The Impact of Thermal Energy on Cross Spectrum PM Noise Measurements

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***Abstract*—Virtually all commercial instruments for the measurement of the oscillator PM noise make use of the Cross Spectrum method (arXiv:1004.5539 [physics.ins-det], 2010). High sensitivity is achieved by correlation and averaging on two equal channels which measure the same input, and reject the background noise. We show that a systematic error always present if the thermal energy of the input power splitter is not accounted for. Such error can result in noise *under estimation* up to a few dB in the lowest-noise quartz oscillators, and in a complete nonsense in the case of cryogenic oscillators.**

I. Introduction and State of the Art

The dual-channel scheme shown in Fig 1 is de facto the standard method for the measurement of the oscillator phase noise, adopted by most manufacturers of instruments. The main reason is that the background noise (reference oscillators and phase detectors) is averaged out. Modern digital electronics provides 'killer' averaging power for cheap, compared to the cost of the RF section.

This method derives from early works in radio astronomy [1] and from the masurement of frequency fluctuations in H masers [2]. It was used for the measurement of PM noise by Walls et al. [3], still using fully analog electronics. However, cross spectrum become practical only after that the availability of commercial FFT analyzers [4].

Since, technical and commercial development come. The method was left aside by the scientific community, and had been absent from the literature for a long time. Our early works on the correlation interferometer [5], [6] are not a true exception, to the extent that we focused only two port components. Yet, we demonstrated a background of -210 dBc/Hz (white) and -175 dBc/Hz at 1 Hz offset (flicker).

Recently, Nelson et al. [7] come up with simulations and a collection of 'ill-looking' spectra, pointing out the presence of a problem. Then, we tackled the problem at two two workshops [8], [9], yet without coming to a clear conclusion.

The fact the thermal noise in the power splitter results in negative correlation is at the hart of radiometry and thermometry [10], [11]. In our early correlation interferometers [5], [6] the phase noise floor is limited by thermal homogeneity and crosstalk, rather than by the room temperature. However, the idea that this applies to the cross-spectrum measurement of oscillators, resulting in the under-estimation of noise, was suggested by Joe Gorin at the 2015 workshop [9], and later analyzed by Hati & al [12]. Working in parallel we come to similar conclusion, yet with significantly different analytical and experimental methods.

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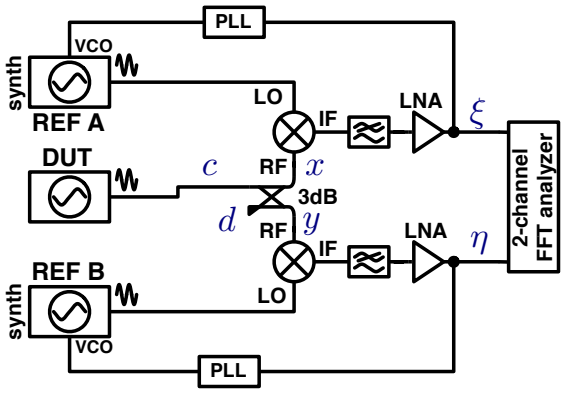

Fig. 1. Dual-channel phase noise measurement system.

A. Short Summary of the Whole Article

We report on a bias error inherent in the thermal energy of the power splitter at the instrument input. In the polynomial law representation of phase noise $S_\varphi = \sum_{n=-4}^{0} \mathsf{b}_n f^n$, the white noise is $S_\varphi = \mathsf{b}_0$. Neglecting the thermal energy of the power splitter, the instrument readout is

$$\mathsf{b}_0 = \frac{k(T_C - T_{\text{bias}})}{P_0} \qquad \text{(biased measure)} \qquad (1)$$

instead of

$$\mathsf{b}_0 = \frac{kT_C}{P_0} \qquad \text{(correct).} \qquad (2)$$

Here k is the Boltzmann constant, T_C is the equivalent temperature as seen at the oscillator output, P_0 is the carrier power, and T_{bias} is the systematic effect of the power splitter. Such bias, generally not accounted for in the laboratory practice, results in the under-estimation of the white PM noise by an amount of kT_{bias}/P_0. The temperature T_{bias} is equal, or of the order of, the physical temperature of the instrument.

