The Magic of Correlation Measurements

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- Theory of the cross spectrum
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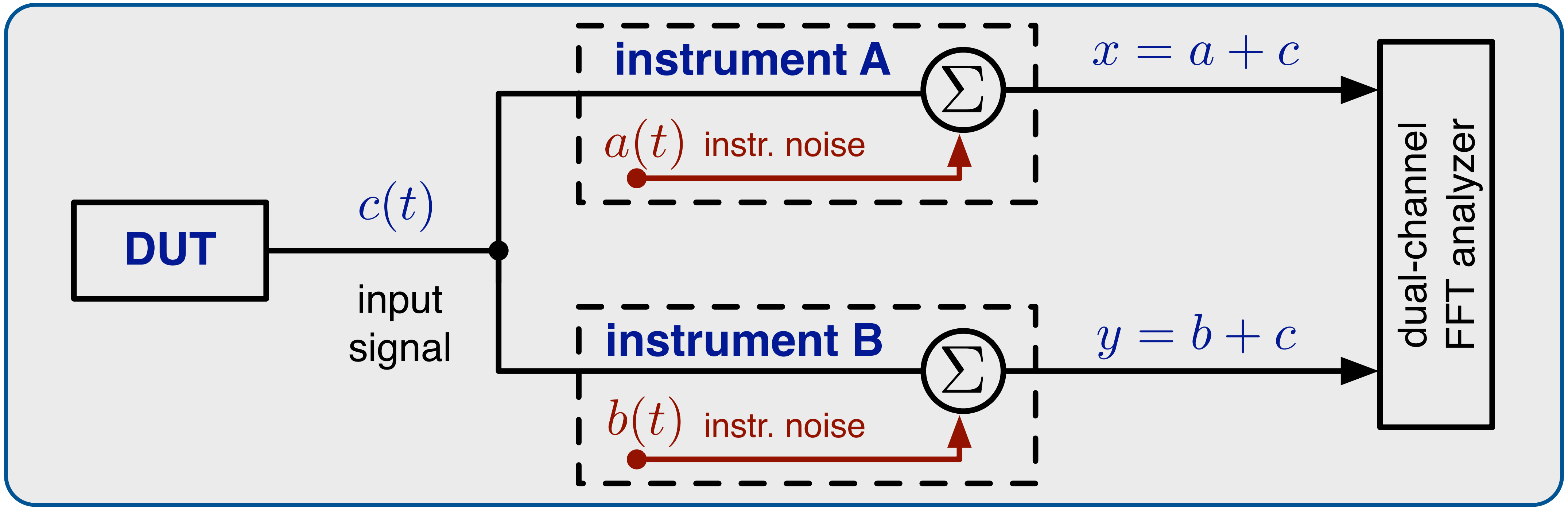
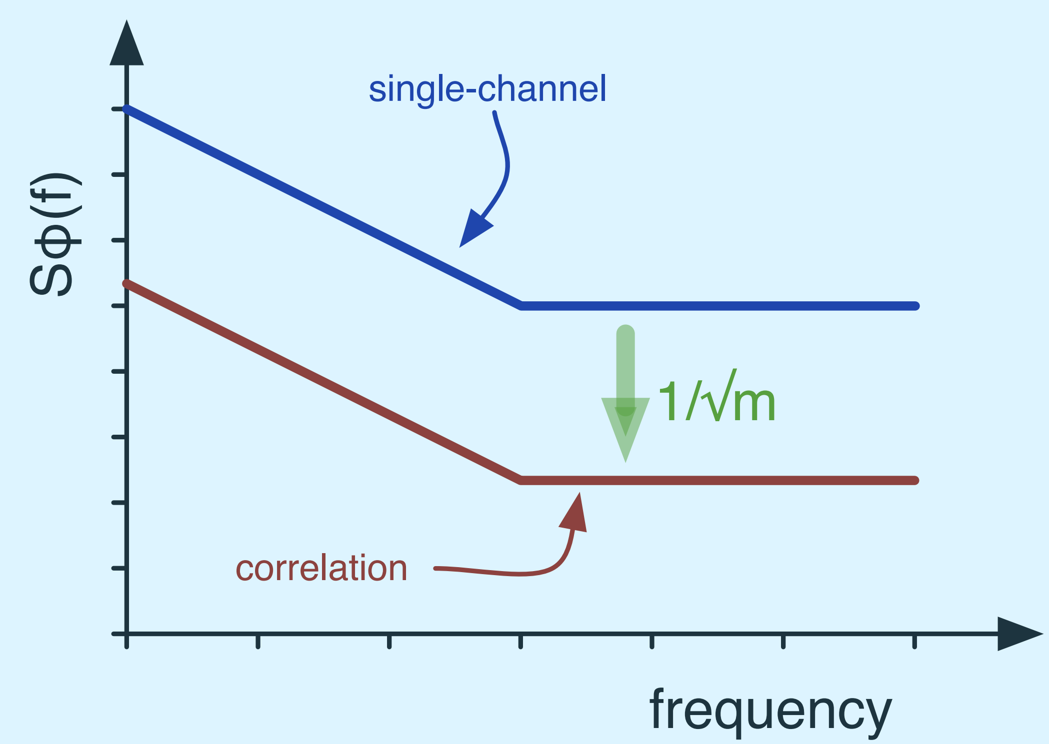
Correlation measurements



