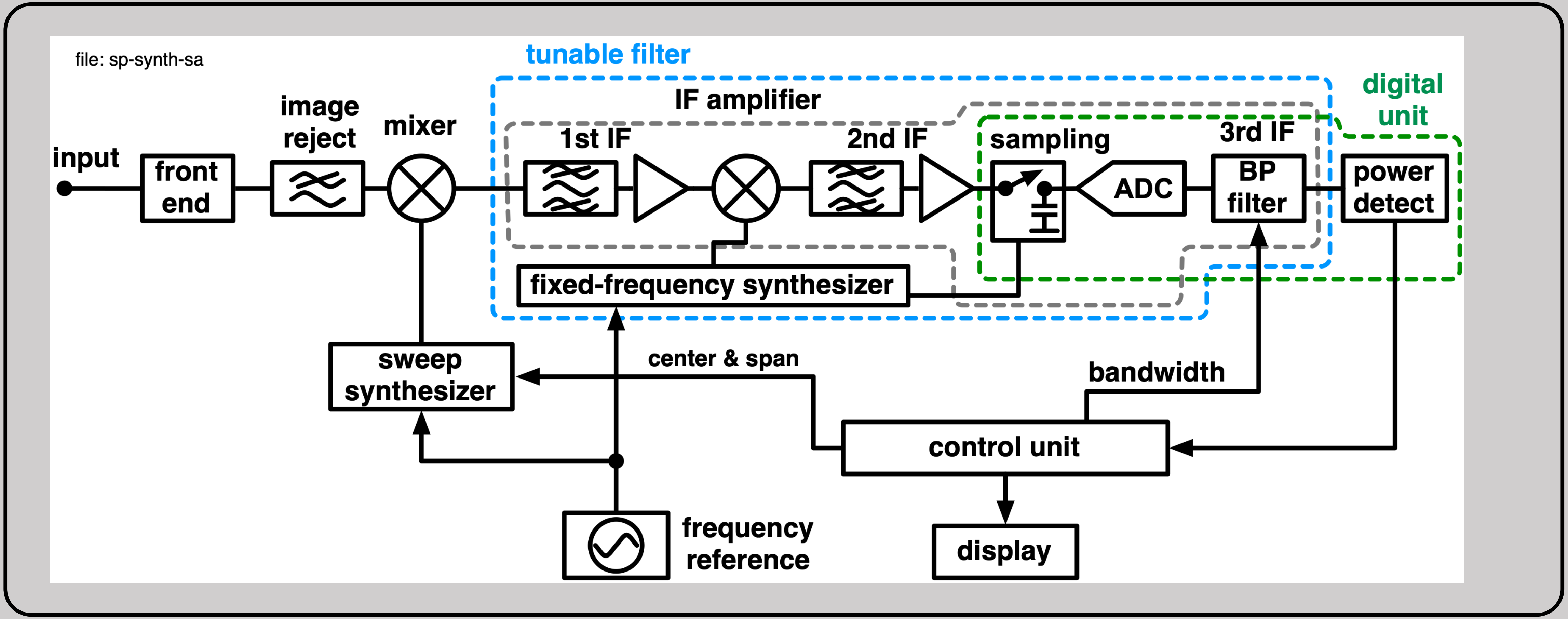
Scanning spectrum analyzer



example:
IF 11 GHz, VCO 6-11 GHz, filter 5.5 GHz
input 0-5 GHz, up-converted to 11 GHz
image 12-17 GHz, down-converted to 11 GHz

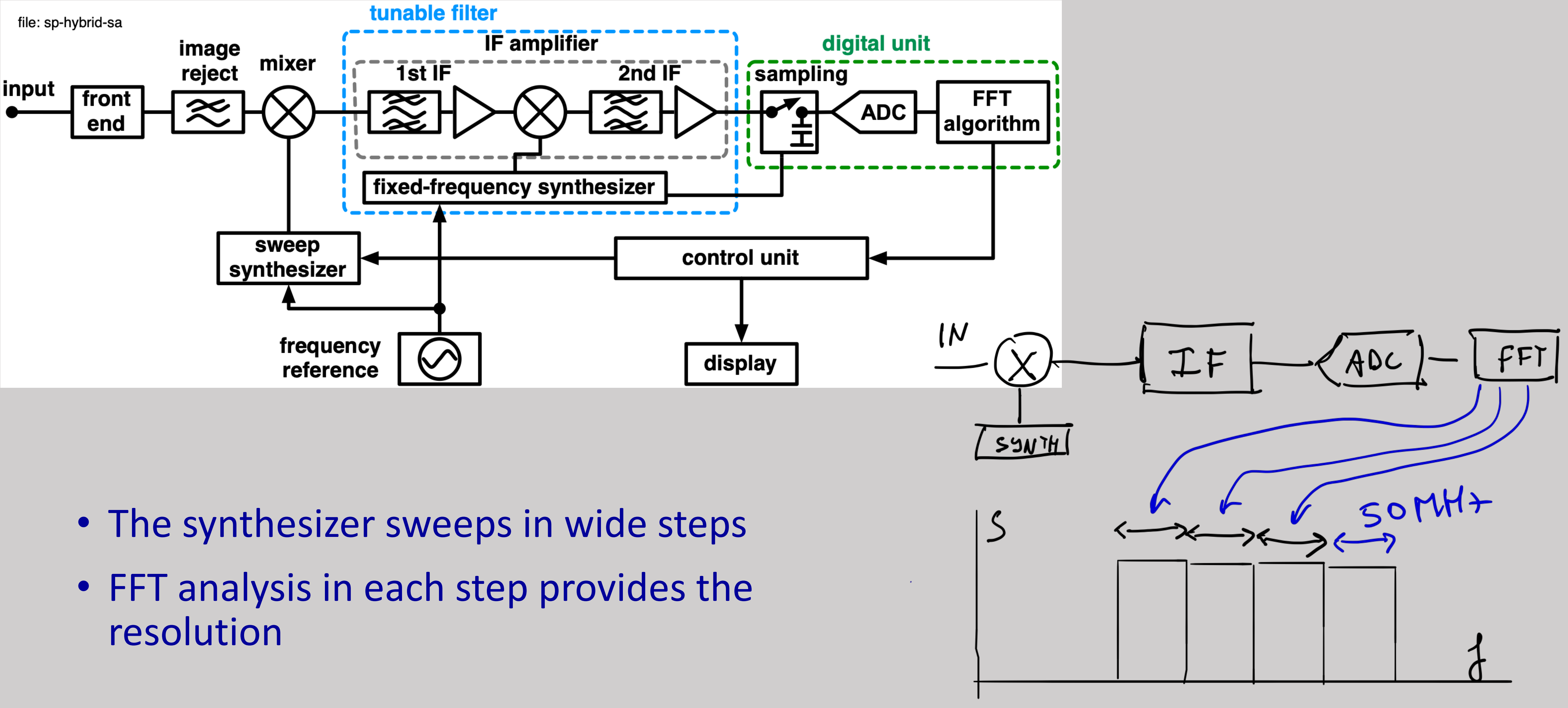
- RF/microwaves
  - The one and only option until the late 1990s
  - Progressively replaced with the hybrid analyzer
- Optics
  - Cannot use IF
  - Analog VCO — tunable laser

Synthesized spectrum analyzer



- The VCO is replaced with a synthesizer
- Otherwise, similar to the scanning SA

Hybrid FFT spectrum analyzer



FFTs taken sequentially and joined

- The synthesizer sweeps in wide steps
- FFT analysis in each step provides the resolution

Lecture 4 ends here

Three years of war

Modern Europe is a paradise
of culture and human rights.

Let’s keep it united, safe and free!

Lecture 5

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Spectrum analyzer
- Lock-in amplifiers and boxcar average
- Frequency-to-digital and time-to-digital converters

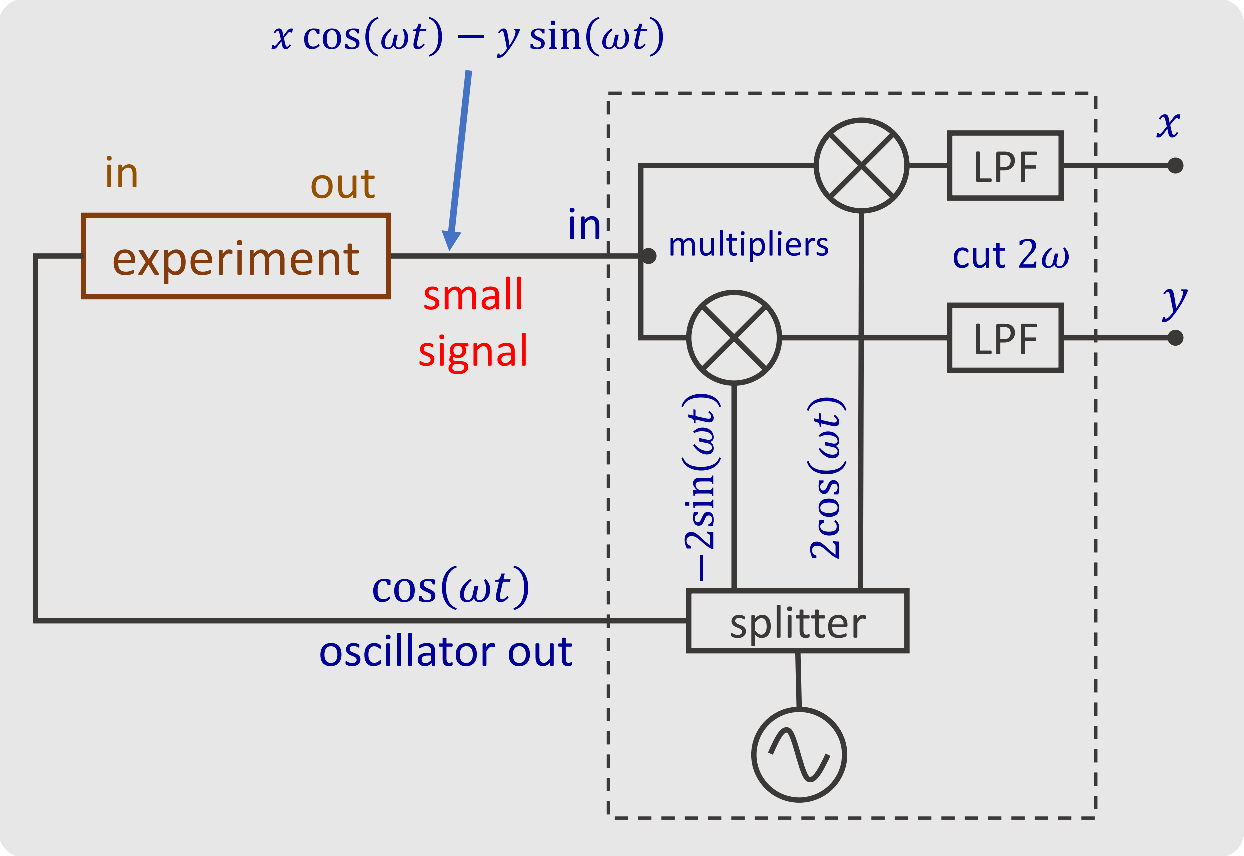
ORCID 0000-0002-5364-1835

home page http://rubiola.org

Lock–in Amplifier

Lock-in Amplifier – main ideas

1. Very small signal
   1. Can be detected if you have the reference
2. AC measurement:
   - Get out of the DC, drift and flicker
3. Differential measurement
   - Oscillator is common mode
   - Fluctuations rejected
4. Transposed filter solves
   - Narrow bandwidth
   - Shape
   - Stability of center frequency and bandwidth

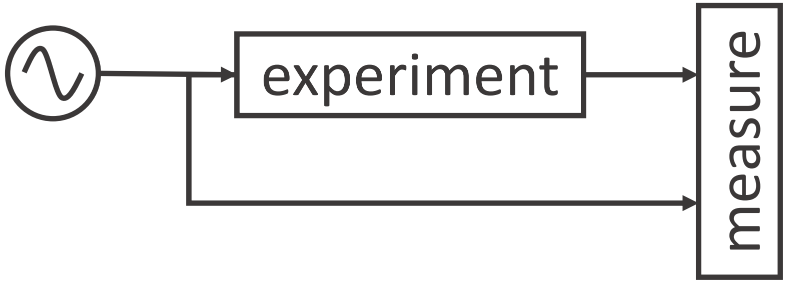


Narrowband $x(t)$ and $y(t)$

$$\{[x(t)\cos(\omega t) - y(t)\sin(\omega t)]\times 2\cos(\omega_t)\} * \mathrm{LPF} = x(t)$$

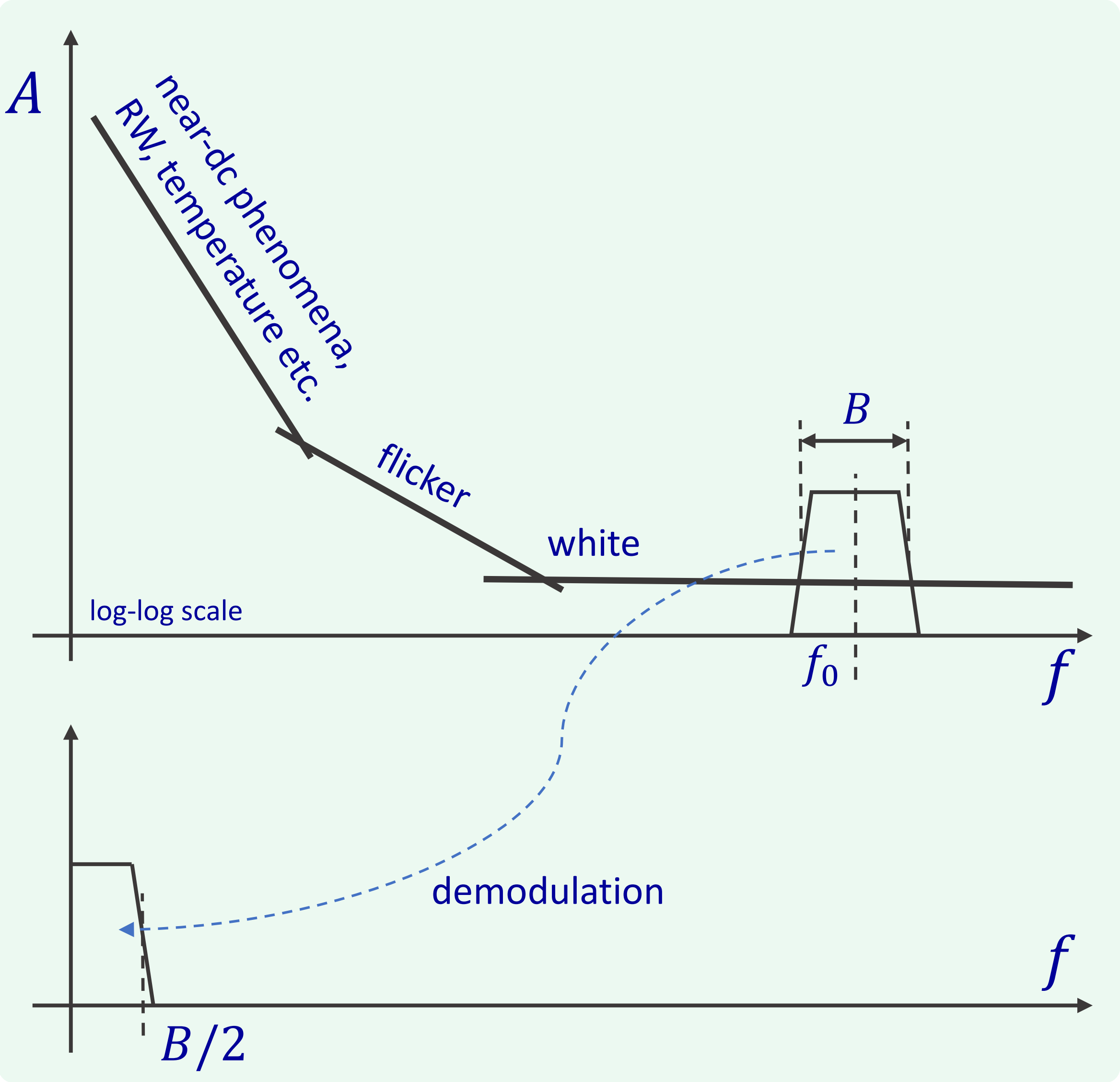
$$\{[x(t)\cos(\omega t) - y(t)\sin(\omega t)]\times[-2\sin(\omega_t)]\} * \mathrm{LPF} = y(t)$$

Synchronous detection

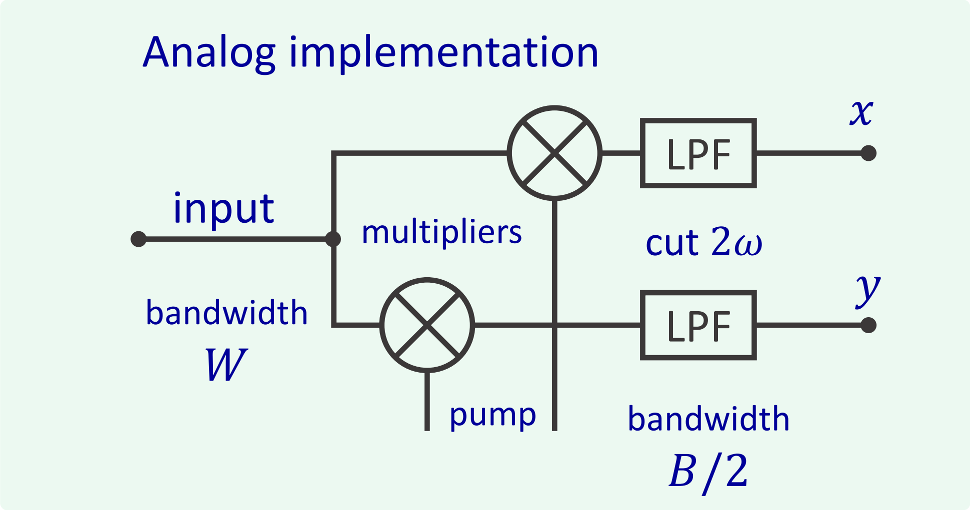


Physical property

- Transparence
- Attenuation
- Resonance
- Molecular absorption
- Capacitance
- Resistance
- etc.