II. Dual-Channel Phase Noise Measurement

A. Simplified Analysis

Two equal and statistically independent channels measure the DUT noise $\varphi(t)$, each adding its own background noise (Fig. 1). The phase-detector outputs are

$$\xi = k_d\varphi + n_1 \qquad \leftrightarrow \qquad \mathcal{X} = k_d\Phi + \mathcal{N}_1 \qquad (3)$$

$$\eta = k_d\varphi + n_2 \qquad \leftrightarrow \qquad \mathcal{Y} = k_d\Phi + \mathcal{N}_2 \qquad (4)$$

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where k_d is the phase-to-voltage gain, the symbol "$\leftrightarrow$" stands for the Fourier transform inverse-transform pair, and time and frequency (t and f) are implied. The single-sided cross PSD is

$$S_{\eta\xi} = \frac{2}{\mathcal{T}} \mathcal{Y}\mathcal{X}^* \qquad \text{(cross PSD)} \tag{5}$$

where the factor '2' accounts for the power at negative frequencies, $\mathcal{T}$ is the acquisition time for each realization (we may let $\mathcal{T} \to \infty$ in theoretical issues), and the superscript '$*$' means complex conjugate. After averaging out the single-channel noise, the cross spectrum is

$$S_{\eta\xi} = k_d^2 \frac{2}{\mathcal{T}} |\Phi|^2 = k_d^2 S_\varphi \tag{6}$$

The equation generally used for the instrument readout is

$$S_\varphi = \frac{1}{k_d^2} S_{\eta\xi} \qquad \text{(readout)}. \tag{7}$$

B. Spectral Estimation

However simple, the proof of (6) provides insight. We introduce the notation $\mathbb{E}\{\,\}$ for the mathematical expectation; $\langle\,\rangle_m$ for the average on m realizations; the 'hat' accent, as in $\hat{S}$, for the estimator; and the superscript 'prime' and 'second' for the real and imaginary part of a variable, as in $\Phi = \Phi' + i\Phi''$. Expanding (5) we find

$$\begin{aligned}\Re\{S_{\eta\xi}\} = \frac{2}{\mathcal{T}}\Big\{ & k_d^2\big(\Phi'^2 + \Phi''^2\big) + \\ & + k_d\big[\Phi'\big(\mathcal{N}_1' + \mathcal{N}_2'\big) + \Phi''\big(\mathcal{N}_1'' + \mathcal{N}_2''\big)\big] + \\ & + \mathcal{N}_1'\mathcal{N}_2' + \mathcal{N}_1''\mathcal{N}_2''\Big\}\end{aligned} \tag{8}$$

$$\begin{aligned}\Im\{S_{\eta\xi}\} = \frac{2}{\mathcal{T}}\Big\{ & k_d\big[\Phi'\big(-\mathcal{N}_1'' + \mathcal{N}_2''\big) + \Phi''\big(\mathcal{N}_1' - \mathcal{N}_2'\big)\big] + \\ & + \mathcal{N}_1'\mathcal{N}_2'' - \mathcal{N}_1''\mathcal{N}_2'\Big\}.\end{aligned} \tag{9}$$

Still not accounting for thermal energy in the power splitter, we can assume that Φ, $\mathcal{N}_1$ and $\mathcal{N}_2$ are statistically independent. Taking the expectation of (8) and (9), we get

$$\mathbb{E}\big\{S_{\eta\xi}\big\} = k_d^2 \frac{2}{\mathcal{T}} \mathbb{E}\big\{|\Phi|^2\big\} + i0 \qquad \text{(expectation)}. \tag{10}$$

The term '$i0$' in (10) emphasizes the fact that all the useful signal (the DUT noise) goes in $\Re\{S_{\eta\xi}\}$.