Two separate instruments measure the same DUT. Only the DUT noise is common

noise measurements		
DUT noise, normal use	a, bc	instrument noise DUT noise
background, ideal case	a, bc = 0	instrument noise no DUT
background, real case	a, bc ≠ 0	c is the correlated instrument noise Zero DUT noise

a(t), b(t) –> instrument noise
c(t) –> DUT noise



Statistics

Exercise set no 1

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Boring but necessary

Vocabulary of statistics

- A ***random process*** **x(t)** is defined through a random experiment e that associates a function x_e(t) with each outcome *e*.
  - The set of all the possible x_e(t) is called *ensemble*
  - The function x_e(t) is called *realization* or *sample function*.
  - The ensemble average is called mathematical expectation $\mathbb{E}\{\ \}$
- A random process is said *stationary* if its statistical properties are independent of time.
  - Often we restrict the attention to some statistical properties.
  - In physics, this is the concept of repeatability.
- A random process x(t) said *ergodic* if a realization observed in time has the statistical properties of the ensemble.
  - Ergodicity makes sense only for stationary processes.
  - Often we restrict the attention to some statistical properties.
  - In physics, this is the concept of reproducibility.

Example: thermal noise of a resistor of value R

- The experiment e is the random choice of a resistor *e*
- The realization x_e(t) is the noise waveform measured across the resistor *e*
- We always measure $<x^2>=4kTRB$, so the process is stationary
- After measuring many resistors, we conclude that $<x^2>=4kTRB$ always holds. The process is ergodic.

Gaussian (normal) distribution

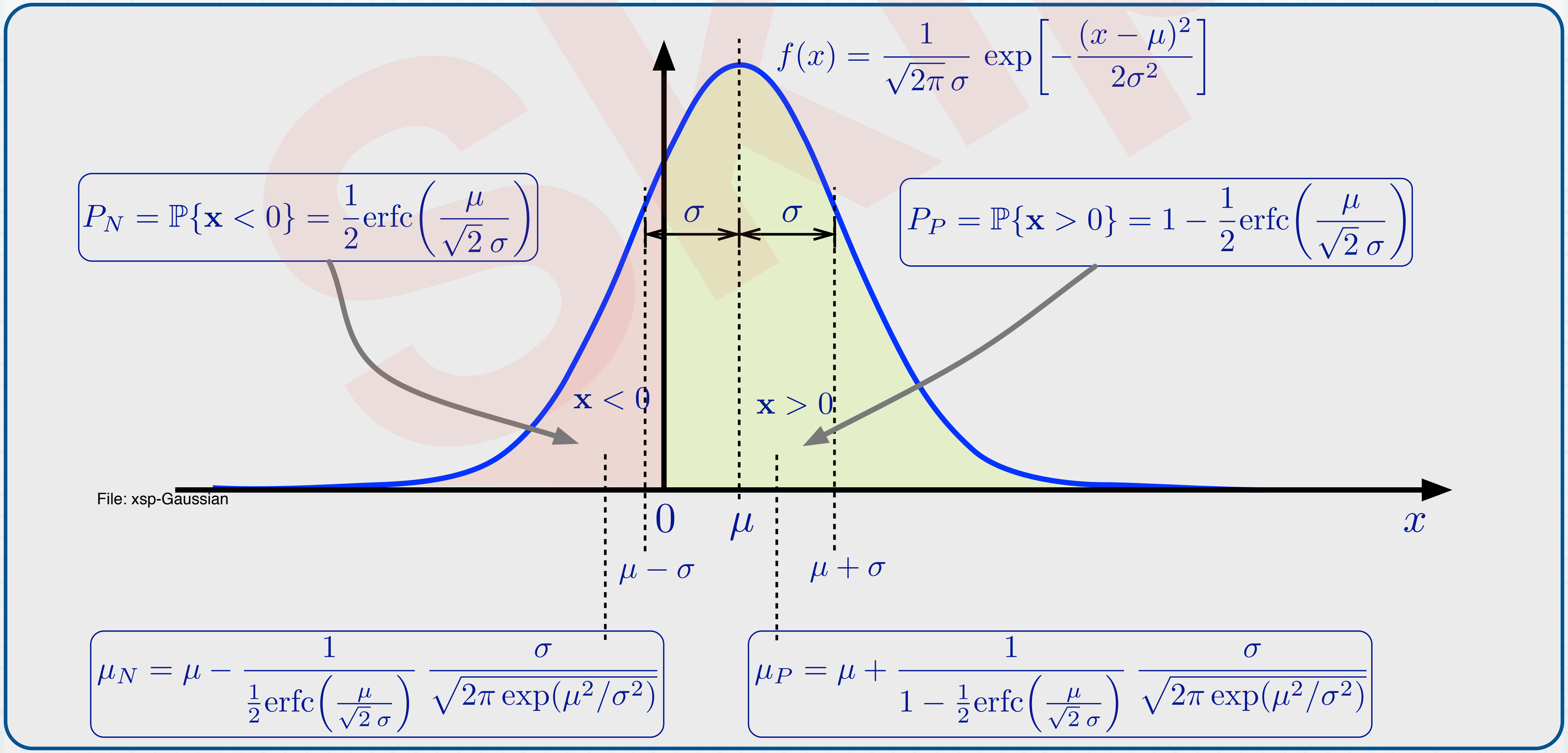
x is normal distributed with zero mean μ and variance σ²