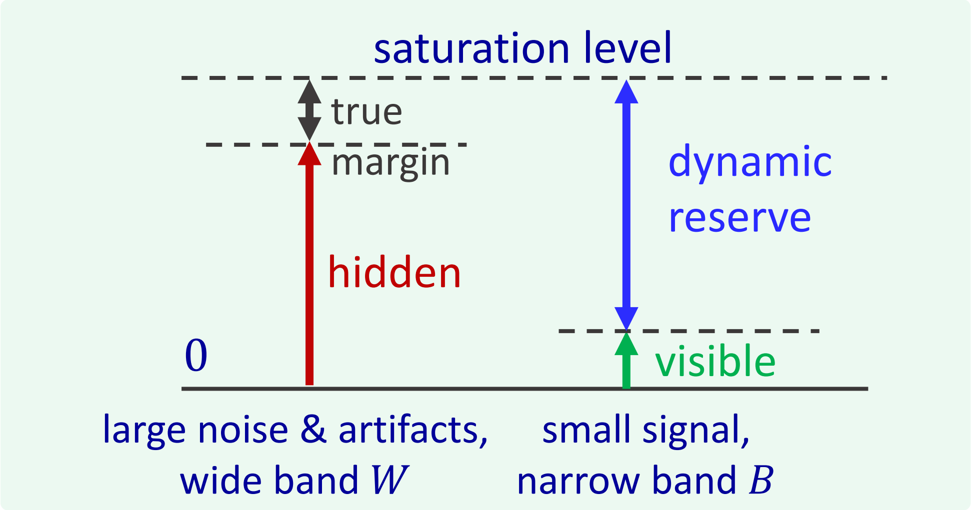


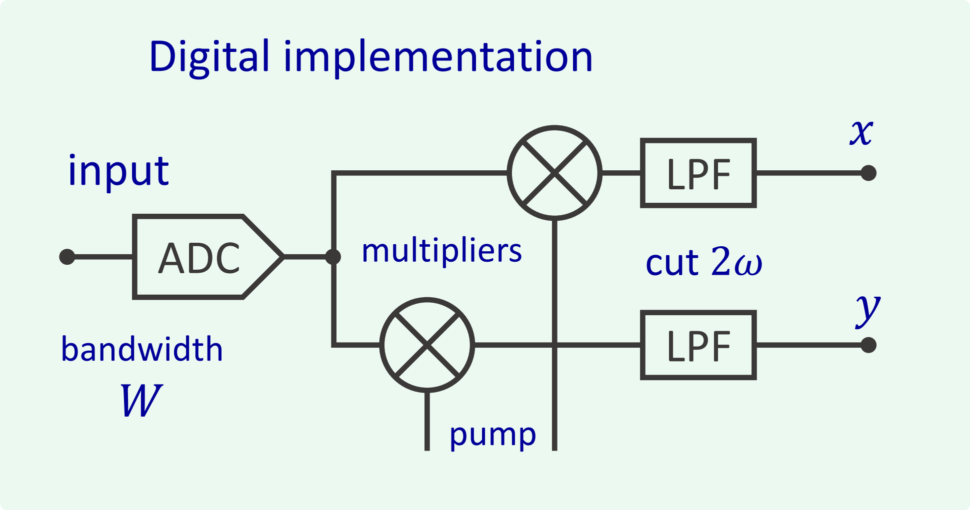
Dynamic reserve



problem

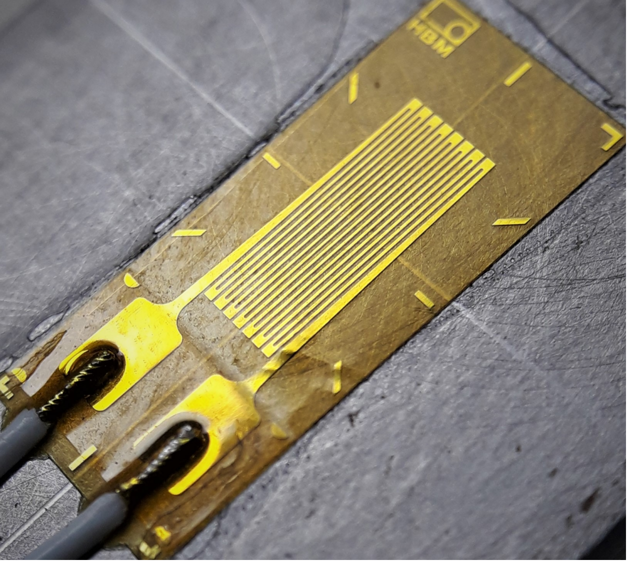
$W \gg B$



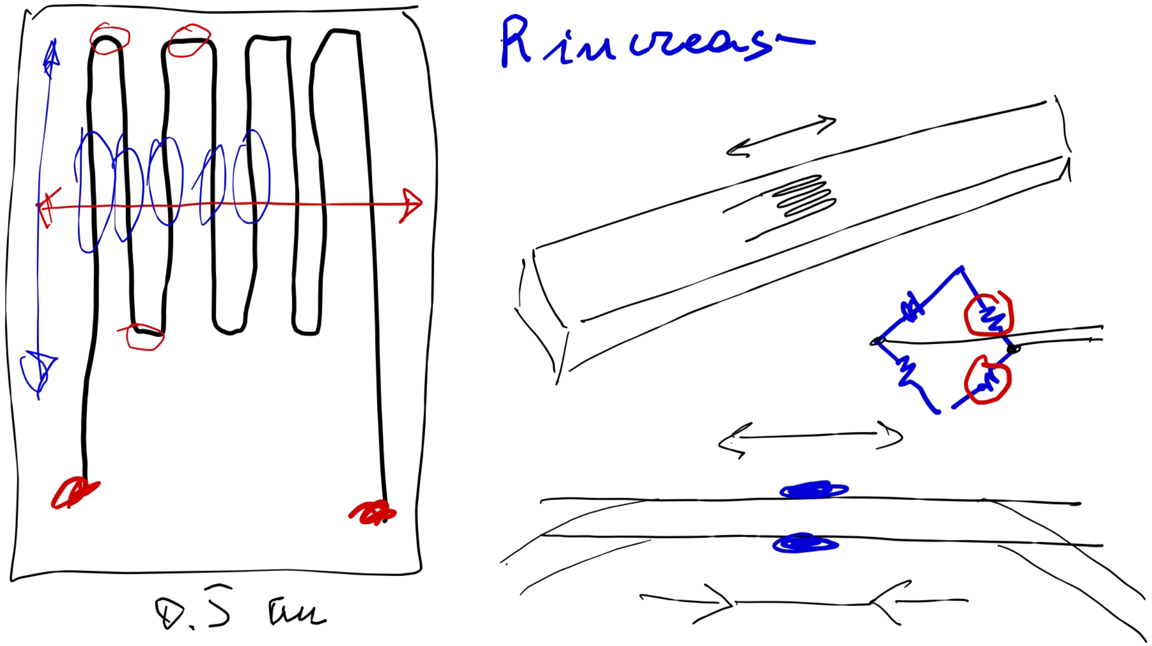


- Analog implementation
  - Multiplier or double-balanced mixer
    - Saturation
  - Passive filters difficult to design
  - Active filters easier to shape, but noisy
- Digital implementation
  - Saturation of the ADC
  - The low-pass filters integrate the signal in its time constant –> Numerical overflow

Example – Strain gauge



Wikimedia, CC-BY-4.0 Cristian V, 2017

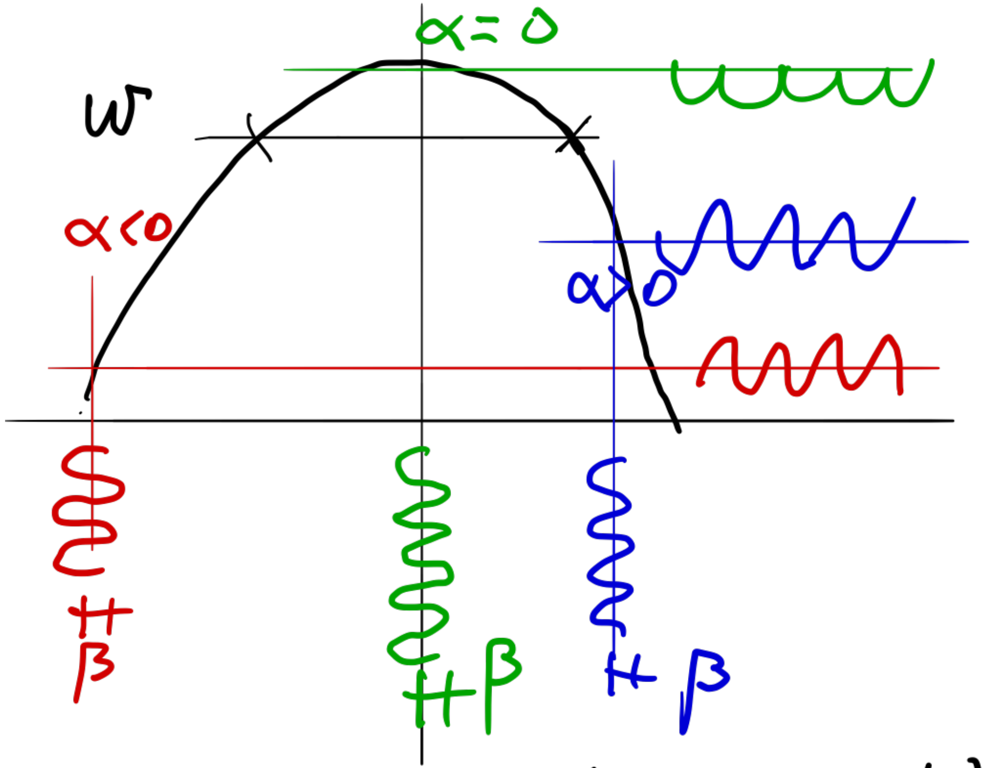


Tricks

- Thermal coefficient dR/RdT matches the material under test
  - Specific strain gauges for steel, concrete, Aluminum, etc.
  - Typical 1 ppm/K residual coefficient
- Beware of the glue
- Two-sensor symmetry doubles the gain and improves the stability
- Wheatstone bridge is magic
- 4-wires connection minimizes the effect of cable resistance
- Virtues of 600 Hz probe
  - multiple of 50 Hz and 60 Hz (EU/USA)
  - Notch filter cancels the pollution from power grid

Application – Spectroscopy

$$y = 1 - x^2/W$$



α signal (freq. offset)

β modulat. index

W 3 dB width

Add a picture with setup or block diagram

$$y = 1 - x^2/W \leftarrow x = \alpha + \beta\cos(\omega_m t)$$

$$y = 1 - \frac{1}{W}\left(\alpha + \beta\cos(\omega_m t)\right)^2$$

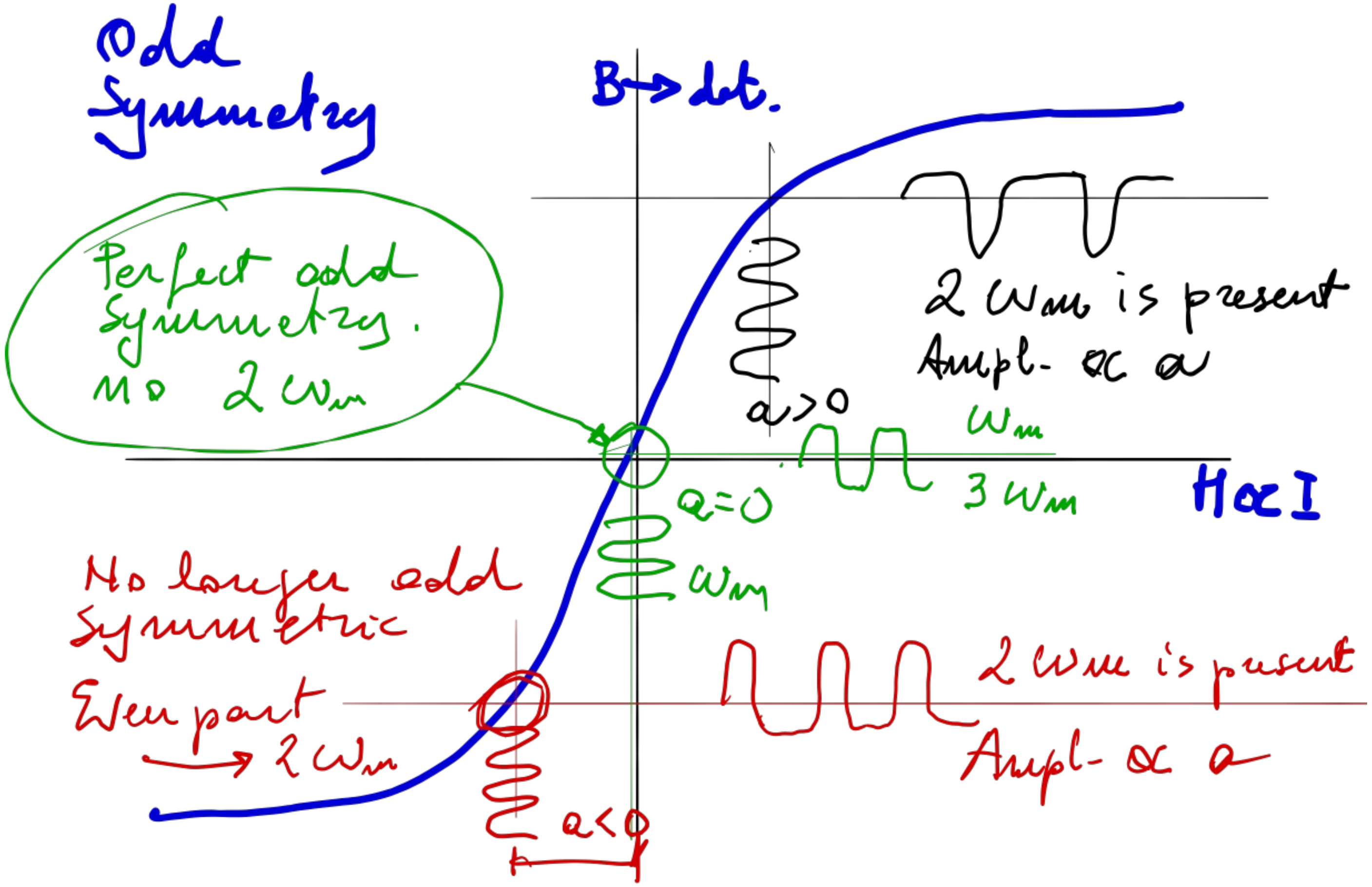
$$= 1 - \frac{1}{W}\left[\alpha^2 + 2\alpha\beta\cos(\omega_m t) + \beta^2\cos^2(\omega_m t)\right]$$

DC $\quad 1 - \frac{1}{W}\left(\alpha^2 + \frac{1}{2}\beta^2\right)$

ω_m $\quad -2\frac{\alpha\beta}{W}\cos(\omega_m t)$ $\quad$ Signal

$2\omega_m$ $\quad -\frac{1}{2}\frac{\beta}{W}\cos(2\omega_m t)$ $\quad$ Validation

Application – Magnetic field



Example:
Magnetic Sat.

Detect
$3\omega_m$

$$y = x^3 \leftarrow x = \alpha + \beta\cos(\omega t)$$

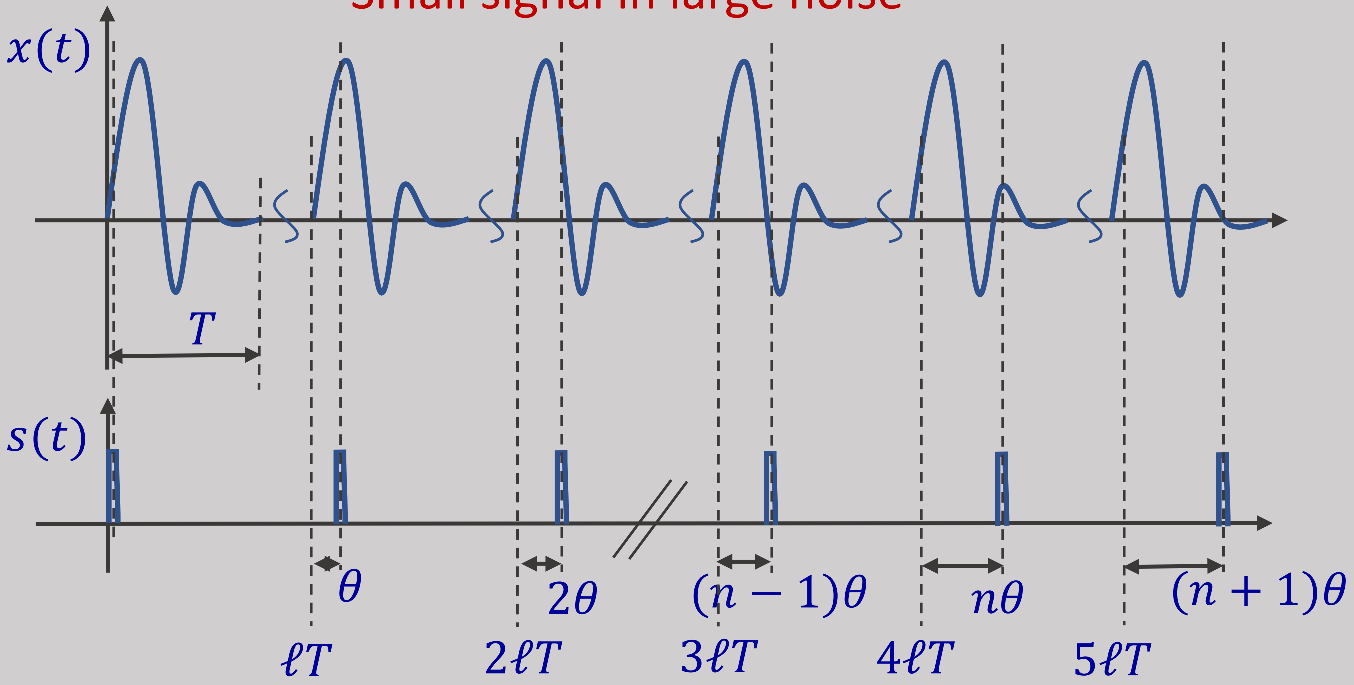
use $\cos^3(\alpha) = \frac{3}{4}\cos(\alpha) + \frac{1}{4}\cos(3\alpha)$

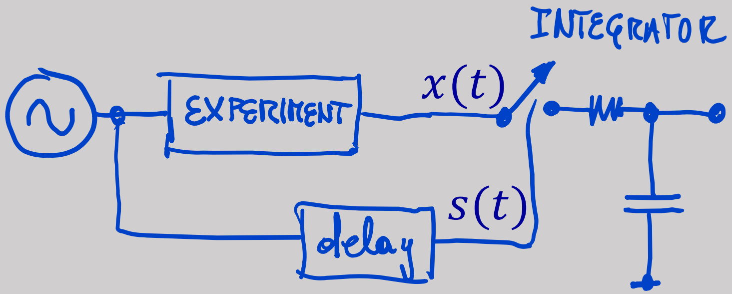
DC	$\alpha^3 + \frac{3}{2}\alpha\beta^2$
ω	$\frac{12\alpha^2 + 3\beta^2}{4}\cos(\omega t)$
2ω	$\frac{3}{2}\alpha\beta^2\cos(2\omega t)$ signal
3ω	$\frac{1}{4}\beta^3\cos(3\omega t)$ Valid.

Boxcar Averager

Boxcar Averager

Small signal in large noise





- Average on m samples for each $\tau = n\theta$, $n = 0 \dots N$
- Takes $N + 1$ integrators
- The integer ℓ is a technical delay

Analog boxcar

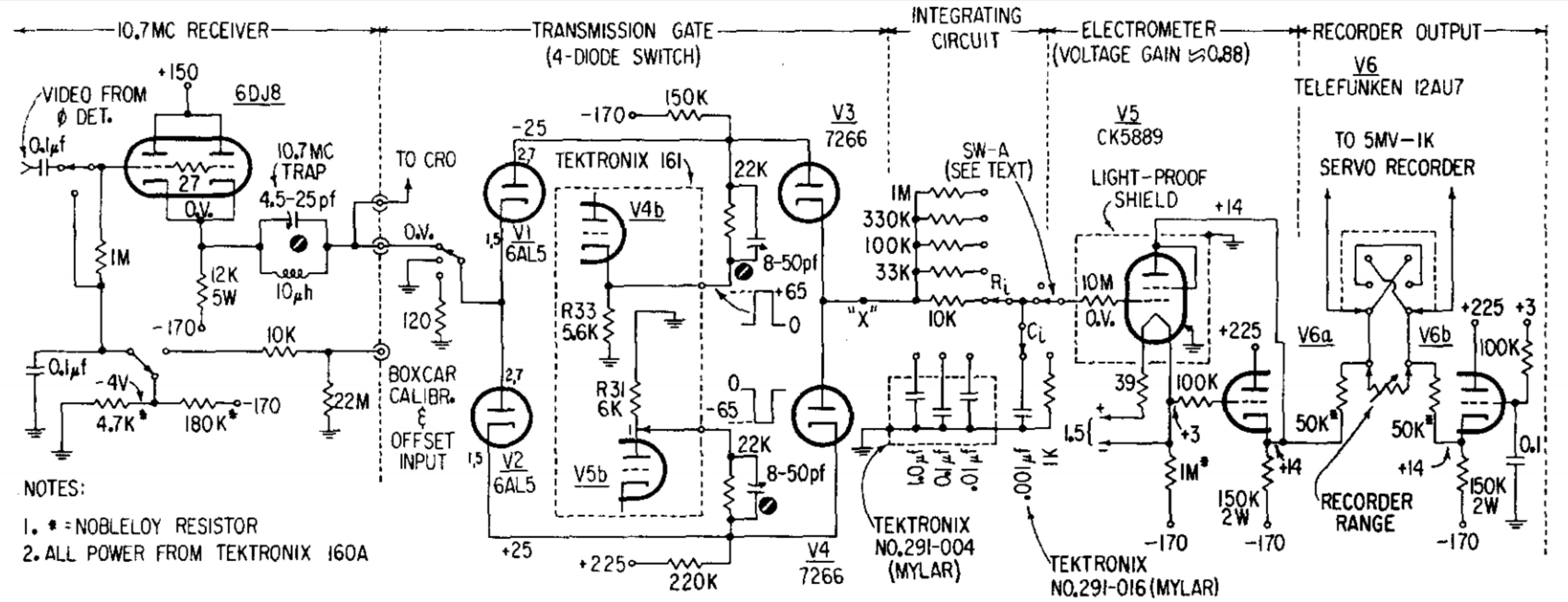
- Early 1950s
- Parallel –> multiple integrators
- Sequential –> one integrator, and slow recorder

Digital boxcar

- Fast electronics
- No need of delay, $\ell = 0$
- Needs large dynamic reserve
  - Use a fraction of ENoB
  - Integrator takes highe no of bits

A Sequential Boxcar in 1960

Blume, Fig. 1



NOTES:

1. * = NOBLELOY RESISTOR
2. ALL POWER FROM TEKTRONIX 160A

R. J. Blume, `Boxcar' Integrator with Long Holding Times, Rev Scient Instrum 32(9) p.1016, Sept 1961

High-Resolution Time-To-Digital & Frequency-To-Digital Converters

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Updated February 21, 2025

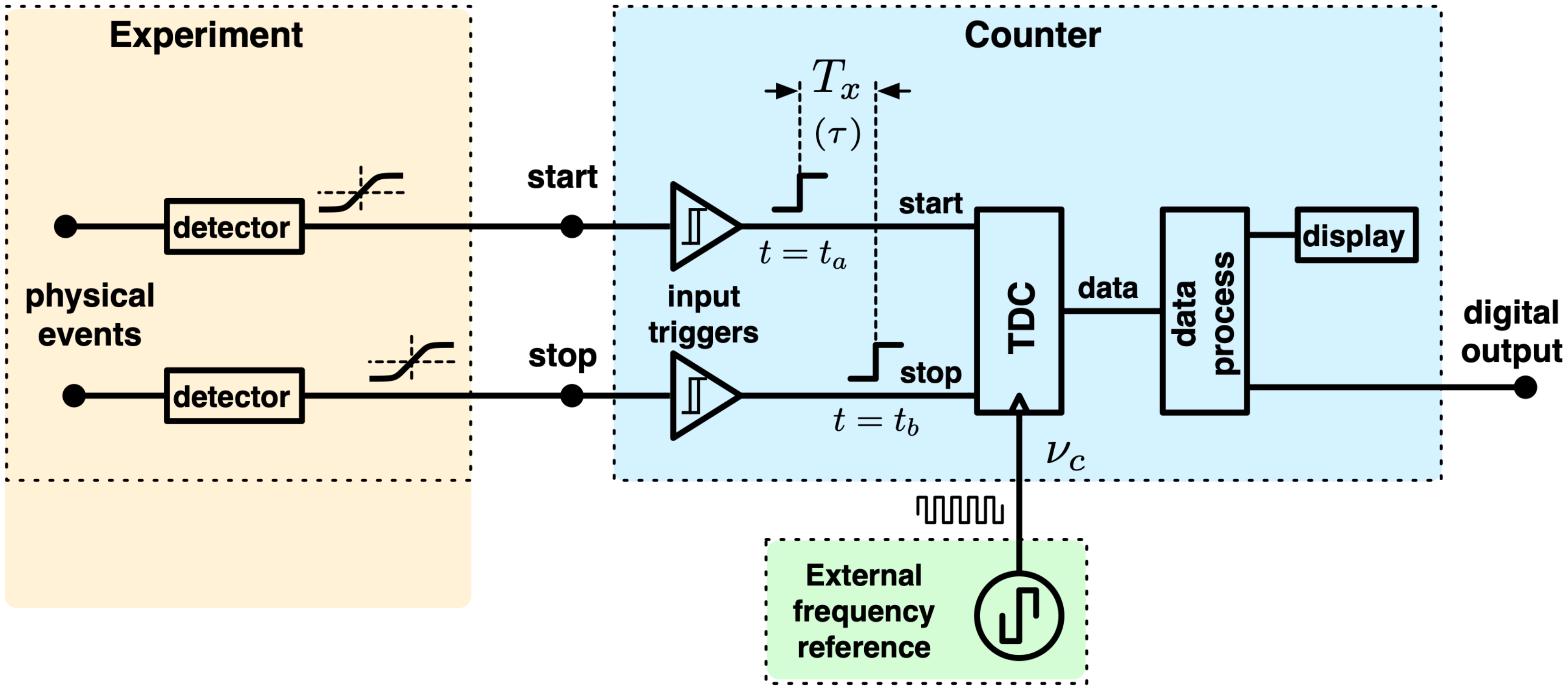
Outline

- Basic counters (RF & microwave)
- The input trigger
- Clock interpolation techniques
- Π, Λ and Ω counter, and statistics

home page http://rubiola.org

Main purposes

Frequency, Period or Time-Interval (TI)