Actual measurements rely on an estimator. After (7) and (10)

$$\hat{S}_\varphi = \frac{1}{k_d^2} \big\langle \Re\{S_{\eta\xi}\} \big\rangle_m \qquad \text{(estimator)} \tag{11}$$

is an obvious choice because this is estimator is unbiased

$$\mathbb{E}\{\hat{S}_{\eta\xi}\} = \mathbb{E}\{\big\langle \Re\{S_{\eta\xi}\} \big\rangle_m\} = \mathbb{E}\,\{S_{\eta\xi}\}$$

III. Thermal Energy in the Input Power Splitter

Starting from this Section, we analyze the thermal noise of the input power splitter, and its consequences on the measurement of S_φ. The simplest way to understand the problem is to focus on thermal noise associated to the RF signals x and y of Fig. 1 (see also Fig. 2, discussed later). The PSD of the available voltage is equal to kTR_0, where $k = 1.38\times10^{-23}$ is the Boltzmann constant, T is the equivalent temperature, and R_0 the characteristic impedance. For reference, the thermal emf in 1 Hz bandwidth e_n across a resistor R is ruled by $\mathbb{E}\{e_n^2\} = 4kTR$, thus e_n is of 0.9 $\mathrm{nV}/\sqrt{\mathrm{Hz}}$ with $R = 50\ \Omega$ at room temperature.

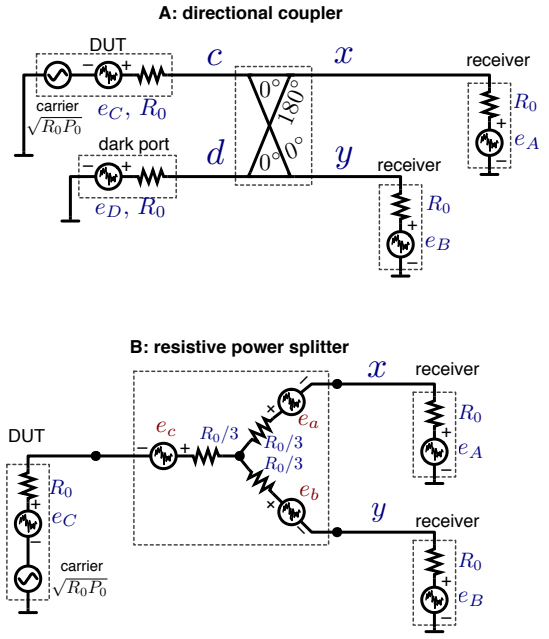

Fig. 2. Power splitters.

Following this approach, the phase detector is seen as a *receiver* described in terms of *back radiation* temperature $T_R^\star$, the same for the two channels. The reason is that the noise radiated back from the output generates crosstalk if the channels are not isolated, and in turn may contribute to the background noise. By contrast, the 'regular' noise temperature T_R give no information of the fraction of noise averaged out or kept through crosstalk. The receiver can be a double balanced mixer as in Fig. 1, or a more complex phase detector, as the interferometer [5], [6].

The problem is therefore to estimate the additive noise $S_c = kT_CR_0$ of signal c from $S_{\eta\xi}$, and then to estimate S_φ using

$$S_\varphi(f) = \frac{kT_C}{P_0} = \frac{S_c}{R_0P_0} \tag{12}$$

where P_0 is the carrier power.

A. Loss-Free Power Splitter

The 4-port directional coupler terminated at one input (dark port) is by far the preferred power splitter (Fig. 2 A). Dropping the carrier, the coupler output signals are

$$x = \tfrac{1}{2\sqrt{2}}\big(e_C - e_D\big) + \tfrac{1}{2}e_A$$

$$y = \tfrac{1}{2\sqrt{2}}\big(e_C + e_D\big) + \tfrac{1}{2}e_B$$

where the terms 'e' are the resistors' thermal emfs [$\mathrm{V}/\sqrt{\mathrm{Hz}}$], all statistically independent. The equivalent temperature is