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\mathbb{E}\{f(x)\} = \mu$$

$$\mathbb{E}\{f^2(x)\} = \mu^2 + \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



One-sided Gaussian distribution

x is normal distributed with zero mean and variance σ²

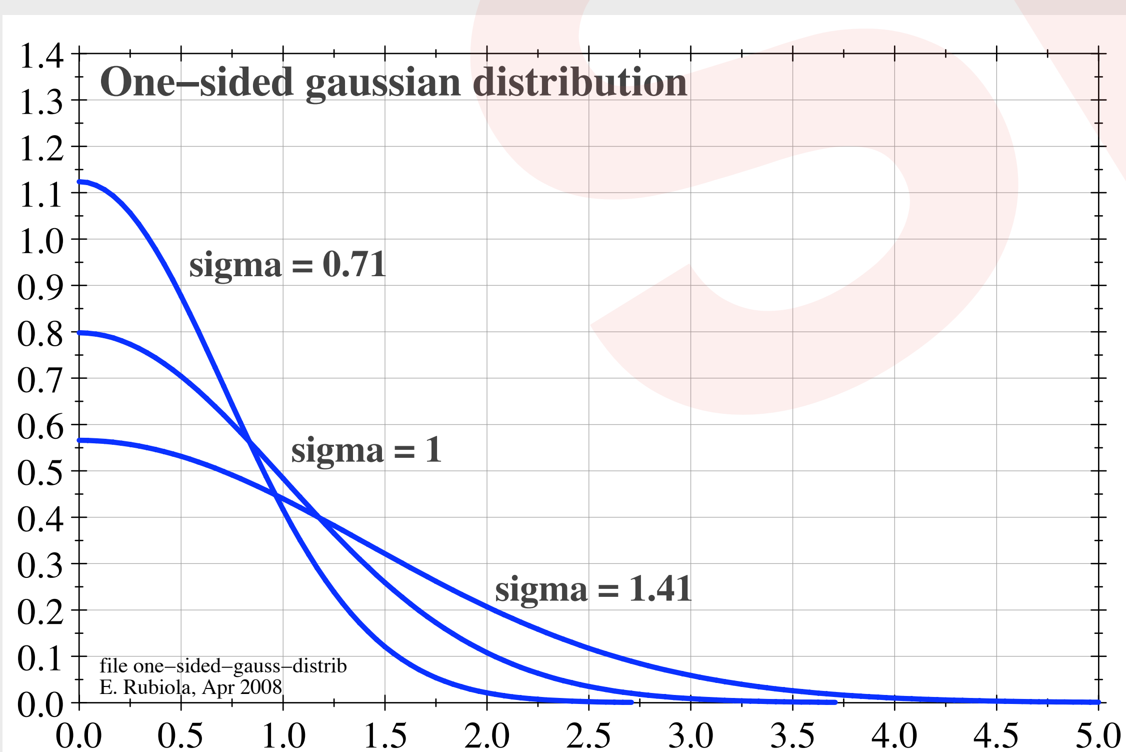
$$y = |x|$$

$$f(x) = 2\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \left(1 - \frac{2}{\pi}\right)\sigma^2$$