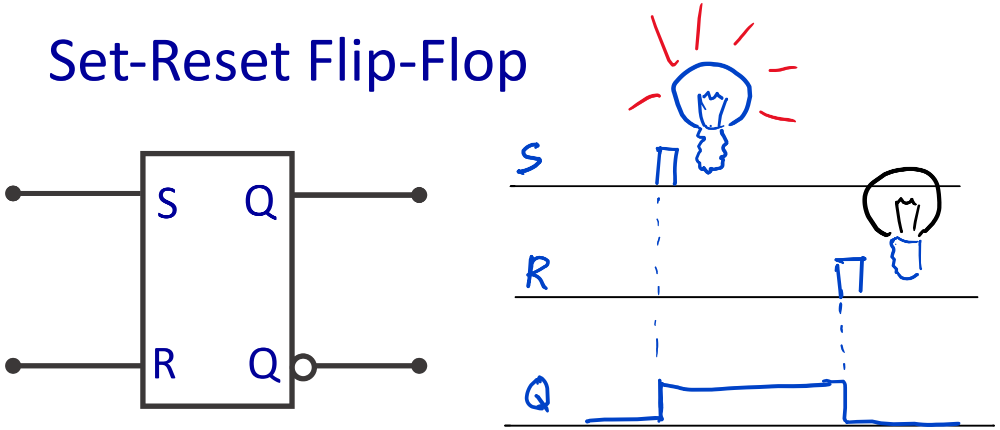


- Compare a physical quantity (frequency, period, time interval) to a frequency reference
- Exploit the full accuracy and precision of the reference, with no degradation

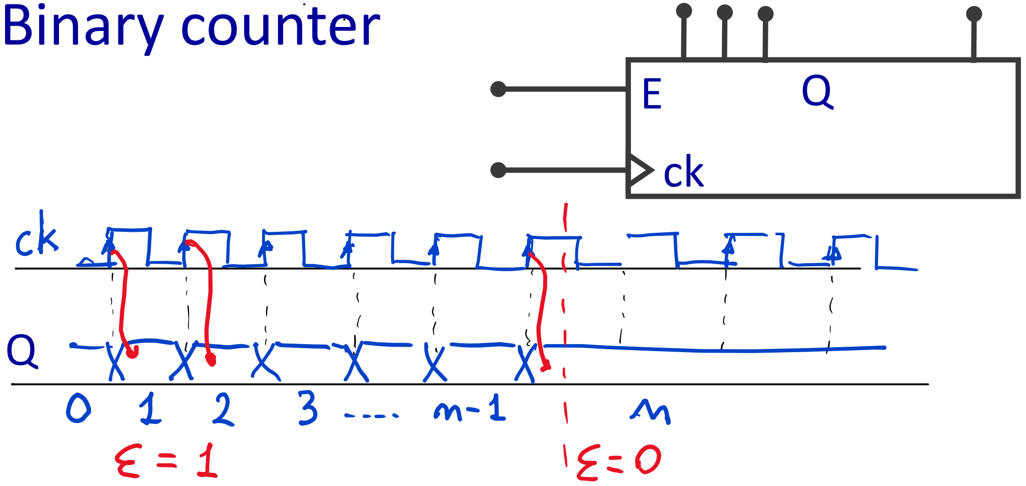
1 – Basic TDCs and FDCs

Digital hardware

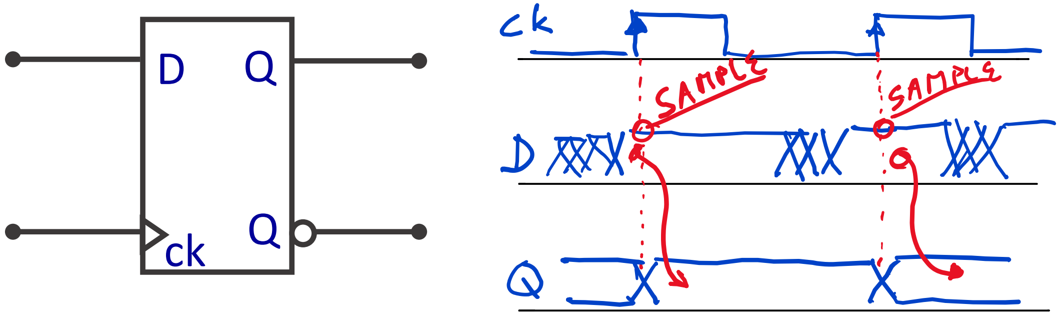
Set-Reset Flip-Flop



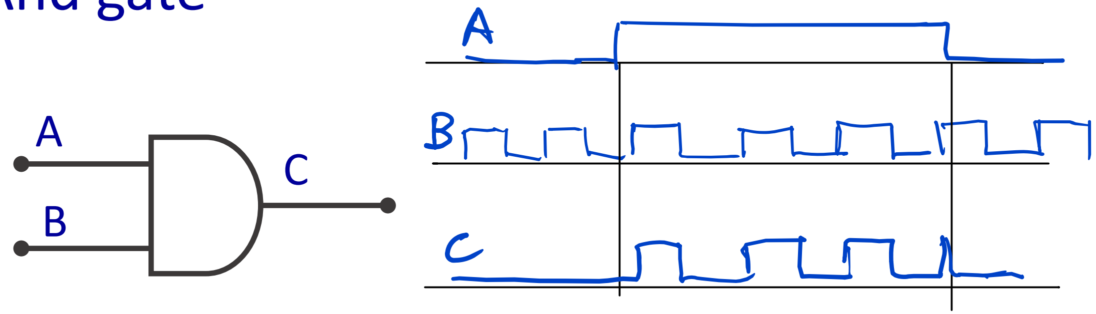
Binary counter



D-Type Flip-Flop (digital sampler)



And gate

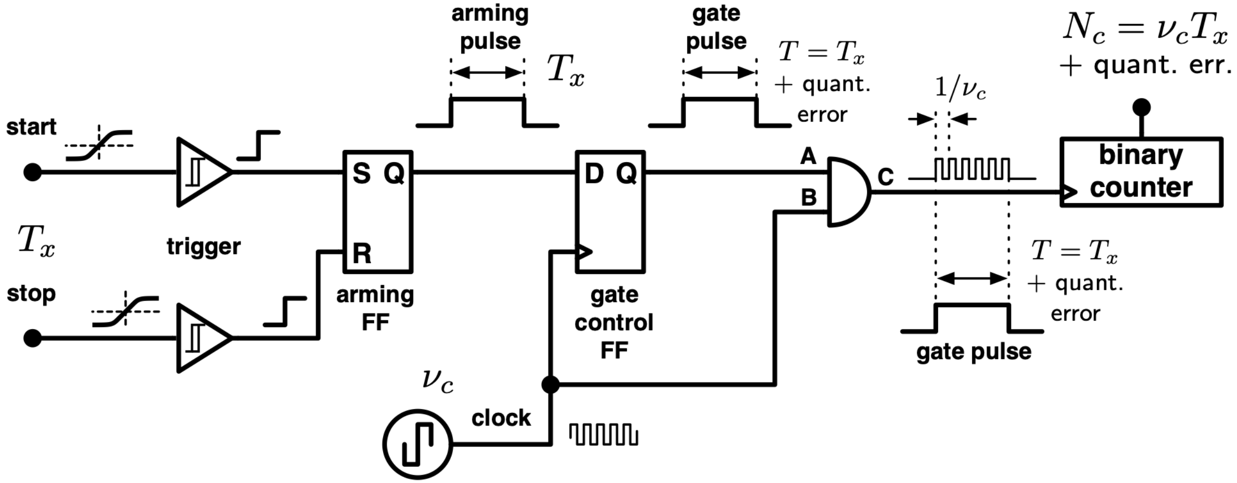


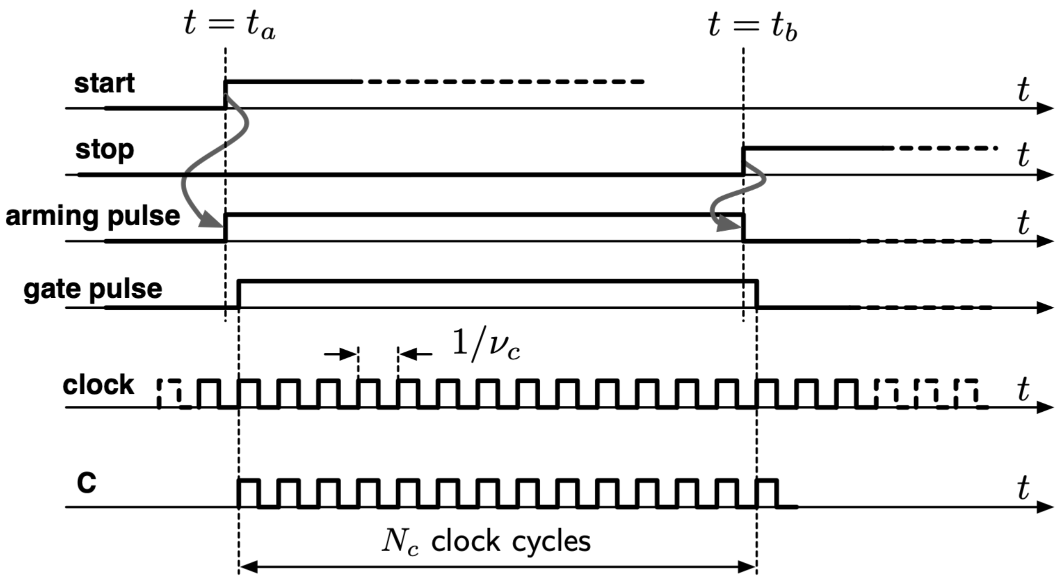
1 & 1 => 1

0 => 0

Time interval

©Timex

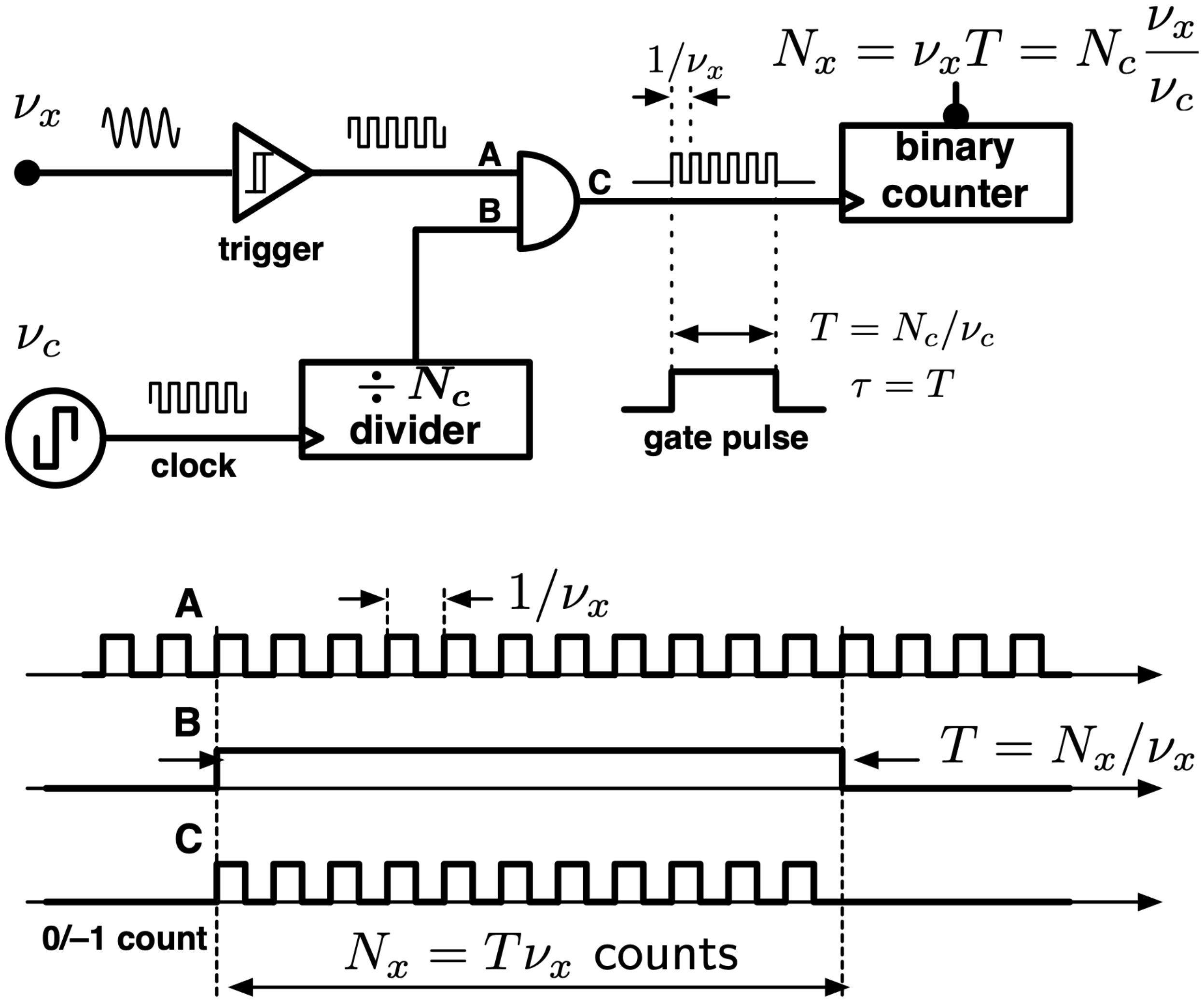




The gate control FF is a trick to synchronize the inputs to the clock

The resolution is set by the clock period $1/\nu_c$

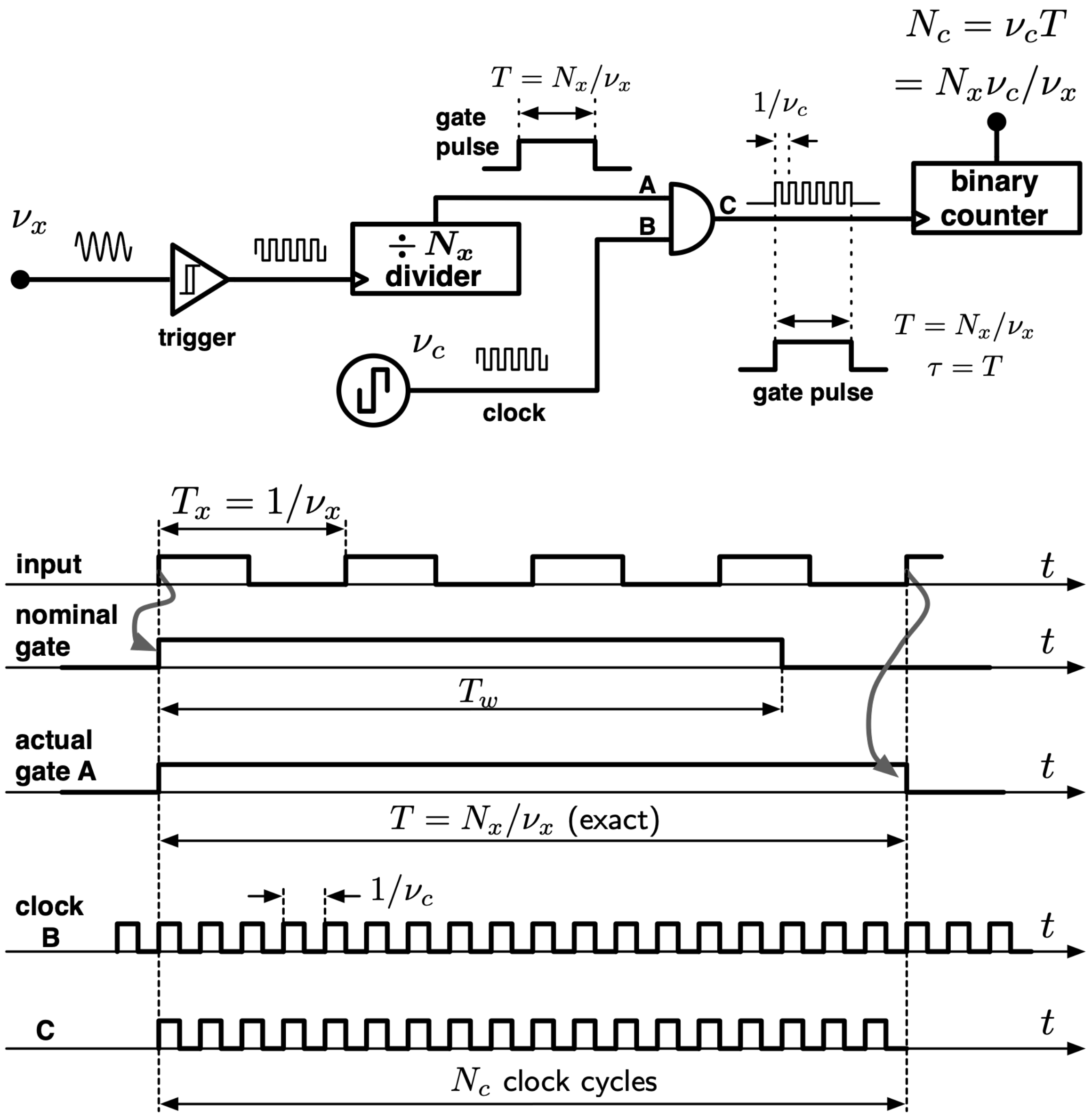
The (old) frequency counter



- Poor resolution, set by the input period $1/\nu x$
- Example, 50 Hz and 1 s measurement time gives 2% resolution