$T_C = PS_\varphi/k$ for the oscillator's noise floor, and T_D for the dark port. Trite calculation gives

$$\mathbb{E}\{S_{yx}\} = \frac{1}{8}\Big[\mathbb{E}\left\{e_C^2\right\} - \mathbb{E}\left\{e_D^2\right\}\Big]$$

$$= \frac{1}{2}\mathbb{E}\big\{S_c\big\} - \frac{1}{2}\mathbb{E}\big\{S_d\big\} \tag{13}$$

$$= \frac{1}{2}k\Big[T_C - T_D\Big]R_0 \tag{14}$$

Equation (14) is the physical principle of the correlation radiometer [10], and also used in Johnson thermometry [11]. In this case, the instrument measures the temperature $\Delta T = T_C - T_D$ using the estimator

$$\widehat{\Delta T} = \frac{2}{kR_0}\big\langle \Re\big\{S_{yx}\big\}\big\rangle_m \qquad \text{(thermometer, radiometer)} \tag{15}$$

Back to phase noise, the rigorous evaluation of phase noise results from (12) and (13), which gives the unbiased estimator

$$\hat{S}_\varphi = \frac{2\hat{S}_{yx}}{R_0P_0} + \frac{kT_D}{P_0}$$

$$= \frac{2\hat{S}_{\eta\xi}}{k_d^2R_0P_0} + \frac{kT_D}{P_0} \qquad \text{(unbiased estimator)} \tag{16}$$

By contrast, discarding the thermal energy of the dark port results in the biased estimator

$$\hat{S}_\varphi = \frac{2\hat{S}_{yx}}{R_0P_0} = \frac{2\hat{S}_{\eta\xi}}{k_d^2R_0P_0} \qquad \text{(biased estimator)} \tag{17}$$

The bias, defined as $\Delta S_\varphi = \hat{S}_\varphi - \mathbb{E}\{S_\varphi\}$ and given by

$$\Delta S_\varphi = -\frac{kT_D}{P_0} \qquad \text{(bias)} \tag{18}$$

results in a systematic *under-estimation* of the DUT noise.

B. Resistive Power Splitter

The Y resistive power splitter (Fig. 2 B) is sometimes used instead of the directional coupler, for example in the Keysight E5500 series [13]. Dropping the carrier, the signals at the splitter output are

$$x = \tfrac{1}{2}e_A + \tfrac{1}{4}e_B + \tfrac{1}{4}e_C - \tfrac{1}{2}e_a + \tfrac{1}{4}e_b + \tfrac{1}{4}e_c \tag{19}$$

$$y = \tfrac{1}{4}e_A + \tfrac{1}{2}e_B + \tfrac{1}{4}e_C + \tfrac{1}{4}e_a - \tfrac{1}{2}e_b + \tfrac{1}{4}e_c \tag{20}$$

The emfs 'e' are all statistically independent, and governed by $\mathbb{E}\{e^2\} = 4kTR$, with $R = R_0$ for the DUT and the receivers, $R = R_0/3$ for the splitter's internal resistors, $T = T_C$ for the DUT, $T = T_R^\star$ for the receivers, and $T = T_S$ for the splitter's internal resistors. The cross PSD at the splitter output is

$$\mathbb{E}\{S_{yx}\} = k\left(\tfrac{1}{4}T_C - \tfrac{1}{4}T_S + T_R^\star\right)R_0 \tag{21}$$

$$= \tfrac{1}{4}S_c + k\left(T_R^\star - \tfrac{1}{4}T_S\right)R_0 \tag{22}$$

The unbiased estimator is obtained combining (12) and (22)

$$\hat{S}_\varphi = \frac{4\hat{S}_{yx}}{R_0P_0} + \frac{k(T_S - 4T_R^\star)}{P_0}$$

$$= \frac{4\hat{S}_{\eta\xi}}{k_d^2R_0P_0} + \frac{k(T_S - 4T_R^\star)}{P_0} \qquad \text{(unbias. estimator)} \tag{23}$$

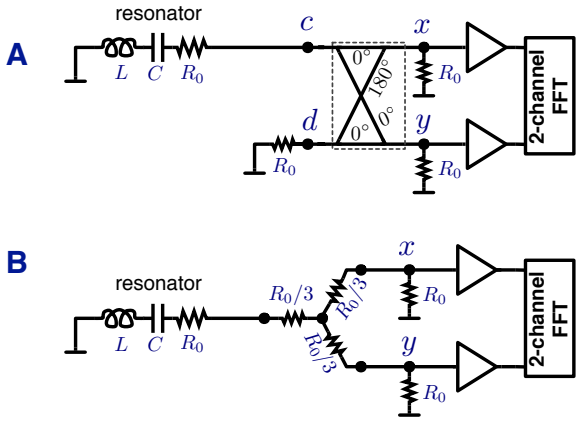

Fig. 3. Experimental configurations.