one-sided Gaussian distribution with $\sigma^2 = 1/2$

quantity with $\sigma^2 = 1/2$	value [$10\log(\ )$, dB]
average $= \sqrt{\dfrac{1}{\pi}}$	0.564 [−2.49]
deviation $= \sqrt{\dfrac{1}{2} - \dfrac{1}{\pi}}$	0.426 [−3.70]
$\dfrac{\text{dev}}{\text{avg}} = \sqrt{\dfrac{\pi}{2} - 1}$	0.756 [−1.22]
$\dfrac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\dfrac{1}{2} - \dfrac{1}{\pi}}$	1.756 [+2.44]
$\dfrac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\dfrac{1}{2} - \dfrac{1}{\pi}}$	0.244 [−6.12]
$\dfrac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \dfrac{1 + \sqrt{1/2 - 1/\pi}}{1 - \sqrt{1/2 - 1/\pi}}$	7.18 [8.56]

Chi-square (χ^2) distribution

DF = degrees of freedom

Definition

x_i are normal distributed variables
zero mean, and variance σ^2

$$\chi^2 = \sum_{i=1}^{r} x_i^2$$

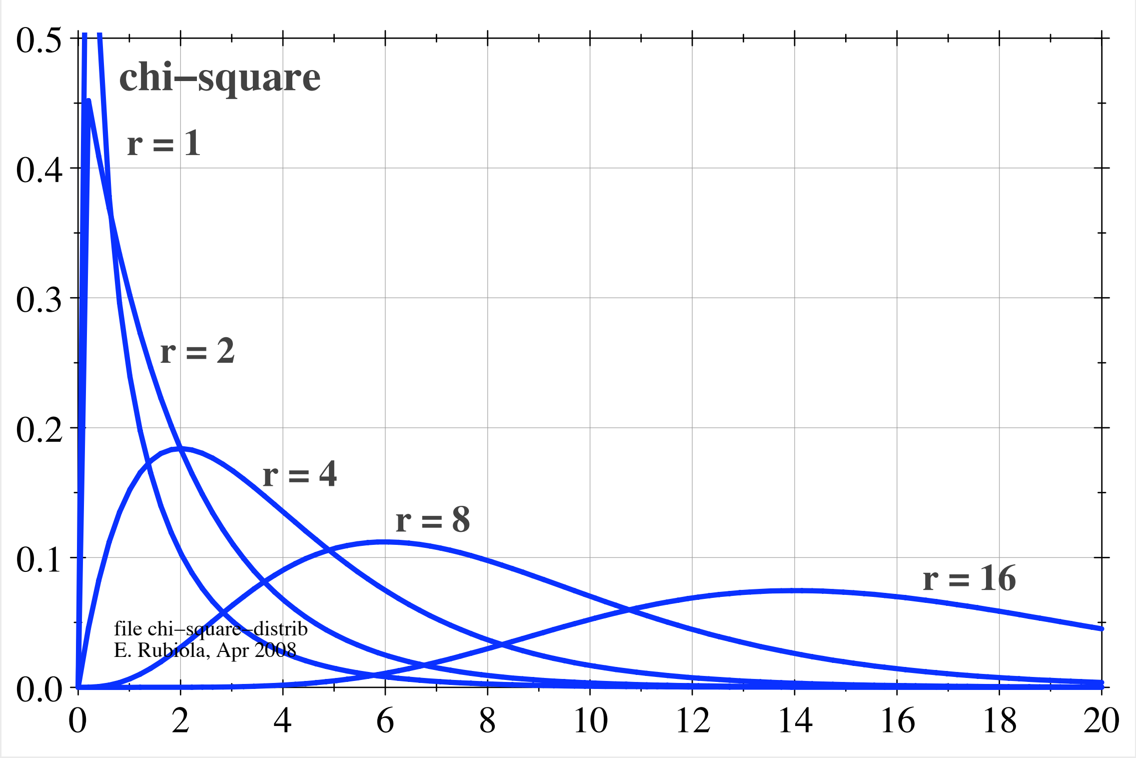
is χ^2 distributed with r DF

Sum

The sum of *m* χ^2-distributed variables