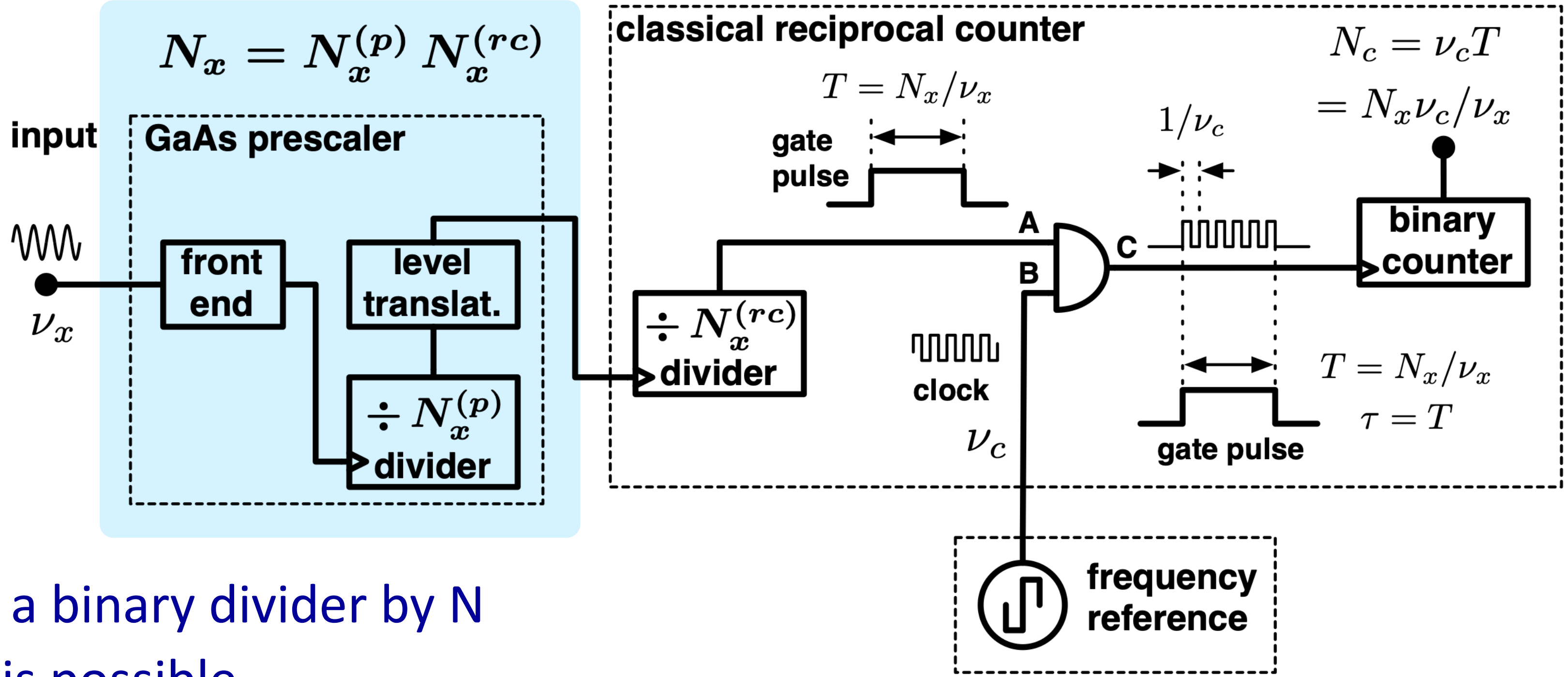
The gate-control FF is not shown

Classical reciprocal counter



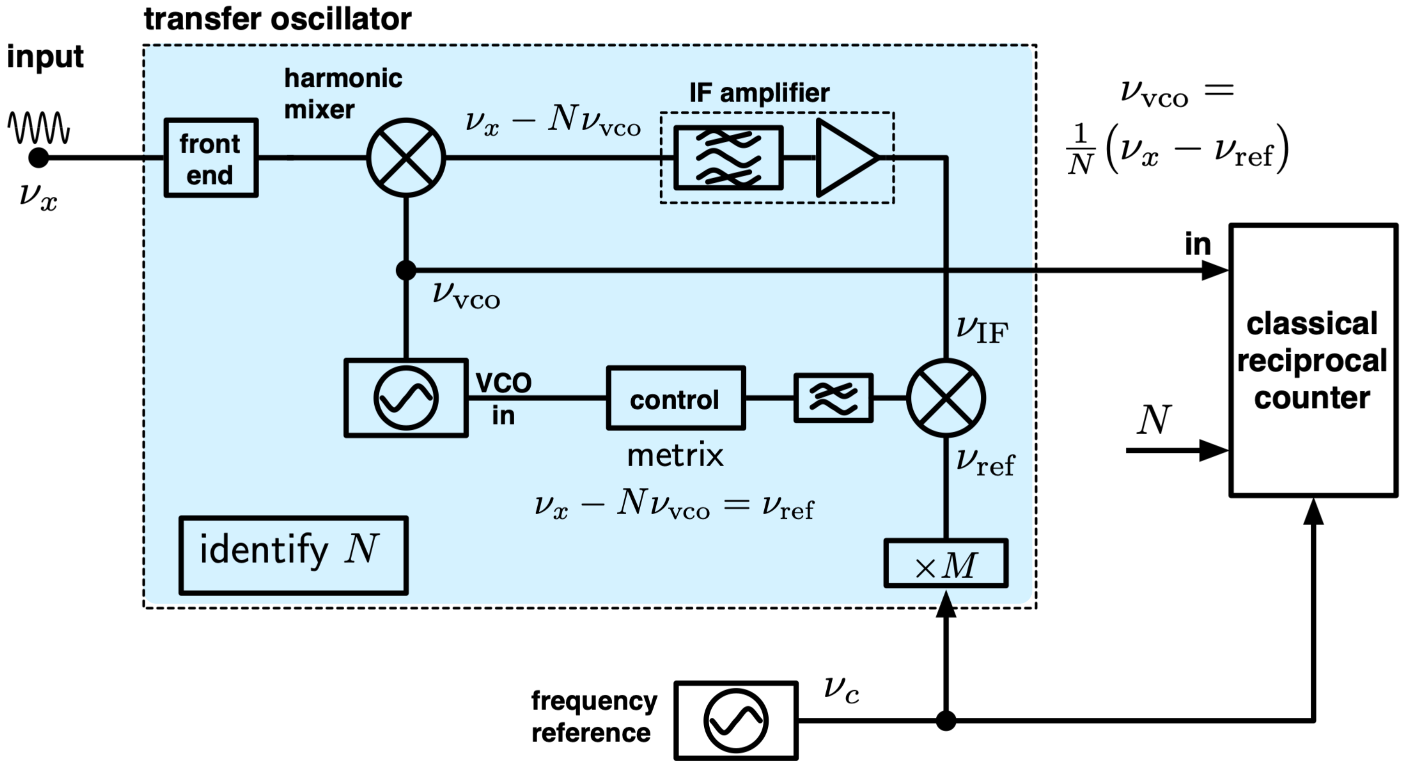
- Use the highest clock frequency permitted by the hardware
- The measurement time is a multiple of the input period
- Example, 50 Hz and 100 MHz clock
  - Resolution 10^{-7} with 100 ms measurement time
  - Measurement time can be …80-100-120… ms

Prescaler



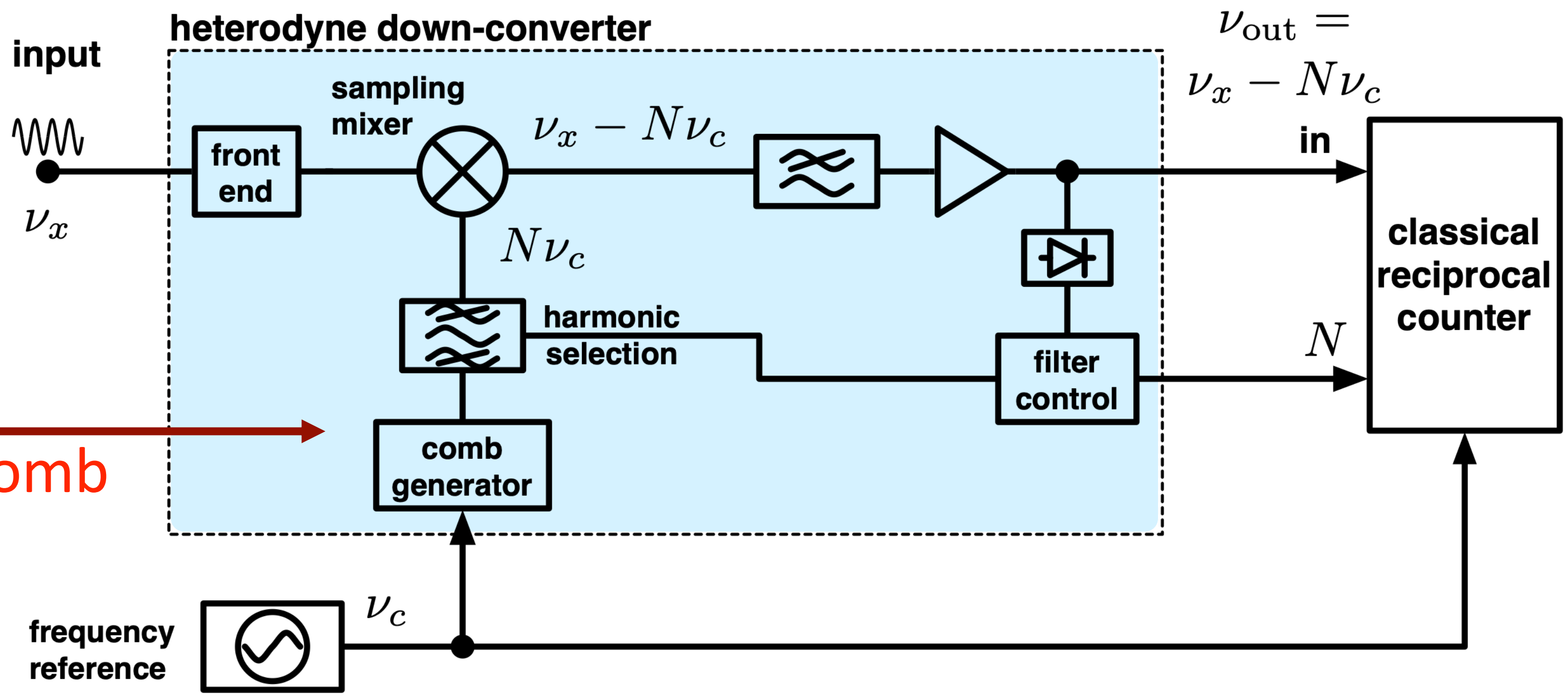
- The prescaler is a binary divider by N
  - Arbitrary N is possible
  - Decimal prescalers are gone
- Reciprocal counter => no resolution loss
- GaAs dividers work up to at least 40 GHz, (20 GHz current, 80 GHz special units)
- Most microwave counters use the prescaler

Transfer oscillator



- The transfer oscillator is a PLL used as a frequency divider
- Harmonics generation takes place inside the mixer
- Harmonics locking condition: $N\nu_{vco} = \nu_x$
- Frequency modulation Δf is used to identify N
- Rather complex scheme,
  $\times N \quad => \quad \Delta\nu \; N\Delta\nu$

Heterodyne counter



now,
FS Comb

- Down-conversion: $\nu_b = |\nu_x - N\,\nu_c\,|$
- ν_b is in the range of a classical counter
- Resolution enhancement because $\delta\nu_x = \delta\nu_b$, so $\dfrac{\delta\nu_x}{\nu_x} = \dfrac{\nu_b}{\nu_x}\,\dfrac{\delta\nu_b}{\nu_b}$
- Used only in some special cases

Example: laser frequency metrology

$\nu_x = 200$ THz (1550 nm)

$\nu_b = 20$ MHz (depends on the experiment)

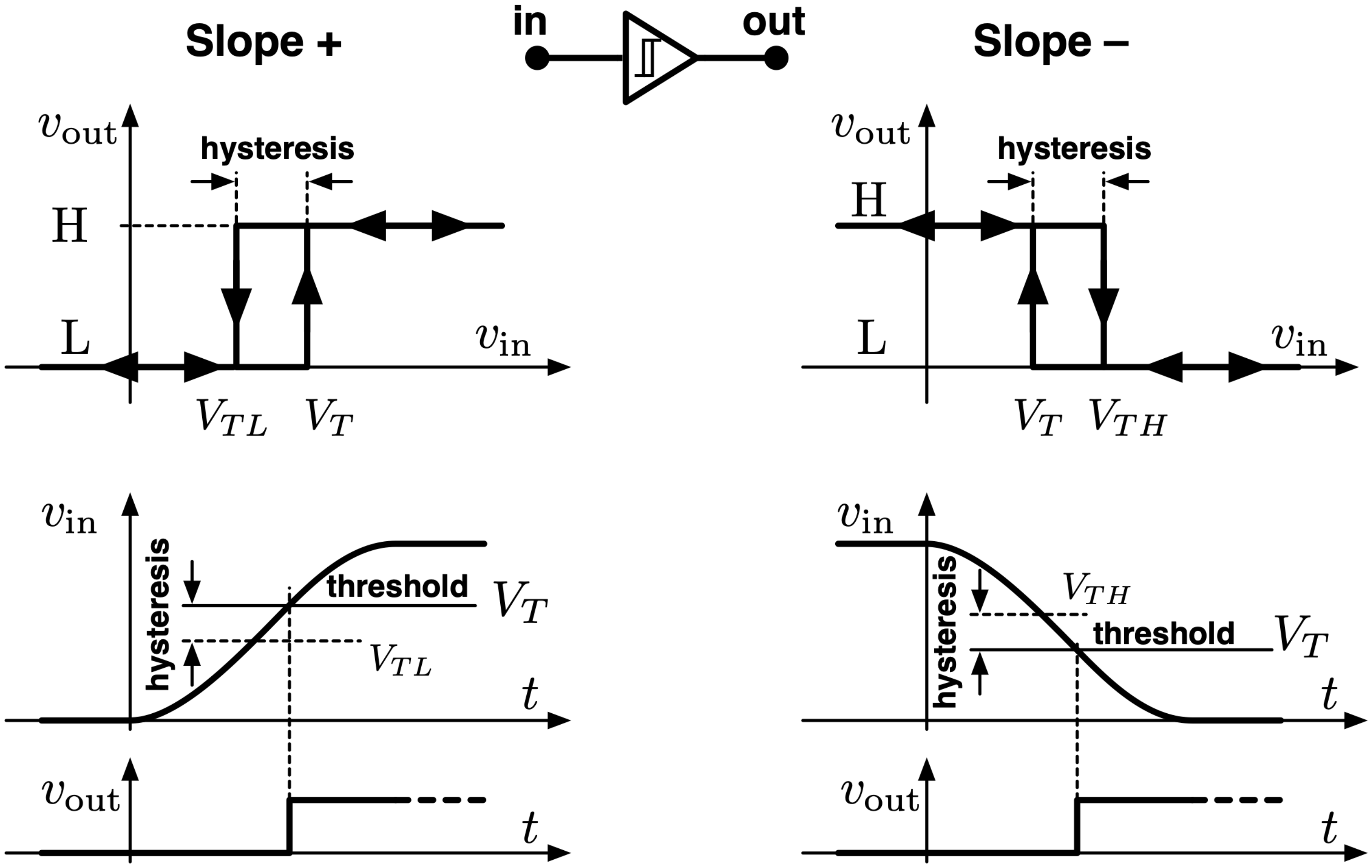
$\nu_b/\nu_x = 10^{-7}$ resolution enhancement

$\delta\nu_b/\nu_b = 10^{-8}$ (RF) gives

$\delta\nu_x/\nu_x = 10^{-15}$ (optics)

2 – Trigger

Trigger hysteresis



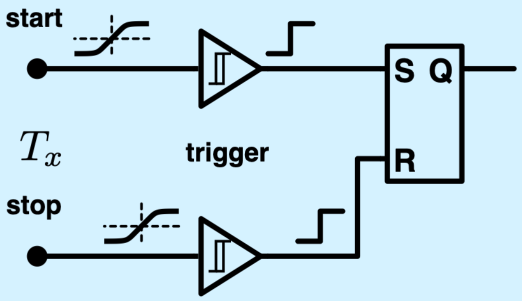
Hysteresis is necessary to avoid chatter in the presence of noise

Threshold fluctuation

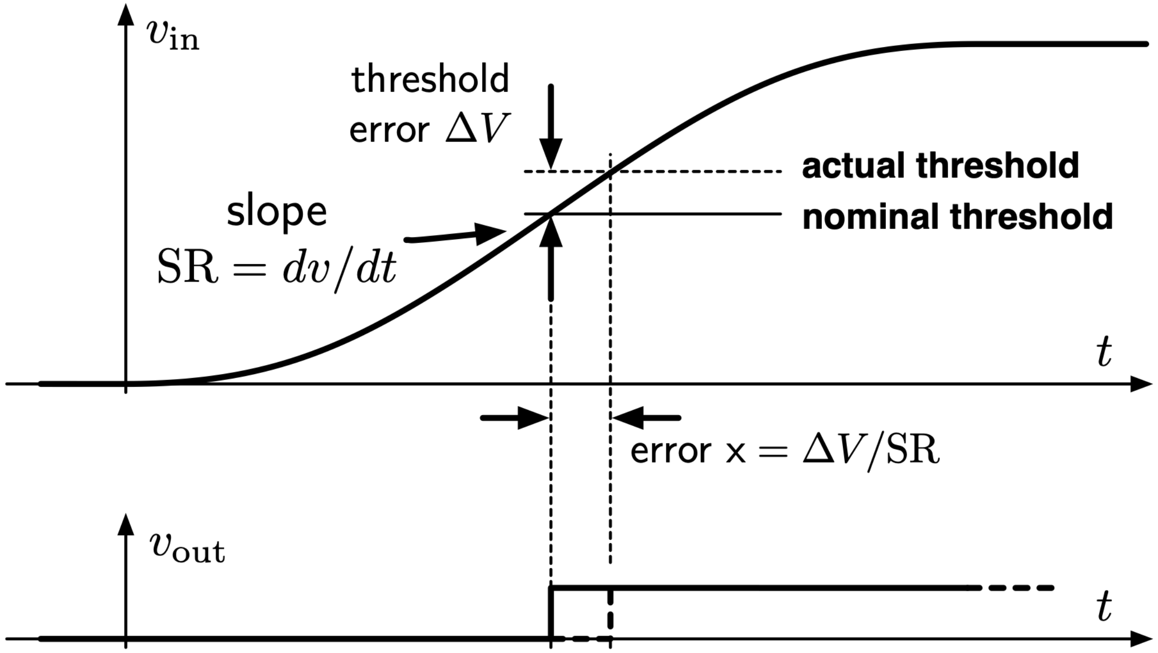
$$\text{'stop'} - \text{'start'}$$

systematic $$\mathsf{x} = \frac{(\Delta V)_b}{\mathrm{SR}_b} - \frac{(\Delta V)_a}{\mathrm{SR}_a}$$

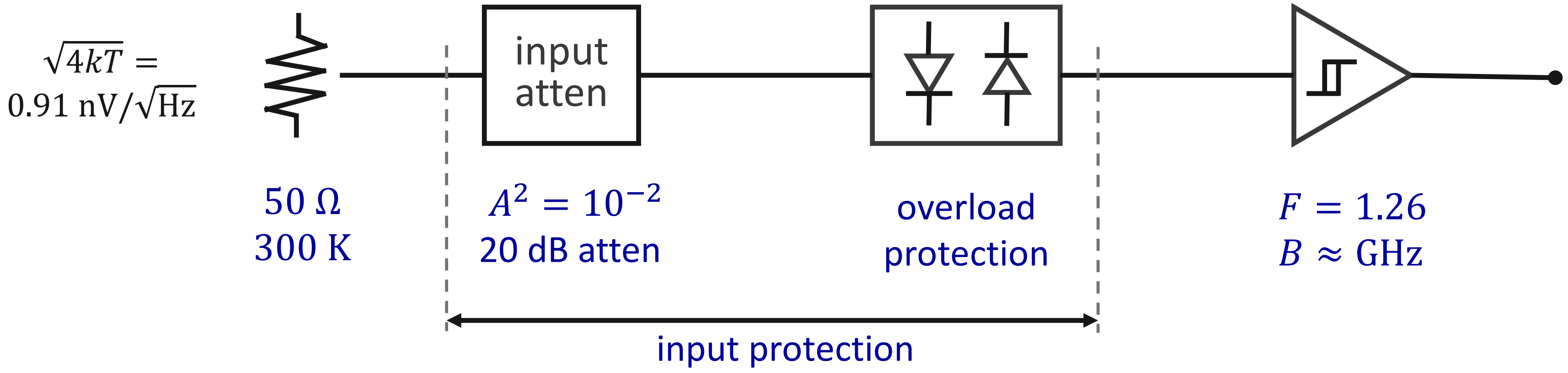
random $$\sigma_{\mathsf{x}}^2 = \frac{(\sigma_V^2)_a}{\mathrm{SR}_a^2} + \frac{(\sigma_V^2)_b}{\mathrm{SR}_b^2}$$



Threshold fluctuation



Don’t blame the trigger



$\sqrt{4kT} =$
$0.91\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$

50 Ω
300 K

$A^2 = 10^{-2}$
20 dB atten

overload
protection

$F = 1.26$
$B \approx \mathrm{GHz}$

input protection

Input noise $\sqrt{4kTB}$ of frequency counters

Type	max freq	Noise BW	e_n
HP 5370	225 MHz	900 MHz	27 µV
SR 620	1.3 GHz	5.2 GHz	66 µV

thumb rule: (noise BW) = 4 (max input f)

Account for 20 dB loss and noise factor 126 (1 dB)
Equivalent noise figure $F = 126$ (21 dB)

Total RMS noise

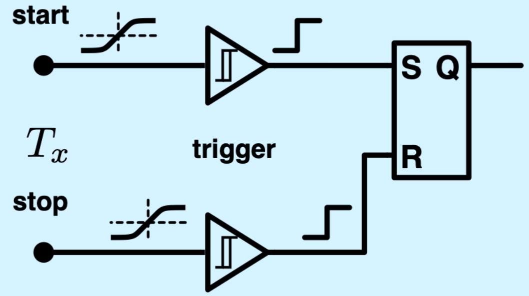
HP 5370	306 µV
SR 620	736 µV

Trigger noise – oversimplified

'stop' − 'start'

systematic $$\mathsf{x} = \frac{(\Delta V)_b}{\mathrm{SR}_b} - \frac{(\Delta V)_a}{\mathrm{SR}_a}$$

random $$\sigma_{\mathsf{x}}^2 = \frac{(\sigma_V^2)_a}{\mathrm{SR}_a^2} + \frac{(\sigma_V^2)_b}{\mathrm{SR}_b^2}$$



SR = Slew Rate (slope)

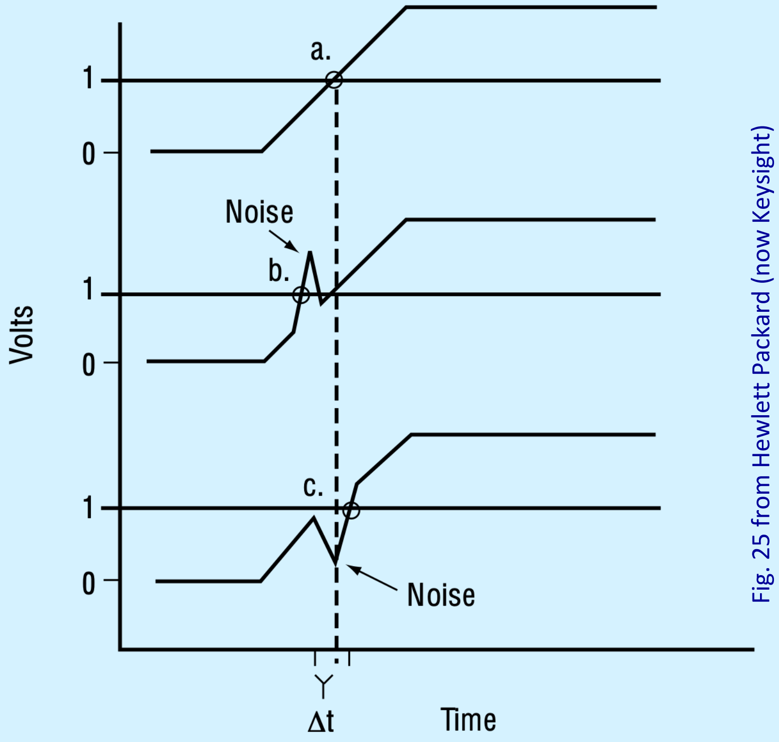


Fig. 25 from Hewlett Packard (now Keysight) Application Note an200-3, 1997

- The effect of noise is often explained with a plot like this
- Yet, the formula holds in the absence of spikes!!!
- To the general practitioner, this explanation looks simple