where $\hat{S}_{\eta\xi}$ is given by (11). Neglecting the splitter's thermal energy results in the biased estimator

$$\hat{S}_\varphi = \frac{4\hat{S}_{yx}}{R_0P_0} = \frac{4\hat{S}_{\eta\xi}}{k_d^2R_0P_0} \qquad \text{(biased estimator)} \tag{24}$$

whose bias is given by

$$\Delta S_\varphi = -\frac{k(T_S - 4T_R^\star)}{P_0} \qquad \text{(bias)}. \tag{25}$$

Unlike the directional coupler, the systematic error can be either positive or negative.

IV. The Challenging Oscillators

Lowest white noise at a given physical temperature is achieved by bandpass filtering between the core oscillator and the output buffer, and with a special design of the output buffer. A single resonator is used as the reference resonator and as the output filter. This design solves the issue of harmonic distortion and circumvents the white phase noise of the oscillator [14] (See also [15, p. 264] for the electrical diagram of a complete oscillator). There results a white floor just above the thermal floor.

The key point is that the the current flowing in the quartz is transferred to the output with a minimum noise contribution of the buffer, which is a common-base amplifier. Out of the quartz bandwidth, the motional resistance is not coupled to the buffer, and the white noise of the sustaining amplifier is not transferred to the buffer.

V. Experiments

Given the additive nature of the white PM noise, the presence of a carrier signal is not necessary, and we can work with the RF noise trusting $\mathsf{b}_0 = kT/P$. Our experiments (Fig 3) are inspired to the internal configuration of the low-noise oscillators, where a resonator is used as the output filter [14].

Having said that, we decided to work on mockup where the frequency is scaled down by a factor of 10^5, i.e., 1 kHz instead of 100 MHz. The reason for this choice is that we have full control on the receiver (regular) noise temperature T_R and on

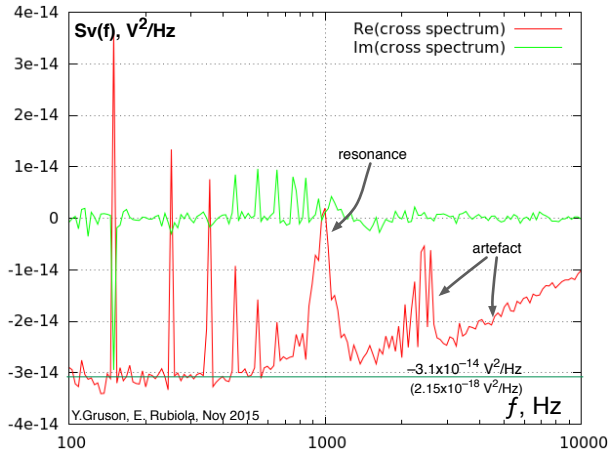

Fig. 4. Cross spectrum measured with the configuration of Fig. 3 A.

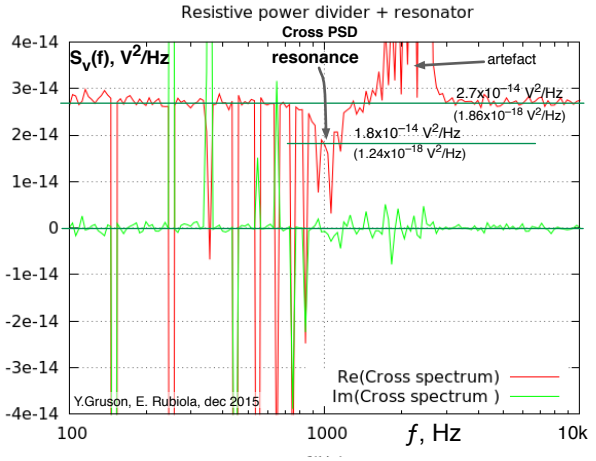

Fig. 5. Cross spectrum measured with the configuration of Fig. 3 B.