$$\chi^2 = \sum_{j=1}^{m} \chi_j^2 \;, \qquad r = \sum_{j=1}^{m} r_j$$

has χ^2 distribution with r = m DF



$$f(x) = \frac{x^{\frac{r}{2}-1}\, e^{-\frac{x^2}{2}}}{\Gamma\left(\frac{1}{2}r\right) 2^{\frac{r}{2}}} \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sigma^2 r$$

$$\mathbb{E}\{[f(x)]^2\} = \sigma^4 r(r+2)$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = 2\sigma^4 r$$

$$z! = \Gamma(z+1), \qquad z \in \mathbb{N}$$

Averaging m χ^2 complex variables

averaging m variables $|X|^2$, complex $X=X'+\imath X''$, yields a χ^2 distribution with r = 2m

$$\frac{1}{m}\chi^2 = \frac{1}{m}\sum_{j=1}^{m}(X_j')^2+(X_j'')^2$$

$$\mathbb{E}\left\{\frac{1}{m}f(x)\right\} = \frac{\sigma^2 r}{m} = 2\sigma^2$$

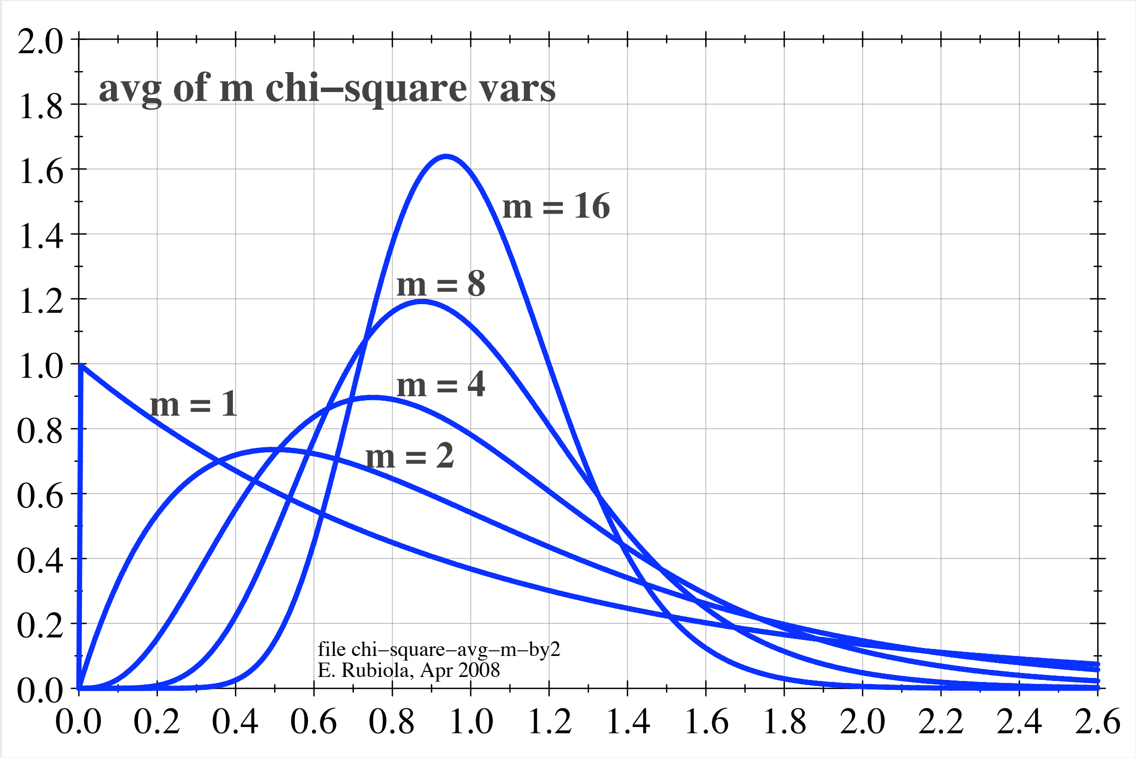
$$\mathbb{E}\left\{\left|\frac{1}{m}f(x)-\mathbb{E}\left\{\frac{1}{m}f(x)\right\}\right|^2\right\} = \frac{2\sigma^4 r}{m^2} = \frac{4\sigma^4}{m}$$