See also:
E. Rubiola & al., Proc. 46
FCS pp.265-269, May 1992

Trigger behavior vs bandwidth

Noise rms slope

$$\mathrm{SR}_n^2 = 4\pi^2 \int_0^B f^2 S_V(f)\, df$$

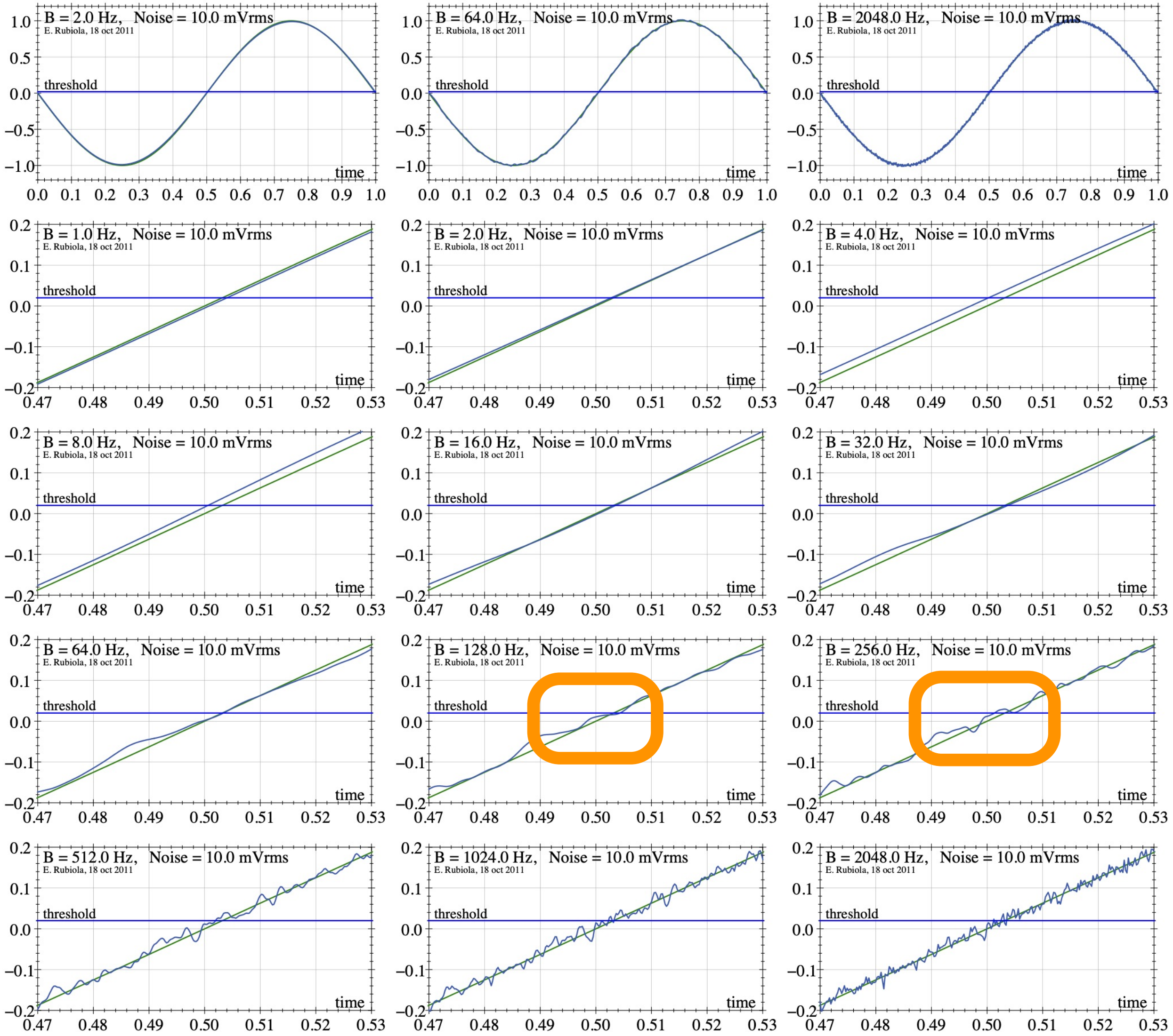
$$\mathrm{SR}_n^2 = \frac{4\pi^2}{3}\sigma_V^2 B^2$$

Critical slope

$$\mathrm{SR}_s^2 = \frac{4\pi^2}{3} S_V B^3$$

$$\mathrm{SR}_s^2 = \frac{4\pi^2}{3}\sigma_V^2 B^2$$

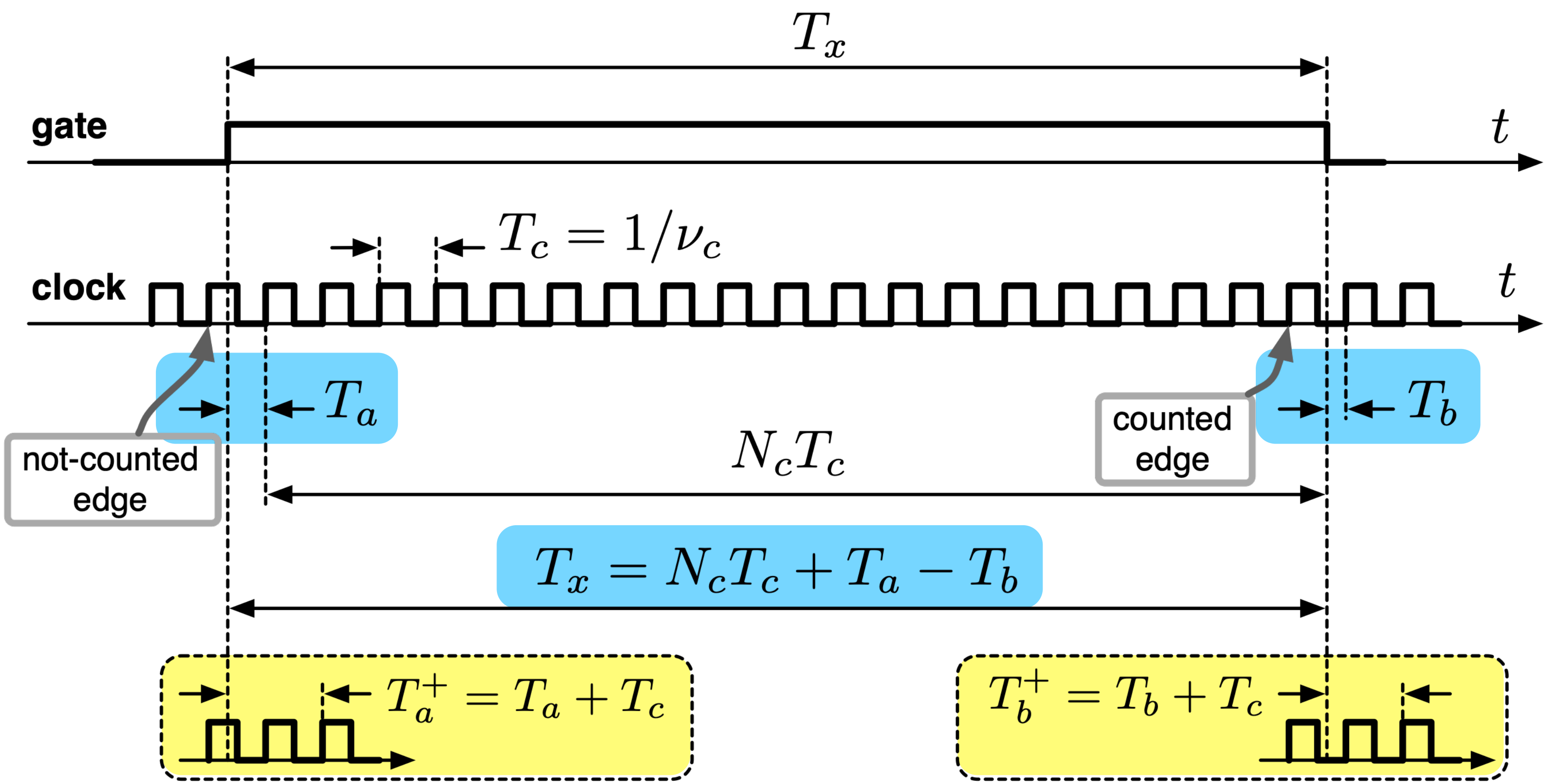
Signal slope equals rms noise slope



- When the noise slope exceeds the clean-signal slope, the total slope changes sign
- There result spikes, and systematic lead error

3 – Interpolation Schemes

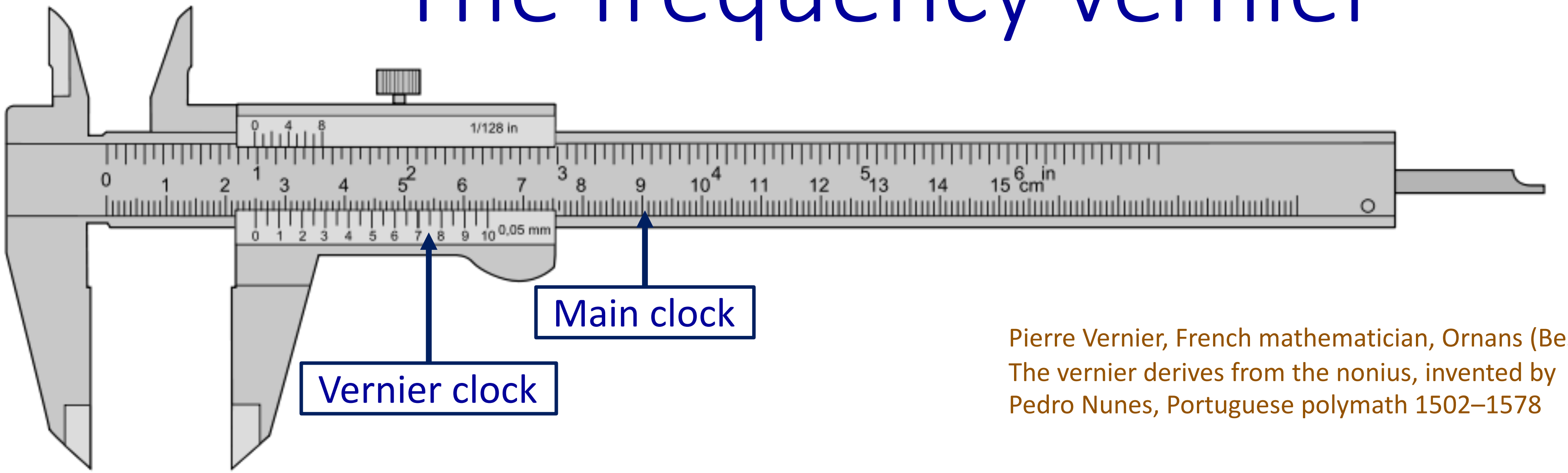
Clock interpolation



Too short T_a and T_b are difficult to measure, so we add one T_c to each

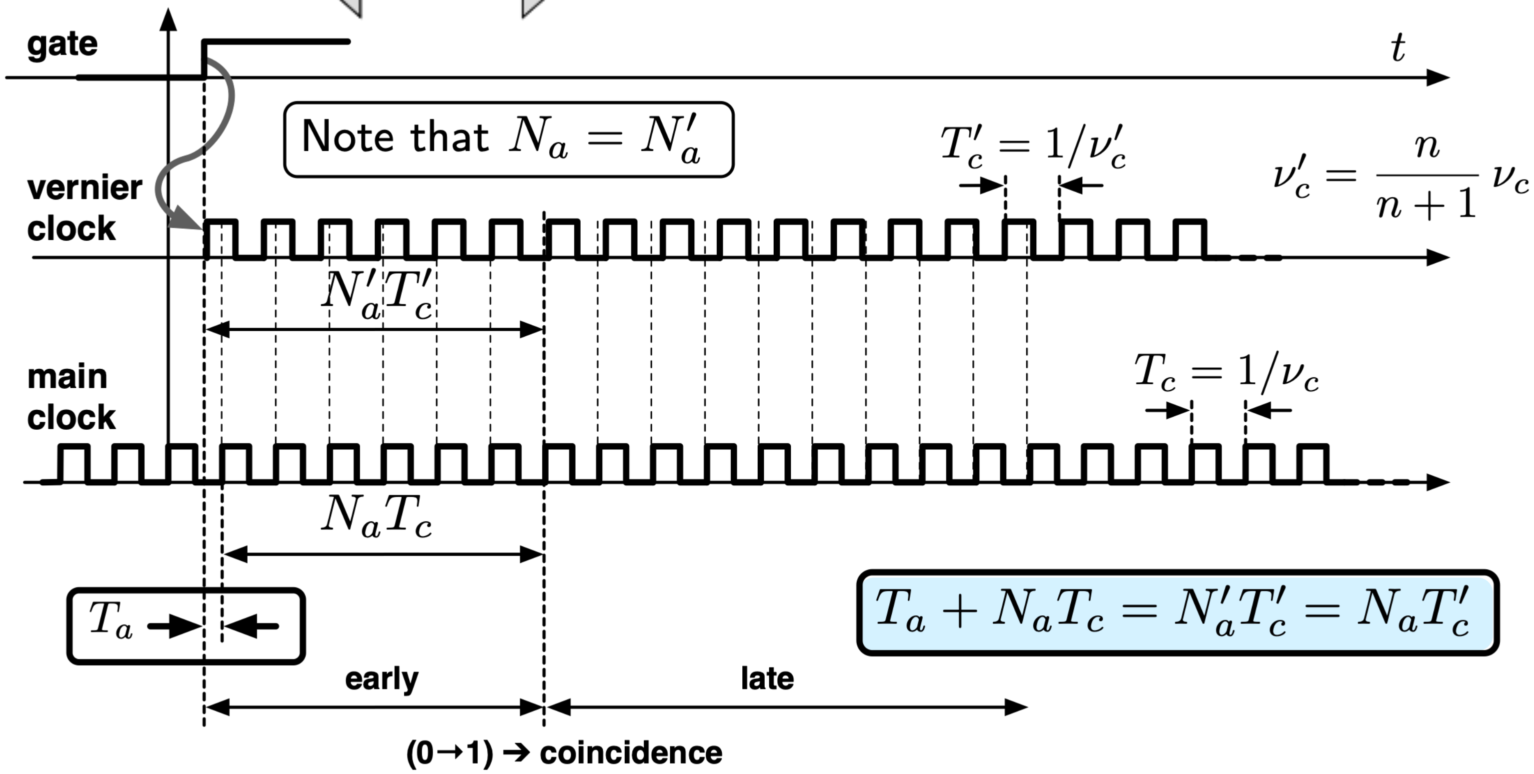
Interpolation is made possible by the fact that
the clock frequency is constant and accurately known

The frequency vernier



Picture Lucasbosch, CC-BY-SA-3.0

Pierre Vernier, French mathematician, Ornans (Besancon), 1580–1637
The vernier derives from the nonius, invented by
Pedro Nunes, Portuguese polymath 1502–1578



Example (HP5370A)

f_c = 200 MHz (5 ns)

n = 256

1/257×5 ns = 20 ps

The key elements

Triggered oscillator hybrid circuit, on a ceramic substrate in a hermetic package

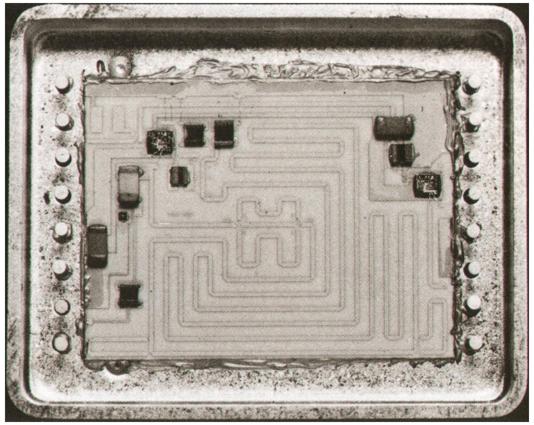
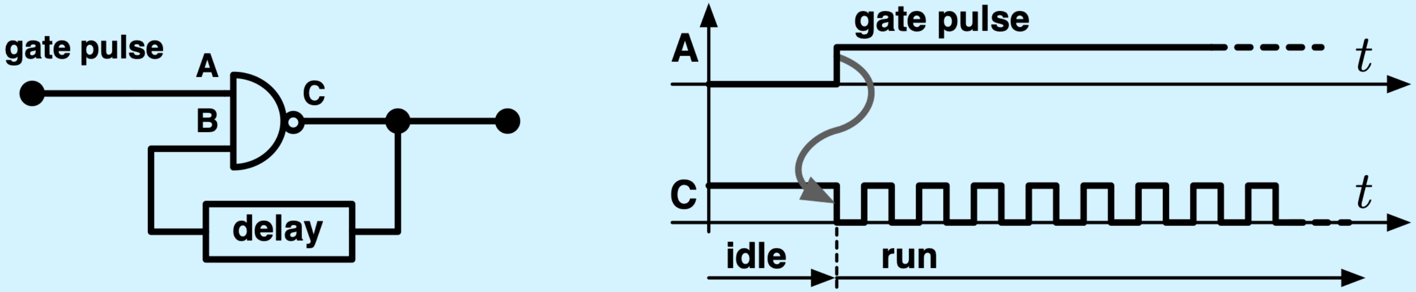
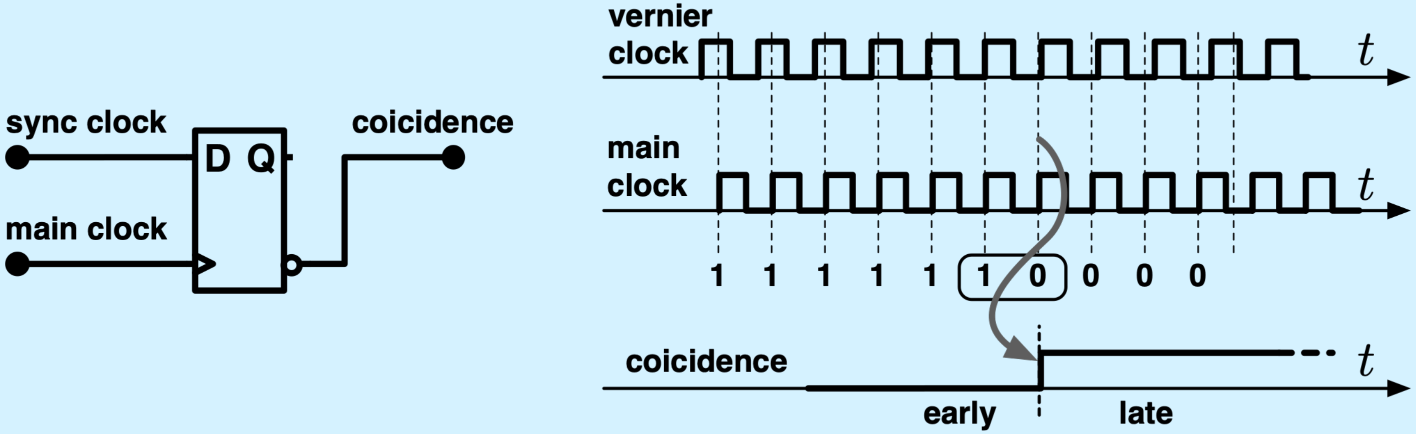


Fig. 3 from D. C. Chu et al., Universal counter resolves picoseconds in time interval measurement, HP Journal, August 1978. ©Hewlett Packard

Triggered oscillator



Coincidence detector



Example: Hewlett Packard 5370A

First commercialized in 1978

- Clock $f_c = 200$ MHz $\Rightarrow$ $\delta T_x = 5$ ns (ECL technology)
- Vernier $n = 256$ $\delta T_a = \delta T_b = \frac{1}{256}\delta T_x = 19.5$ ps
- It takes max 257 cycles of f_c for the two clocks to coincide
- Conversion time $T = nT_c = 1.283$ µs
- Resolution, compared to the speed of light
  - free space, $\delta\ell = c\delta T_a = 6$ mm
  - cable, $v = 0.67\ c$, $\delta\ell = 4$ mm
- Actual resolution is ≈ 35 ps, due to noise

D. C. Chu, M. S. Allen, A. S. Foster, Universal counter resolves picoseconds in time interval measurement, HP Journal, August 1978