the back-scatter temperature T_R^*. In fact, working with JFET operational amplifiers (AD 743) at low R_0 (< 10 kΩ) as the receiver, the input current noise is negligible, and gives rise to no crosstalk even with large-size averaging. Thus it holds that $T_R^* = T_{\text{ph}} \approx 300$ K, the physical temperature of the amplifier. The noise temperature T_R is of the order of 1500 K, which is rejected by averaging the cross spectrum.

All the experiments are done in a shielded chamber with proportional-integral control of temperature and humidity, and the critical circuits are further shielded in a mumetal enclosure borrowed from an atomic-clock experiment.

A. First Experiment

In the first experiment (Fig. 3 A) we use a custom directional coupler based on traditional transformers with laminated silicon steel core. Recycling surplus parts, we ended up with a trivial $1:\sqrt{2}$ voltage ratio, hence R_0 is of 300 Ω on the left-hand side and of 600 Ω on the right-hand side. The resonator is implemented with a 470 mH olla ferrite inductor and a mylar capacitor, resonating at 1 kHz with $Q = 5$ (loaded) and $R_0 = 300$ Ω.

The spectrum, obtained after a few thousands of FFT acquisitions for convergence, is shown on Fig. 4. At the resonance, the coupler is loaded to two equal resistors R_0 at the same temperature. The cross-spectrum S_{yx} is equal to zero, as predicted by Eq. (14). Off resonance, the resonator is seen as open circuit ($\Gamma = 1$), for there is no thermal noise to x and y. Conversely, the signal d is anti-correlated at the two outputs, as seen from the phase relationships 0° and 180° on the coupler. The real part $\Re\{S_{yx}\}$ is of about -2.2×10^{-18} V^2/Hz. This is in fairly good agreement with the value of -2.5×10^{-18} predicted by Eq. (14) with $R_0 = 600$ Ω (at the right-hand side of the coupler). Notice that $\Im\{S_{yx}\}$ is close to zero at all frequencies, as expected.

B. Second Experiment

The second experiment (Fig. 3 B) uses a Y power splitter implemented with three $R_0/3 = 100$ Ω metal-film resistors, while the resonator is the same as in the first experiment.

At the resonance, Eq. (21) predicts that $S_{yx} = kT_{\text{ph}}R_0$ when the whole system is at the physical temperature T_{ph}. The value of 1.25×10^{-18}, seen on Fig. 5, is in a close agreement to Eq. (21) with $T_{\text{ph}} = 300$ K and $R_0 = 300$ Ω.

Off resonance, e_c turns into open circuit, and the system changes configuration. From Fig. 2 B, we get

$$S_{yx} = k\left[\frac{15}{8}T_R^* - \frac{3}{8}T_S\right]R_o$$

assuming that back-scatter temperature of the two receivers is the same and equal to T_R^*, and that the power splitter is at the temperature T_S. When the whole system is at the physical temperature T_{ph}, it holds that

$$S_{yx} = \frac{3}{2}kT_{\text{ph}}R_0$$

The value observed on Fig. 5, $S_{yx} = 1.86\times10^{-18}$ V^2/Hz is in a close agreement to the above formula, again evaluated with $T_{\text{ph}} = 300$ K and $R_0 = 300$ Ω. Notice that $\Im\{S_{yx}\}$ is close to zero at all frequencies, as expected. And of course $\Im\{S_{yx}\}$ is close to zero at all frequencies.

VI. Conclusion

The internal termination of the input power splitter introduces a systematic error ΔS_φ given by Eq. (18) for the loss-free directional coupler terminated at one end, and by Eq. (25) for the Y resistive splitter.

In most practical cases it holds that $T_C \gg T_{\text{ph}}$, thus the result is correct for any practical purpose. However, in the case of the low-noise oscillators using the filter, the the result can be grossly underestimated because T_C is just above T_{ph}.

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