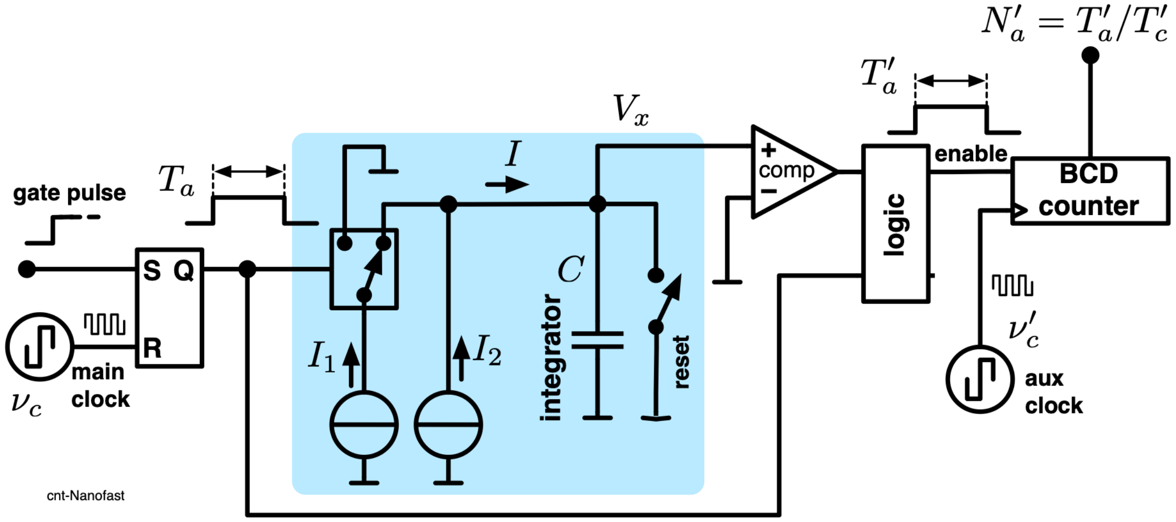
Keysight 53230A has same resolution

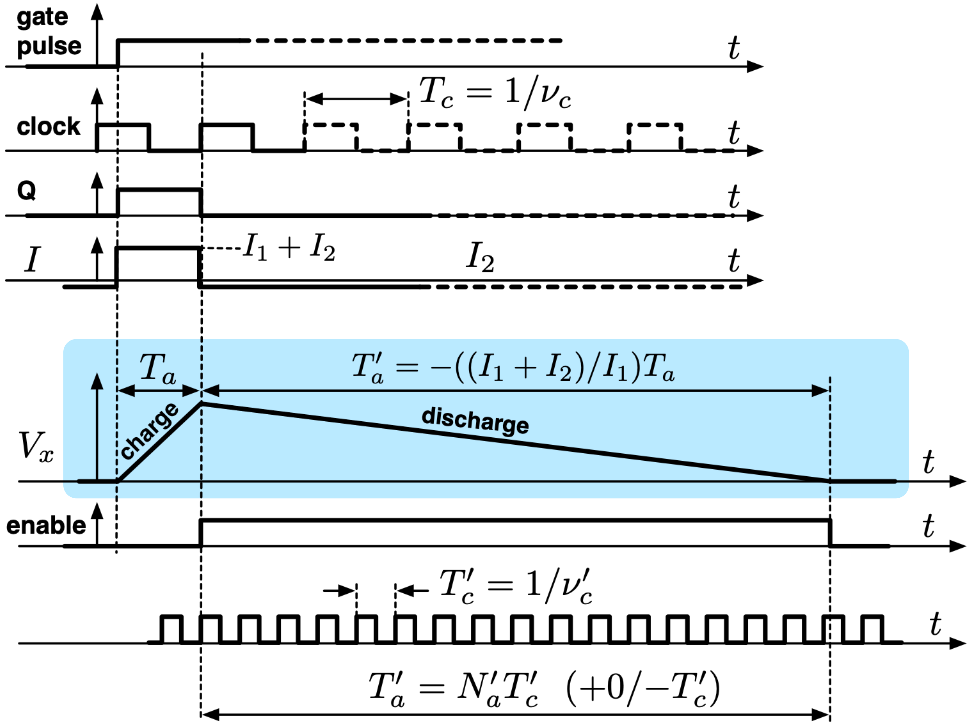
Image © Keysight Technologies


The original HP 5370A

Image from the instruction and operation manual, ©Hewlett Packard

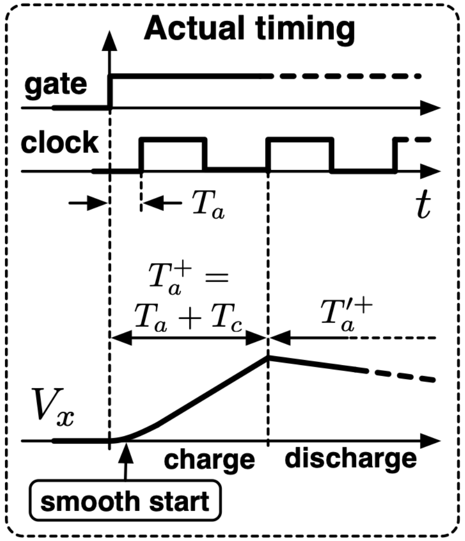
The Nutt’s dual-slope interpolator





Similar to the dual-slope voltmeter

R. Nutt, Digital time intervalometer, RSI 39(9) p.1342-1345, September 1968



Example: Nanofast 536 B

Semi-commercial product designed in the early 1970s at the Smithsonian Astrophysical Laboratory

- Main clock $\nu_m = 10\ \mathrm{MHz}$
- Auxiliary clock (internal, for short time intervals)

$$\nu_c = 20\ \mathrm{MHz} \longrightarrow T_c = 50\ \mathrm{ns}$$

- Interplay of currents

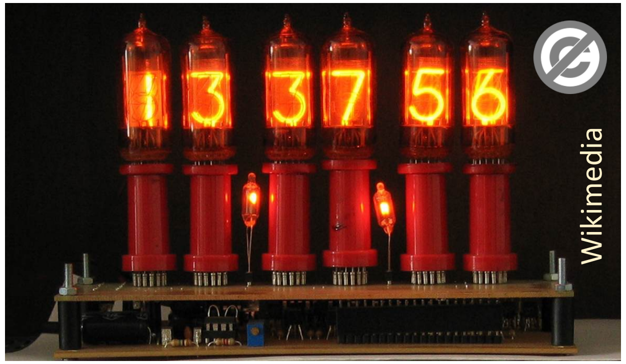
$$\frac{I_1 + I_2}{I_2} = 4096$$

- Time resolution

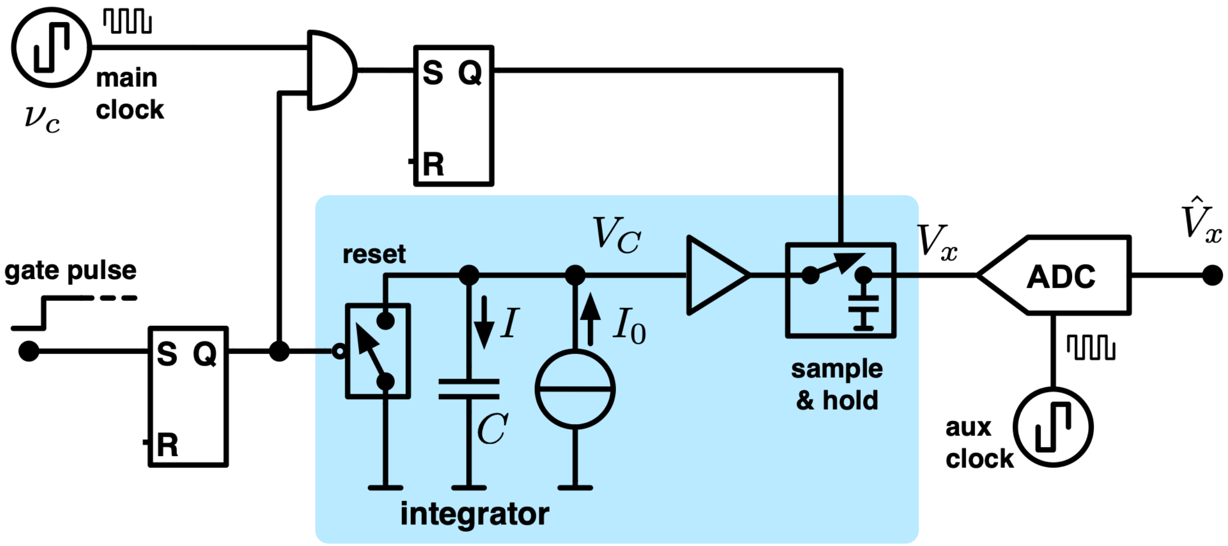
$$\Delta t = T_c \frac{I_2}{I_1 + I_2} = 12.2\ \mathrm{ps}$$

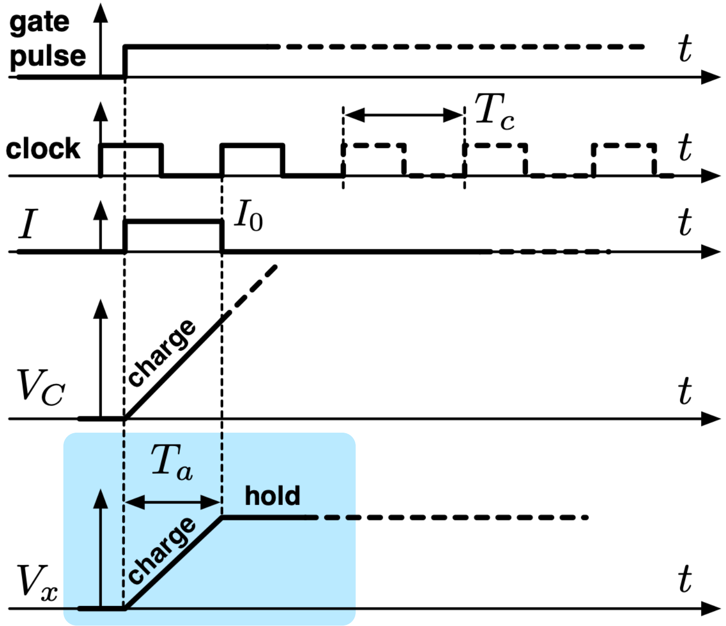
- Early TTL technology, and nixie display
- Was used in the Mark IV VLBI system
- For reference, in 12.2 ps a pulse propagates 2.74 mm in a coax cable (speed 0.66 c)

Nixie display



The ramp interpolator



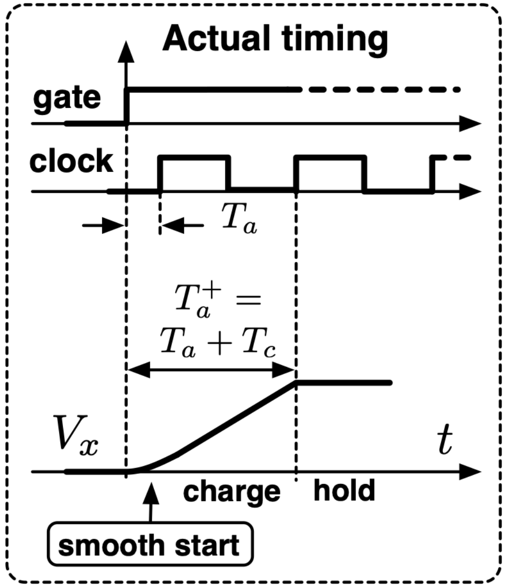


Example (Stanford SR620)

f_c = 90 MHz (T_c = 11.1 ns)

11 bits

$T_c / 2^{11}$ = 5.4 ps



This costs 1 bit ADC resolution loss

Example: Stanford SR 620

- Clock $\nu_c = 90$ MHz. —> $T_c = 11.1$ ns (Locked to the 10 MHz reference multiplied by 9)
- Successive approximation ADC, 12 bits
- One bit is lost due to the extra T_c (minor technical detail)
- Resolution

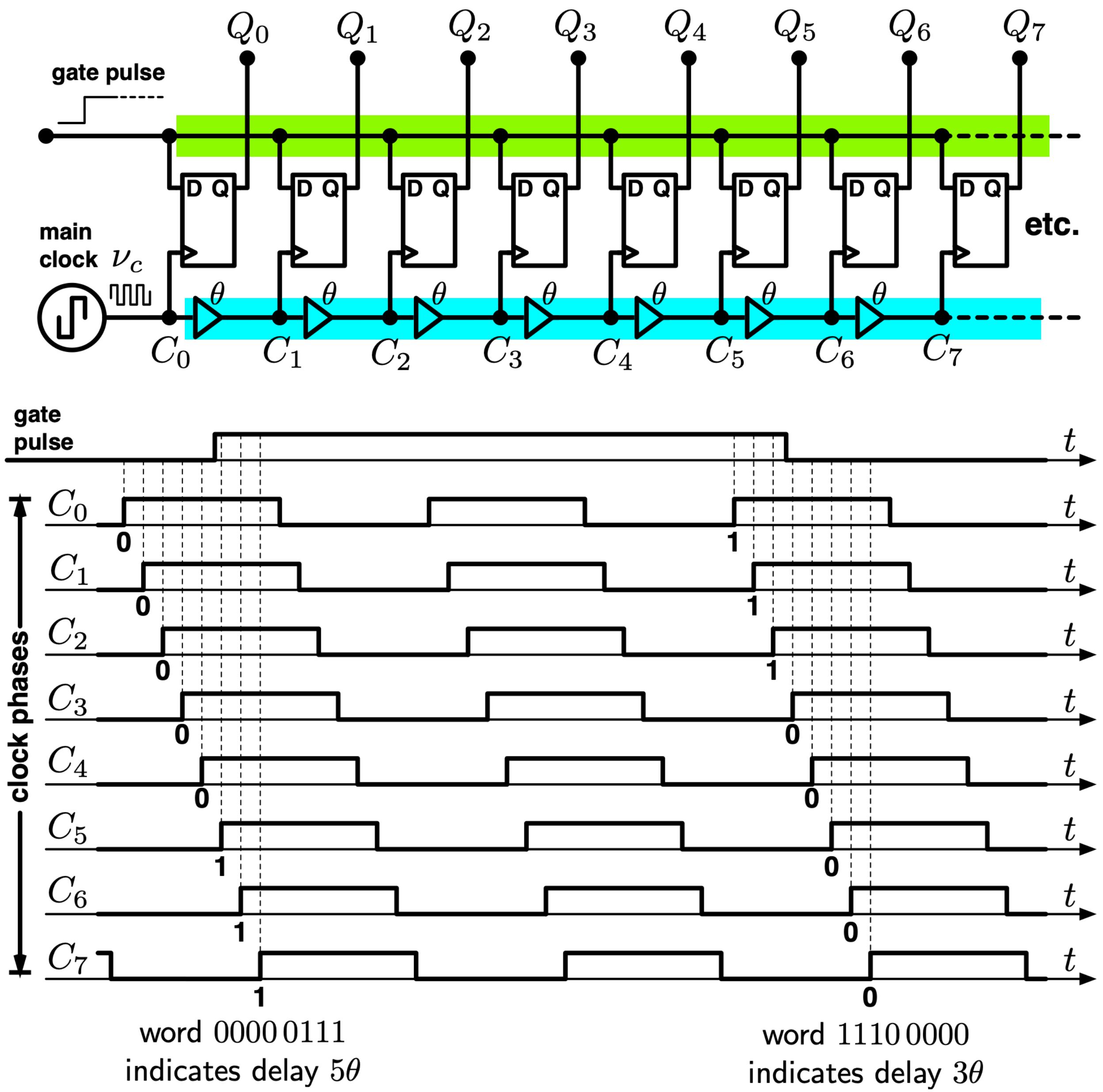
$$\Delta T = \frac{T_c}{2^{11}} = 5.4 \text{ ps}$$

- Actual resolution ≈ 50 ps, due to noise

Image © Stanford Research Systems

Thermometer-code interpolator

Also called Multi-tapped delay-line interpolator



FPGA implementation

- Needs full layout control
- The pipeline may not fit in a cell

Great for ASIC implementation

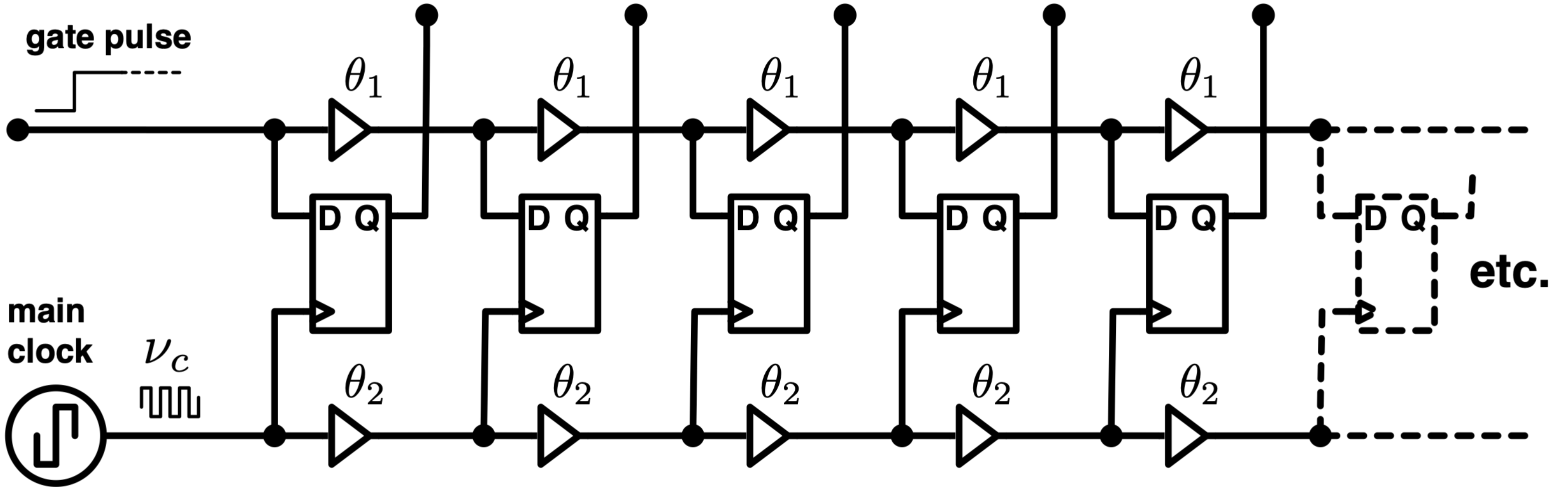
Vernier (enhanced resolution) version

- Delay is on both lines is inevitable
- Just exploit it

$\theta_{eq} = \theta_{ck} - \theta_{in}$

Review article:
J. Kalisz, Metrologia 41 (2004) 17–32

Vernier thermometer-code interpolator



$$\theta_{\mathrm{eq}} = \theta_2 - \theta_1$$

Owing to physical size, both θ_1 and θ_2 are always present

Featured review article: J. Kalisz, Metrologia 41 (2004) 17–32

Ring oscillator

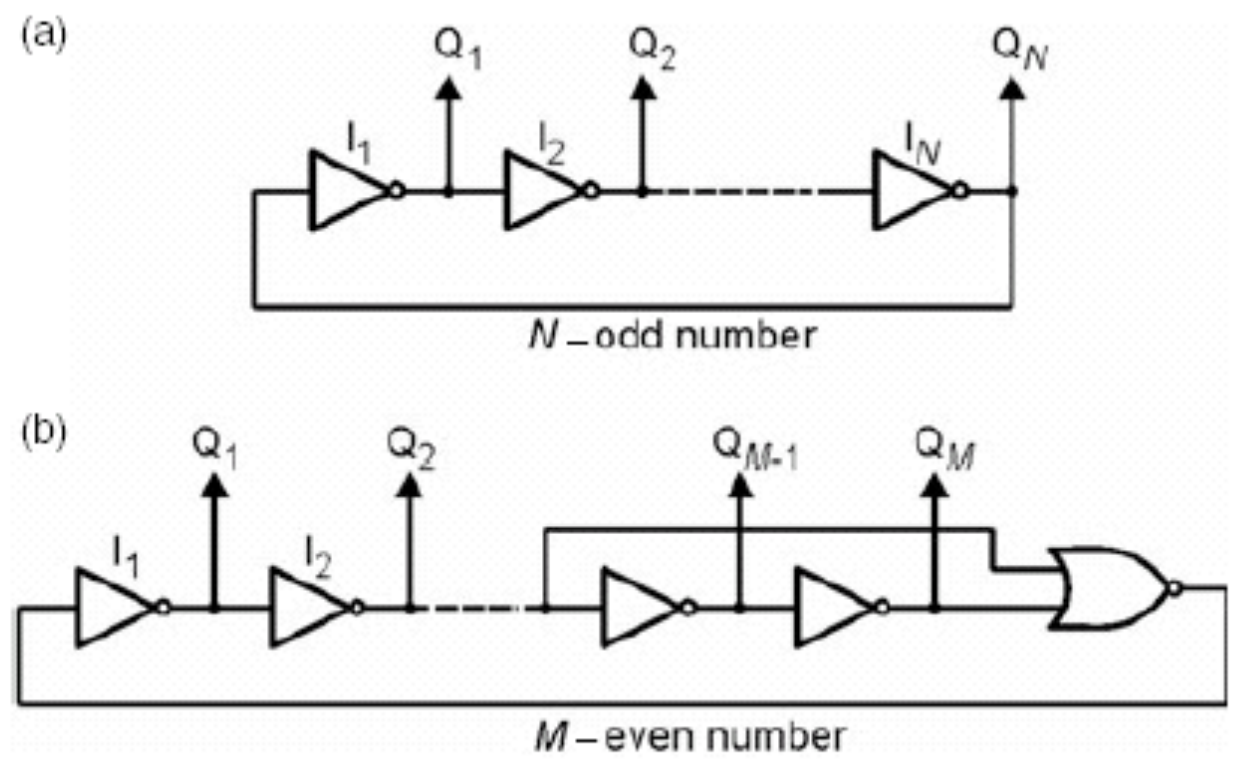
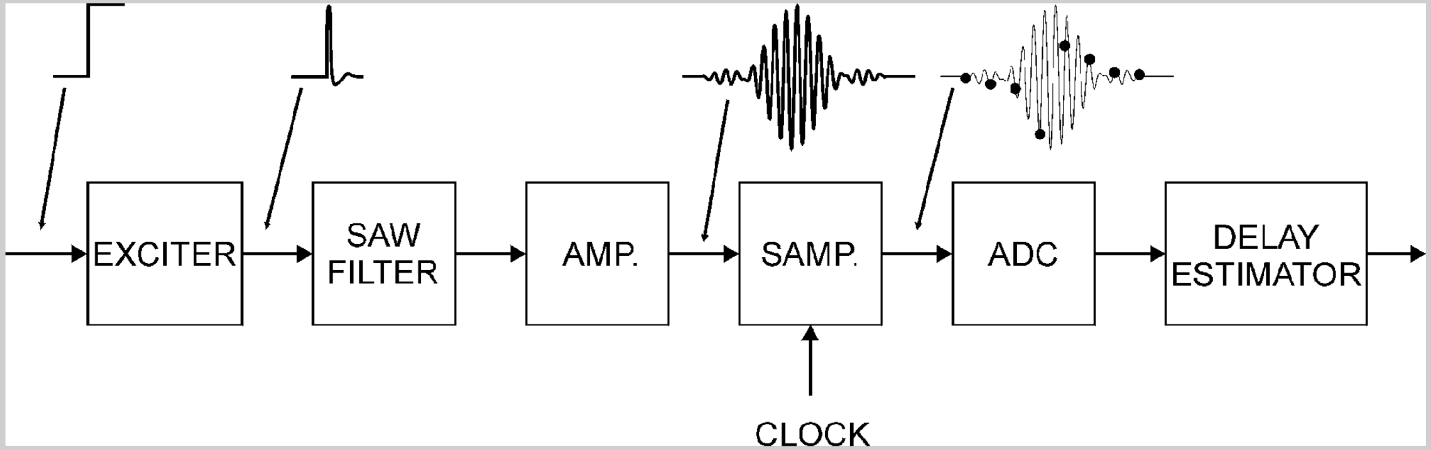


Figure from J. Kalisz, Metrologia 41 (2004) 17–32

Also used in PLL circuits for clock-frequency multiplication

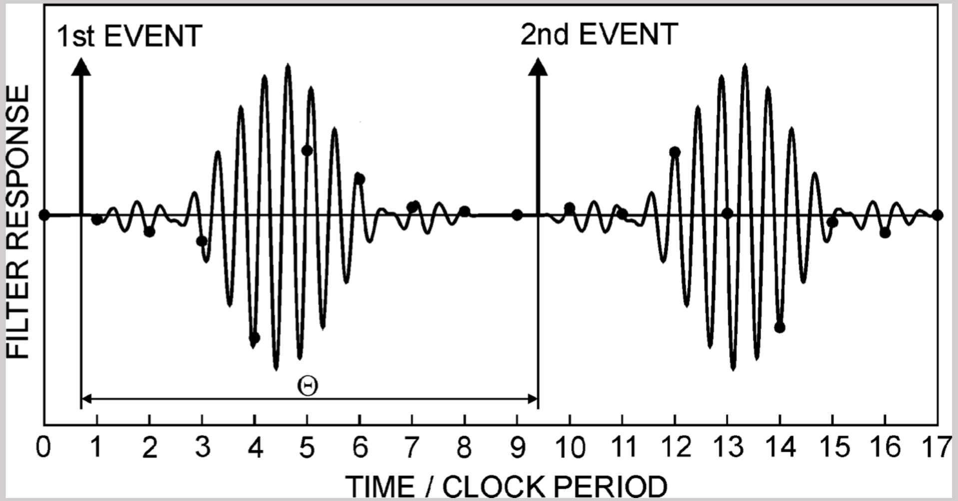
SAW delay-line interpolator

A – Block diagram



Panek & Prochazka, Fig.1

B – Pulse waveforms

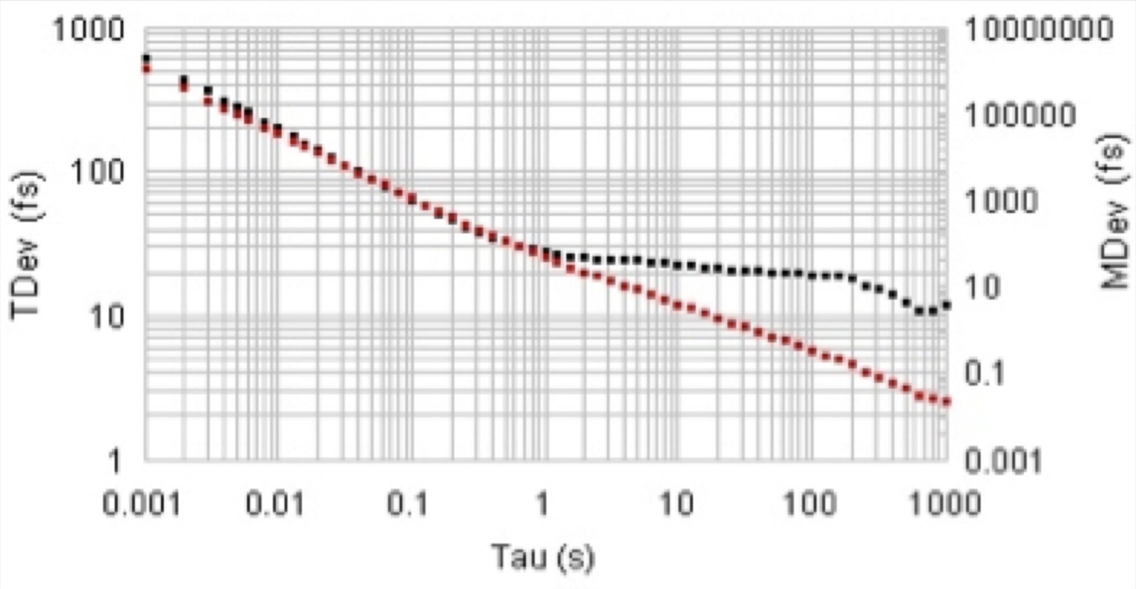


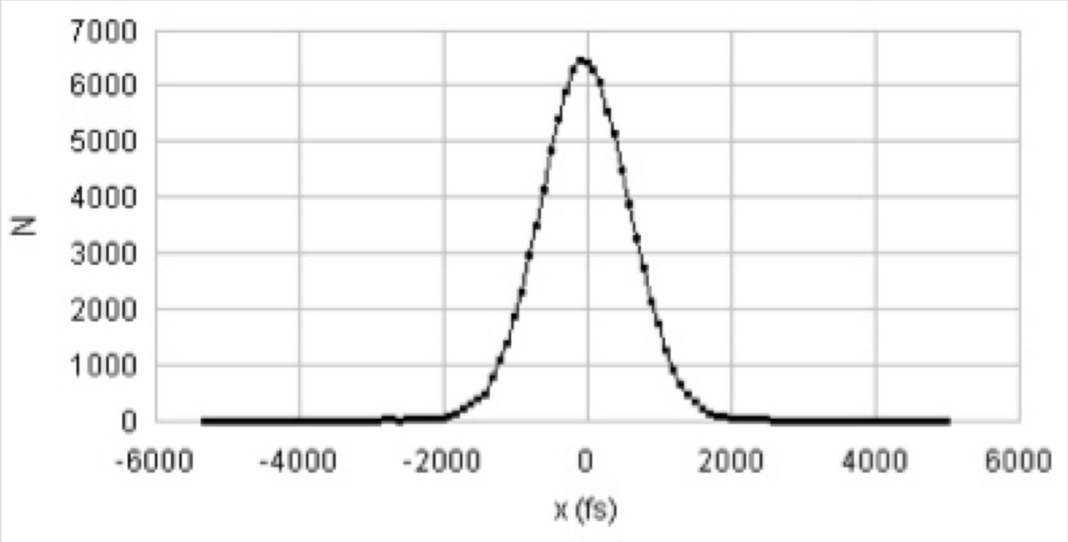
Panek & Prochazka, Fig.2

- Dispersion stretches the input pulse
- Sub-sampling and identification of the alias

P. Panek, I. Prochazka, Rev. Sci. Instrum. 78(9) 094701, 2007

Sigma Time STX301





- Rumors are that this is none of the methods shown
- No information at all, I'm unable to reverse-engineer

All figures are from the data sheet

Commercial equipment

Brand	Country	Type	Resol.	Type	Method
Carmel	USA	NK732	3 ps	PCI/PXI time stamp	Ramp
Eventech	Latvia	ESTT 704	1.3 ps	Lab instrument	
Guide Tech	USA	GT667/668	1 ps	PCI/PXI time stamp	Ramp
K+K Messtechnik	Germany	FXE Series	12 ps	PXI	Ramp
Keysight	USA	53230A	20 ps	Lab instrument	Frequency Vernier
Lange Electronic	Germany	KL-3360	50 ps	Π / Λ, special purpose	Ramp
Lumat				PCI card	Thermometer code
Stanford	USA	SR620	25 ps	Lab instrument	Ramp
Serenum		TDC	6 ps rms	PCB module	FPGA Thermometer code
AMS Group		TDC GPX	22 ps	Chip	
MAXIM	USA	MAX35101	3.8 ps	Chip	
SPAD Lab	Italy	TDC Module		Packaged module	Vernier ASIC. Markovich, RSI 2012
Texas	USA	THS788	8 ps	Chip	Thermometer code

4 – Basic Statistics

– After all, not that basic! –

Basics

Average

$$\langle \nu \rangle = \int_{-\infty}^{+\infty} \nu(t) w(t)\, dt$$

weight function $w(t)$

normalization

$$\int_{-\infty}^{+\infty} w(t)\, dt = 1$$

Variance

$$\sigma_y^2 = \int_{-\infty}^{+\infty} y^2(t) w(t)\, dt$$

weight function $w(t)$

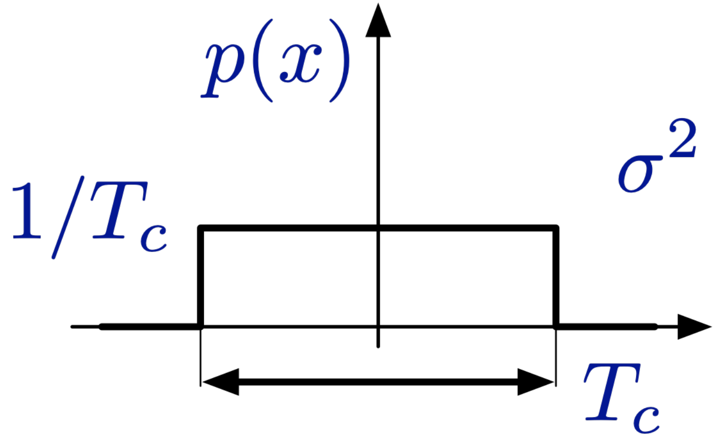
normalization

$$\int_{-\infty}^{+\infty} w(t)\, dt = 1$$

$\langle \cdots \rangle$ = average

$\mathbb{E}\{\cdots\}$ = expectation

Quantization uncertainty



$$\sigma^2 = \frac{T_c^2}{12}$$

$$1/\sqrt{12} = 0.29$$

Example: 100 MHz clock

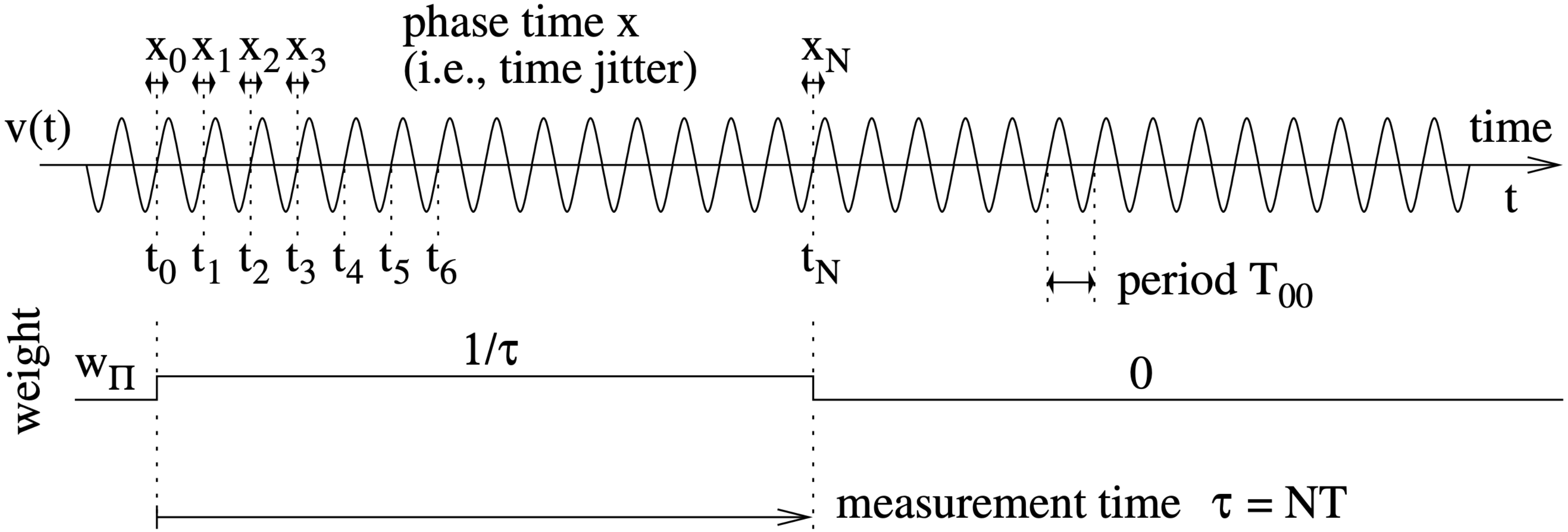
T_x = 10 ns

σ = 2.9 ns

Π (classical) counter

E. Rubiola, RSI 76(5) 054703, May 2005

E. Rubiola, Fig.2



the measure is a scalar product

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Pi(t)\, dt$$

Π estimator

$$w_\Pi(t) = \begin{cases} 1/\tau & 0 < t < \tau \\ 0 & \text{elsewhere} \end{cases}$$

weight

$$\int_{-\infty}^{+\infty} w_\Pi(t)\, dt = 1$$

normalization

variance

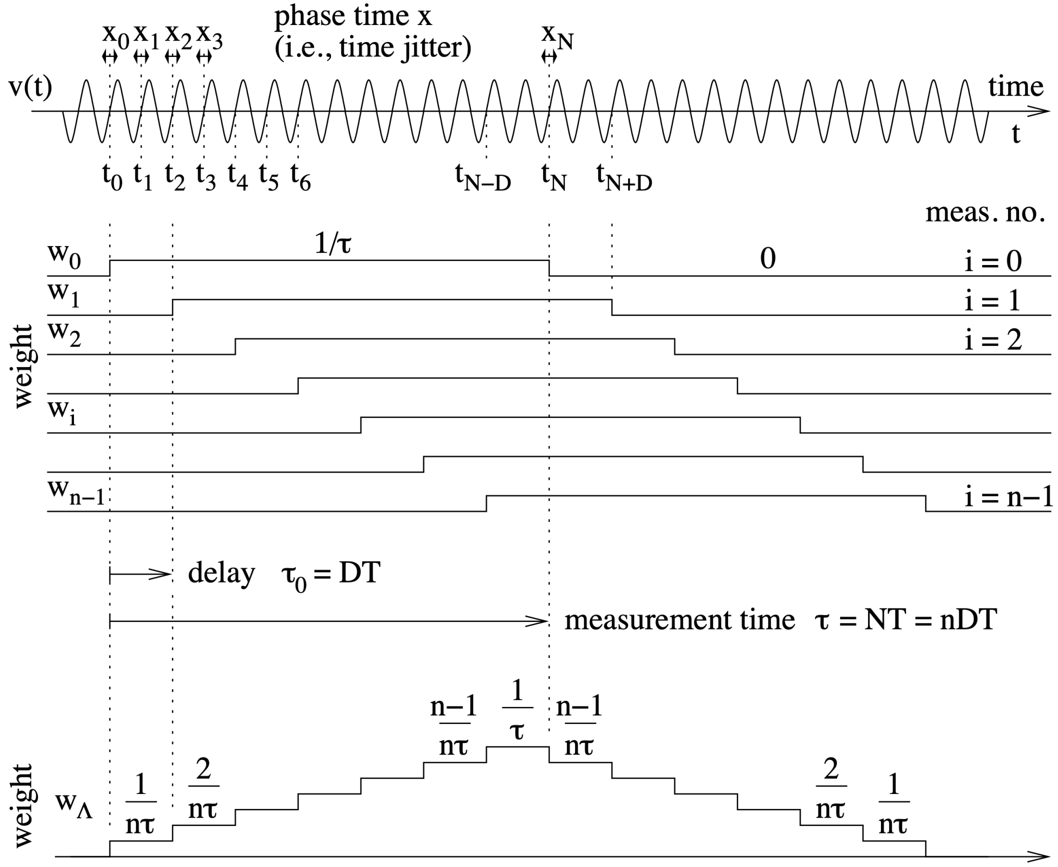
$$\sigma_y^2 = \frac{2\sigma_x^2}{\tau^2}$$

classical variance

Λ counter

E. Rubiola, RSI 76(5) 054703, May 2005

E. Rubiola, Fig.3



$$\mathbb{E}\{\nu\} = \frac{1}{n}\sum_{i=0}^{n-1}\overline{\nu}_i \qquad \overline{\nu}_i = N/\tau_i$$

Λ estimator

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t)\, w_\Lambda(t)\, dt$$

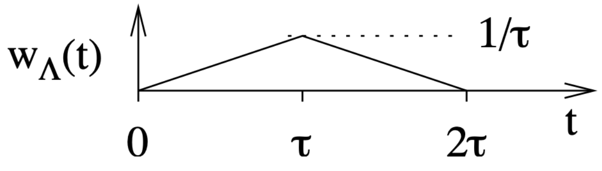
weight

$$w_\Lambda(t) = \begin{cases} t/\tau & 0 < t < \tau \\ 2 - t/\tau & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases}$$

normalization

$$\int_{-\infty}^{+\infty} w_\Lambda(t)\, dt = 1$$

limit τ₀ -> 0 of the weight function



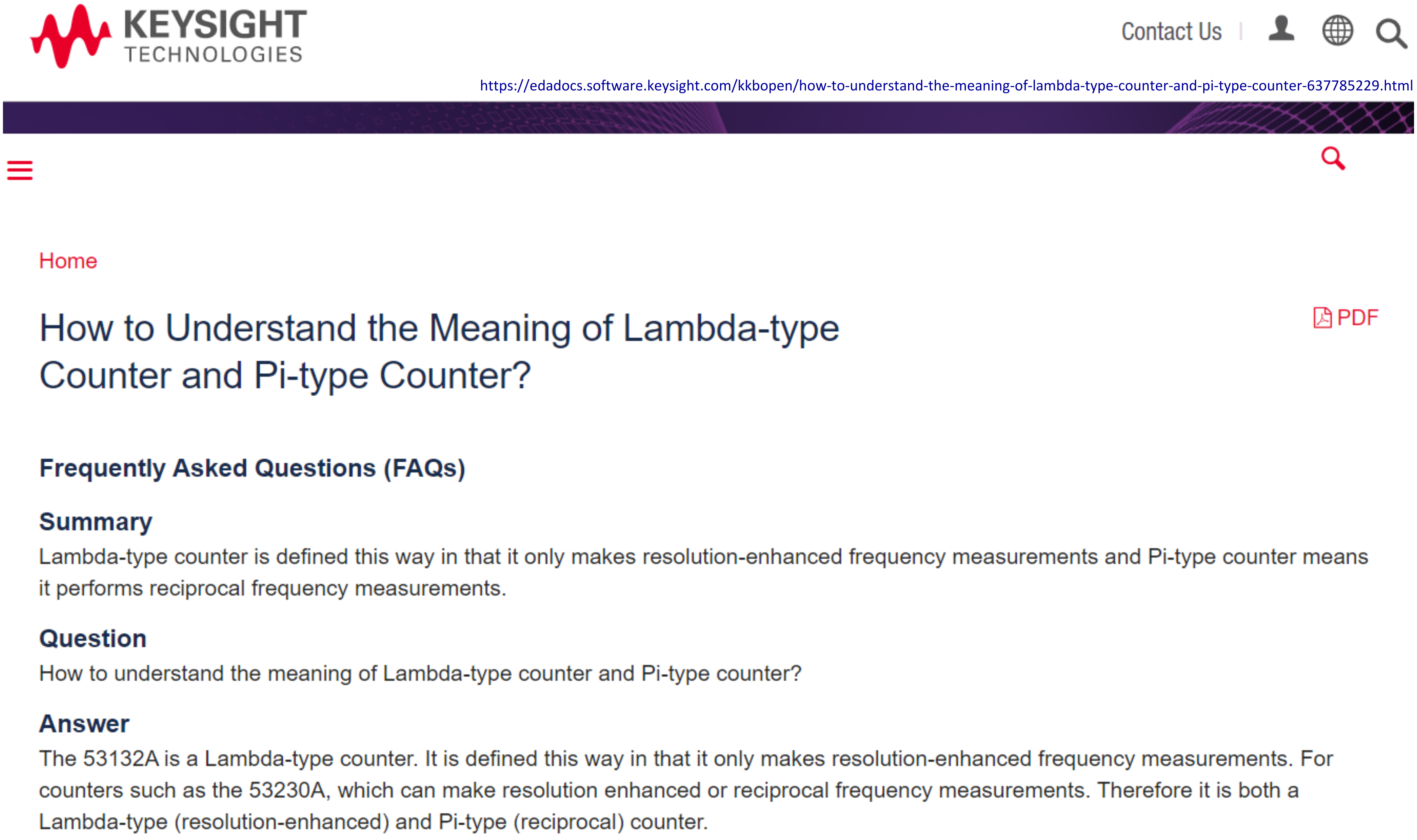
white noise: the autocorrelation function is a narrow pulse, about the inverse of the bandwidth

the variance is divided by n

$$\sigma_y^2 = \frac{1}{n}\,\frac{2\sigma_x^2}{\tau^2}$$

classical variance

https://edadocs.software.keysight.com/kkbopen/how-to-understand-the-meaning-of-lambda-type-counter-and-pi-type-counter-637785229.html



How to Understand the Meaning of Lambda-type Counter and Pi-type Counter?

Frequently Asked Questions (FAQs)

Summary

Lambda-type counter is defined this way in that it only makes resolution-enhanced frequency measurements and Pi-type counter means it performs reciprocal frequency measurements.

Question

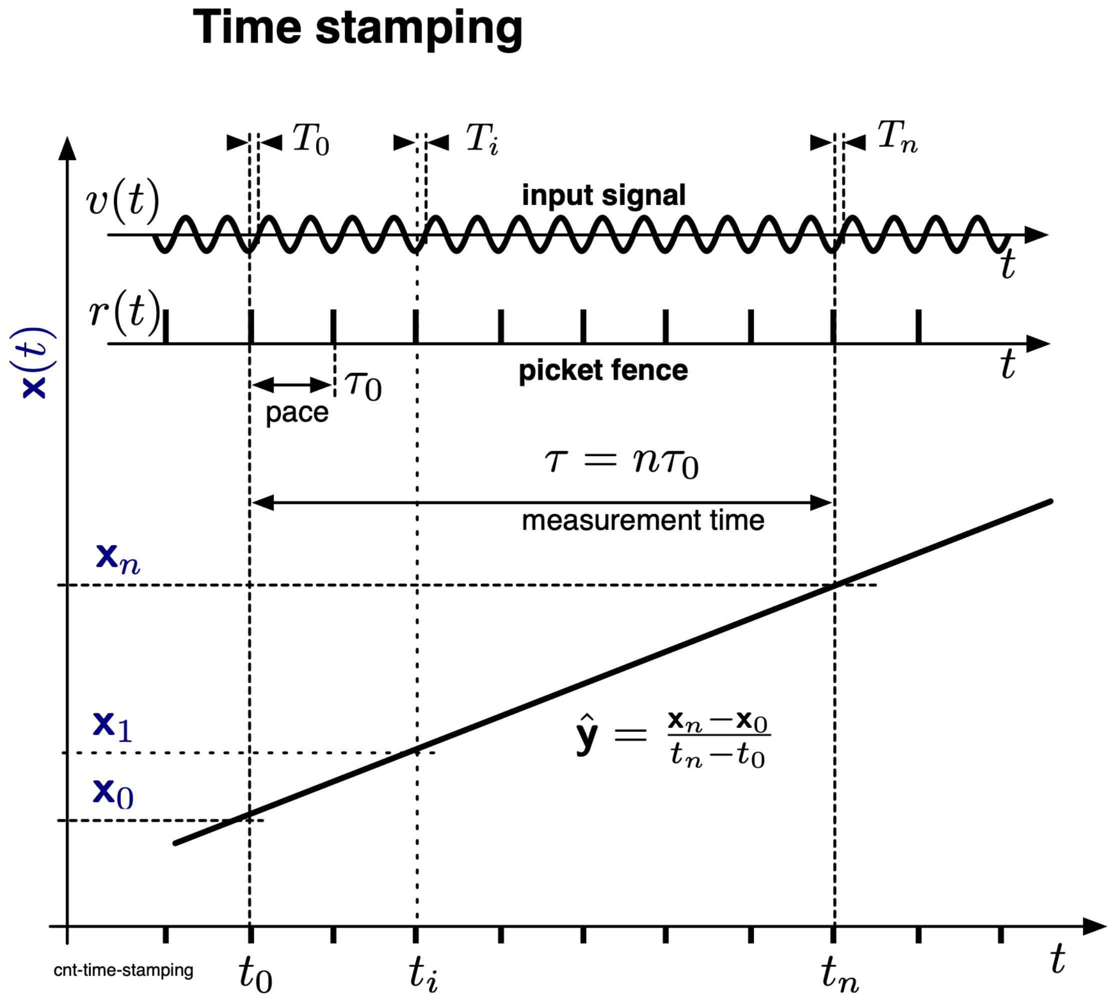
How to understand the meaning of Lambda-type counter and Pi-type counter?

Answer

The 53132A is a Lambda-type counter. It is defined this way in that it only makes resolution-enhanced frequency measurements. For counters such as the 53230A, which can make resolution enhanced or reciprocal frequency measurements. Therefore it is both a Lambda-type (resolution-enhanced) and Pi-type (reciprocal) counter.

Ω (linear-regression) counter

E. Rubiola & al, IEEE Transact. UFFC 63(7) pp.961–969, July 2016



$\mathbf{x}(t) = t + \mathrm{x}(t)$ phase time

$\mathbf{y}(t) = 1 + \mathrm{y}(t)$ fractional frequency

$\mathrm{x}(t) = \varphi(t)/2\pi\nu_0$ fluctuation

$\mathbf{y}(t) = \dot{\mathbf{x}}(t)$

y is estimated with a linear regression on the **x** series

$$\hat{\mathbf{y}} = \frac{\sum_i \left(\mathbf{x}_i - \langle \mathbf{x} \rangle, t_i - \langle t \rangle\right)}{\sum_i \left(t_i - \langle t \rangle\right)^2}.$$

Linear regression on a sequence of time stamps provides accurate estimation of frequency and best rejection of white PM noise

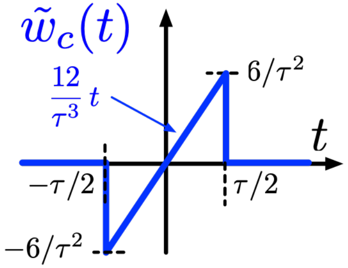
Ω counter

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Omega(t)\, dt$$

$$\mathbb{E}\{\mathbf{y}\} = \int_{-\infty}^{+\infty} \mathbf{x}(t) \tilde{w}_\Omega(t)\, dt$$

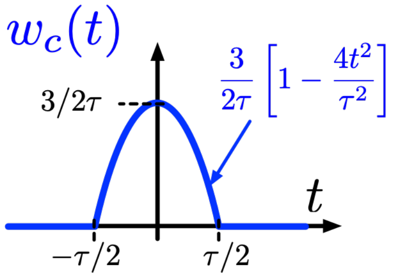
Using phase data

Apologize for inconsistent notation, subscript “c” <–> “Ω”



$$\tilde{w}_c(t) = \begin{cases} \frac{12}{\tau^3}\, t & \text{for } t \in (-\frac{\tau}{2}, \frac{\tau}{2}) \\ 0 & \text{elsewhere} \end{cases}$$

Using frequency data



$$w_c(t) = \begin{cases} \frac{3}{2\tau}\left[1 - \frac{4t^2}{\tau^2}\right] & \text{for } t \in (-\frac{\tau}{2}, \frac{\tau}{2}), \\ 0 & \text{elsewhere.} \end{cases}$$

Lecture 5 ends here

Formulae found in manuals

$$(\Pi) \quad \sigma_y = \frac{1}{\tau}\sqrt{2(\delta t)^2_{\text{trigger}} + 2(\delta t)^2_{\text{interpolator}}}$$

$$(\Lambda) \quad \sigma_y = \frac{1}{\tau\sqrt{n}}\sqrt{2(\delta t)^2_{\text{trigger}} + 2(\delta t)^2_{\text{interpolator}}}$$

$$n = \begin{cases} \nu_0\tau & \nu_{00} \le \nu_I \\ \nu_I\tau & \nu_{00} > \nu_I \end{cases}$$

Understanding technical data

classical reciprocal counter

$$\sigma_y^2 = \frac{2\sigma_x^2}{\tau^2}$$ classical variance

enhanced-resolution counter

$$\sigma_y^2 = \frac{1}{n}\,\frac{2\sigma_x^2}{\tau^2}$$ classical variance

low frequency:
full speed

$$\tau_0 = T \quad \Longrightarrow \quad n = \nu_{00}\tau$$

$$\sigma_y^2 = \frac{1}{\nu_{00}}\,\frac{2\sigma_x^2}{\tau^3}$$ classical variance

high frequency:
housekeeping takes time

$$\tau_0 = DT \quad \text{with } D{>}1 \quad \Longrightarrow \quad n = \nu_{00}\tau$$

$$\sigma_y^2 = \frac{1}{\nu_I}\,\frac{2\sigma_x^2}{\tau^3}$$ classical variance

the slope of the classical variance tells the whole story

$$1/\tau^2 \quad \Longrightarrow \quad \Pi \text{ estimator (classical reciprocal)}$$

$$1/\tau^3 \quad \Longrightarrow \quad \Lambda \text{ estimator (enhanced-resolution)}$$

look for formulae and plots in the instruction manual

Examples

Stanford SRS-620

$$\left[\begin{matrix}\text{RMS}\\ \text{resolution}\\ \text{(in Hz)}\end{matrix}\right] = \frac{\text{frequency}}{\text{gate time}}\sqrt{\frac{(25\text{ ps})^2 + \left[\begin{pmatrix}\text{short term}\\ \text{stability}\end{pmatrix} \times \begin{pmatrix}\text{gate}\\ \text{time}\end{pmatrix}\right]^2 + 2\times\left[\begin{matrix}\text{trigger}\\ \text{jitter}\end{matrix}\right]^2}{\text{N}}}$$

RMS resolution	$\sigma_\nu = \nu_{00}\sigma_y$
frequency	ν_{00}
gate time	τ

Agilent 53132A

$$\left[\begin{matrix}\text{RMS}\\ \text{resolution}\end{matrix}\right] = \begin{pmatrix}\text{frequency}\\ \text{or period}\end{pmatrix} \times \left[\frac{4\times\sqrt{(t_{\text{res}})^2 + 2\times(\text{trigger error})^2}}{(\text{gate time})\times\sqrt{\text{no. of samples}}} + \frac{t_{\text{jitter}}}{\text{gate time}}\right]$$

$$t_{\text{res}} = 225\text{ ps}$$

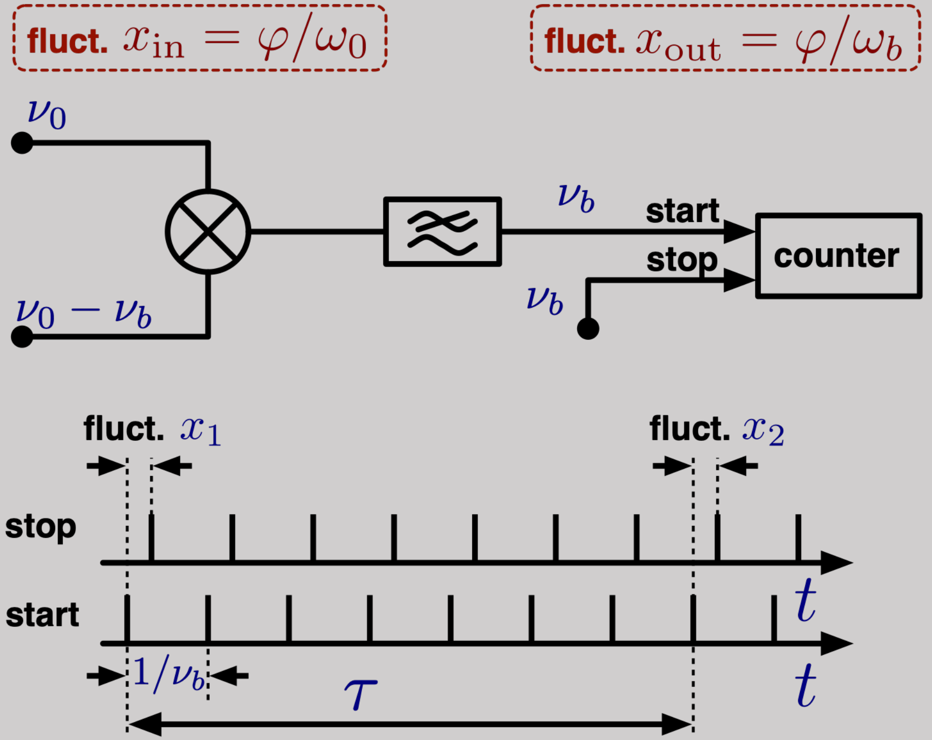
$$t_{\text{jitter}} = 3\text{ ps}$$

$$\text{number of samples} = \begin{cases}(\text{gate time})\times(\text{frequency}) & \text{for } f < 200\,\text{kHz}\\ (\text{gate time})\times 2{\times}10^5 & \text{for } f \geq 200\,\text{kHz}\end{cases}$$

RMS resolution	$\sigma_\nu = \nu_{00}\sigma_y$ or $\sigma_T = T_{00}\sigma_y$
frequency	ν_{00}
period	T_{00}
gate time	τ
no. of samples	$n = \begin{cases}\nu_{00}\tau & \nu_{00} < 200\text{ kHz}\\ \tau\times 2{\times}10^5 & \nu_{00} \geq 200\text{ kHz}\end{cases}$

5 – Measurement and Beat Note

The beat-note method



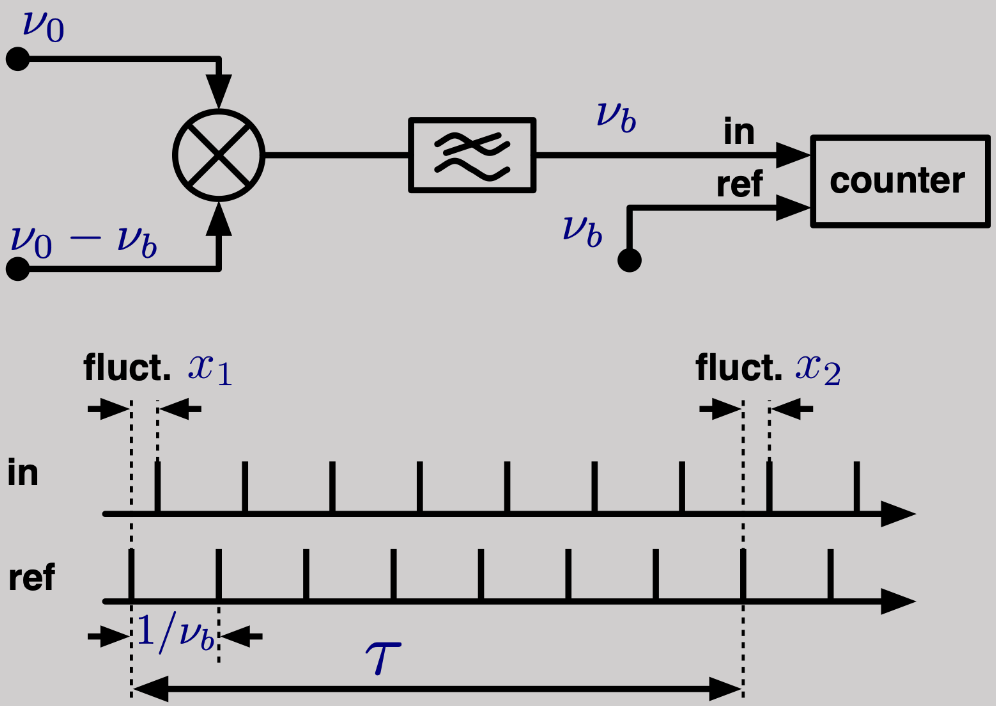
The beat stretches the phase-time fluctuation *x* by a factor $\kappa = \omega_0 / \omega_b$

In RF/µwaves, we can get $\kappa = 10^4 \ldots 10^{10}$

$$\varphi_{\text{out}} = \varphi_{\text{in}} \quad \Rightarrow \quad \mathrm{x}_{\text{out}} = \frac{\nu_0}{\nu_b}\,\mathrm{x}_{\text{in}} = \kappa\,\mathrm{x}_{\text{in}}$$

the phase-time *x* is the phase fluctuation ϕ expressed in seconds (instead of rad)

White noise



Classical variance

$$\sigma_y^2 = \frac{1}{N}\;\frac{\nu_b^2}{\nu_0^2}\;\frac{2\sigma_{\mathsf{x}}^2}{\tau^2}$$

$$\sigma_{\mathsf{x}}^2 = \sigma_q^2 + \frac{\sigma_v^2}{(dv/dt)^2}$$

quantization

additive noise

slew rate

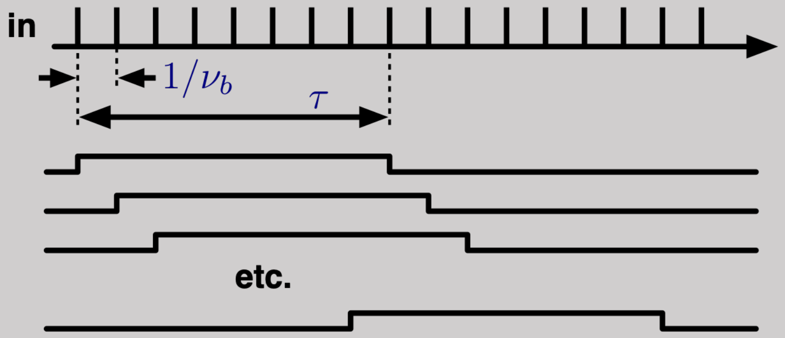
measurement time

average on N (overlapped) measures

beat-method amplification

Quantization noise

Allow overlapped measurements (Λ-type counter)



max. no. of overlapped measures (Λ-type counter) $N = \tau\nu_b$

Classical variance

$$(\sigma_y^2)_q = \frac{1}{N}\;\frac{\nu_b^2}{\nu_0^2}\;\frac{2\sigma_x^2}{\tau^2}$$

quantization

equivalent clock frequency

$$\sigma_q^2 = \frac{T_q^2}{12} = \frac{1/\nu_{\text{ck}}^2}{12}$$

$$(\sigma_y^2)_q = \frac{2\,\sigma_q^2\,\nu_b}{\nu_0^2}\;\frac{1}{\tau^3}$$

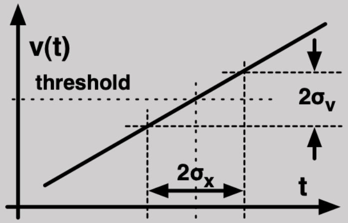
Note the $1/\tau^3$ law (Λ counter & white noise)

Additive white noise

$$\sigma_x = \frac{\sigma_v}{dv/dt}$$

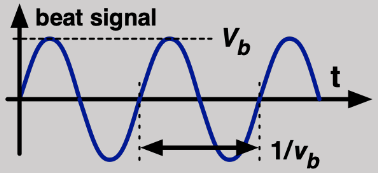
additive noise

slew rate



sinusoidal signal of peak amplitude V_b

$$\sigma_x = \frac{\sigma_v}{2\pi V_b\, \nu_b}$$



max. no. of overlapped measures (Λ-type counter) $N = \tau\nu_b$

$$(\sigma_y^2)_a = \frac{1}{N}\,\frac{\nu_b^2}{\nu_0^2}\,\frac{2\sigma_x^2}{\tau^2}$$

Classical variance

$$(\sigma_y^2)_a = \frac{2\,\sigma_v^2}{4\pi^2 V_b\, \nu_0^2\, \nu_b}\,\frac{1}{\tau^3}$$

Note the $1/\tau^3$ law
(Λ counter & white noise)

Optimum beat frequency – A

This is about white noise -- do not forget flicker

Classical variance

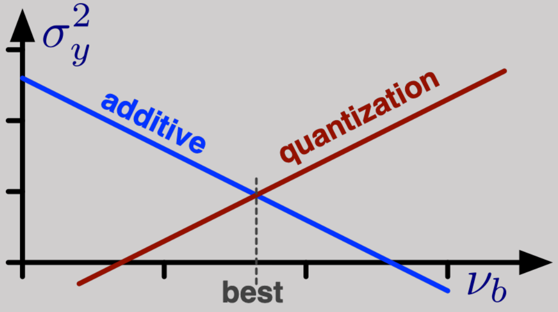
$$(\sigma_y^2)_a = (\sigma_y^2)_q$$

additive quantization

Case (A): σ_v is given
(cannot select the bandwidth)

$$\frac{2\,\sigma_v^2}{4\pi^2 V_b \nu_0^2\,\nu_b}\,\frac{1}{\tau^3} = \frac{2\,\sigma_q^2\,\nu_b}{\nu_0^2}\,\frac{1}{\tau^3}$$

$$(\nu_b)_{\text{best}} = \frac{1}{2\pi V_b}\,\frac{\sigma_v}{\sigma_q}$$

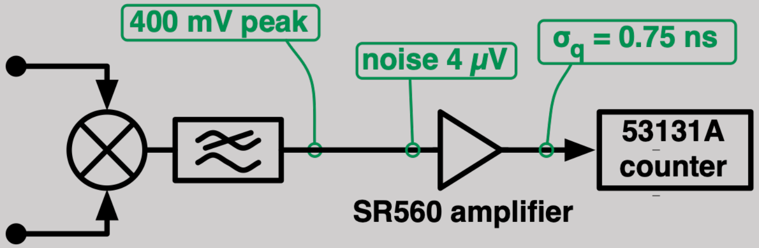


Example

SR560 amplifier (Stanford)
e_n = 4 nV/√Hz, B = 1 MHz

$\sigma_v = e_n \times \sqrt{B}$ = 4 µV
53131A counter (Agilent)

σ_q = 750 ps
V_b = 400 mV (mixer output)



$(\nu_b)_{\text{best}}$ = 2.1 kHz

Optimum beat frequency – B

Case (B): the noise PSD is given (we can select the bandwidth)

This is about white noise -- do not forget flicker

Variance

$$(\sigma_y^2)_a = \frac{2\,\sigma_v^2}{4\pi^2 V_b\,\tau^3 \nu_0^2\,\nu_b}$$

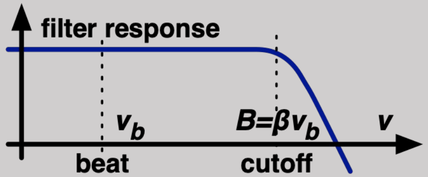
$$\sigma_v^2 = FkT_0\,R_0\,B$$

$$B = \beta\nu_b$$

B proportional to ν_b

$$V_b^2 = 2R_0P_b$$

sinusoidal beat note



$$(\sigma_y^2)_a = \frac{2\;\;FkT_0R_0\;\;\beta\nu_b}{4\pi^2\;2R_0P_b\;\nu_0^2\,\nu_b}\;\frac{1}{\tau^3}$$

Classical variance

$$(\sigma_y^2)_a = \frac{\beta}{4\pi^2\nu_0^2}\;\frac{FkT_0}{P_b}\;\frac{1}{\tau^3}$$

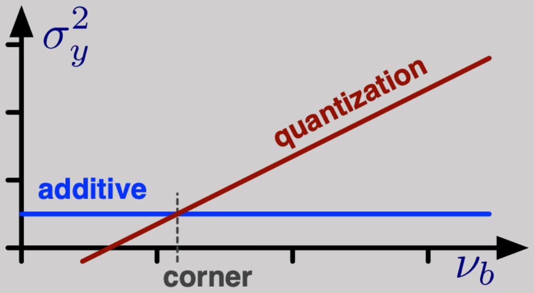
Corner frequency

$$\underset{\text{additive}}{(\sigma_y^2)_a} = \underset{\text{quantization}}{(\sigma_y^2)_q}$$

$$(\sigma_y^2)_q = \frac{2\,\sigma_q^2\,\nu_b}{\nu_0^2\,\tau^3}$$

$$\frac{\beta}{4\pi^2\nu_0^2}\;\frac{FkT_0}{P_b}\;\frac{1}{\tau^3} = \frac{2\,\sigma_q^2\,\nu_b}{\nu_0^2}\;\frac{1}{\tau^3}$$

$$(\nu_b)_{\text{corner}} = \frac{\beta}{4\pi^2}\;\frac{FkT_0}{P_b}\;\frac{1}{2\sigma_q^2}$$



Example

53131A counter
(Agilent)

σ_q = 750 ps

ν_0 = 100 MHz
P_b = 1 mW
F = 2 (3 dB)
β = 10

$(\sigma_y)_a = 1.42 \times 10^{-17}$ @ τ = 1 s

$(\nu_b)_{\text{corner}}$ = 1.8 Hz

Linear regression vs. Λ estimator