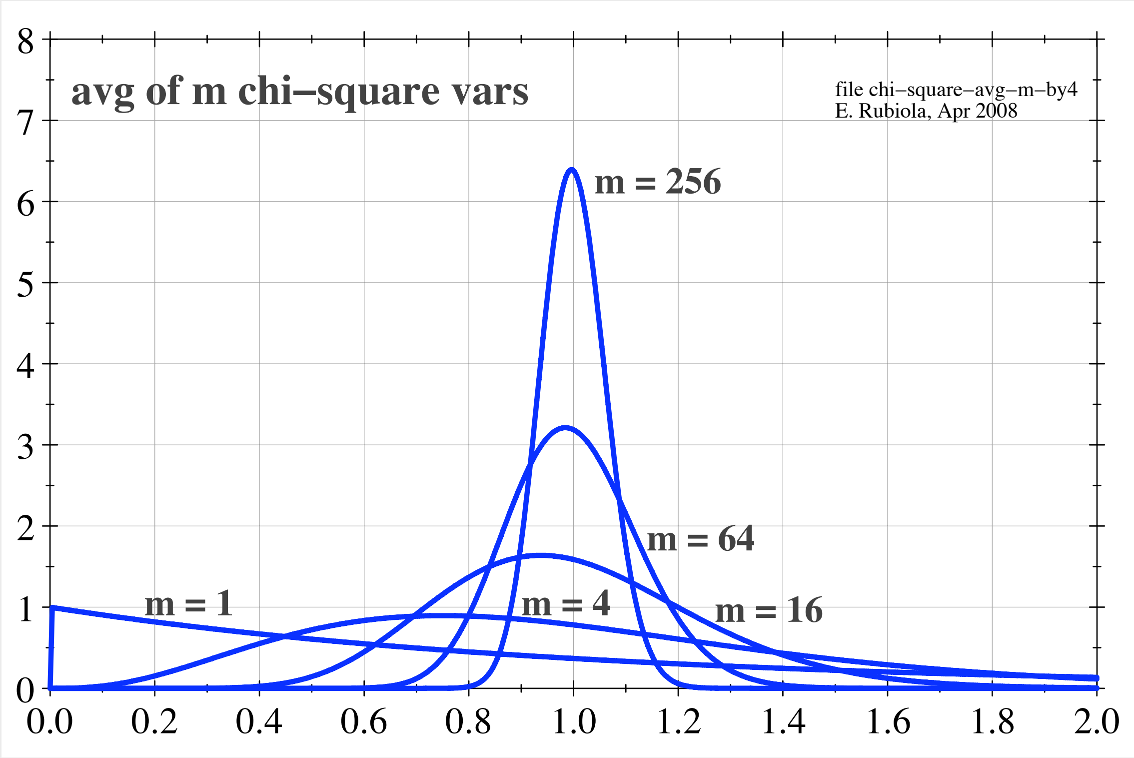
$$\frac{\text{dev}}{\text{avg}} = \frac{1}{\sqrt{m}}$$

relevant case: $\sigma^2 = 1/2$

$\text{avg} = 1$

$\text{dev} = \dfrac{1}{\sqrt{m}}$

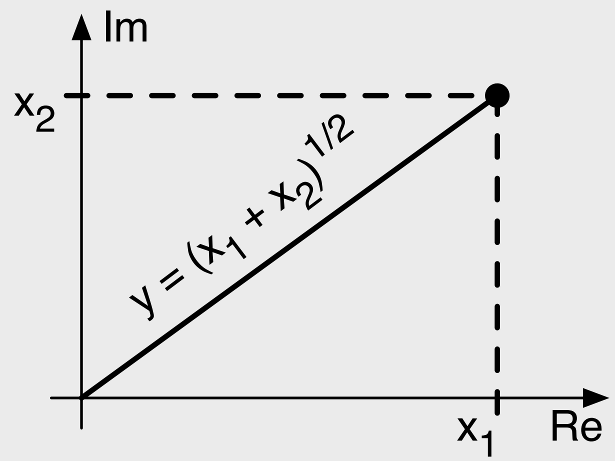




Rayleigh distribution

x_1 and x_2 are normal distributed with zero mean and equal variance σ^2

$$x = \sqrt{x_1^2 + x_2^2}$$



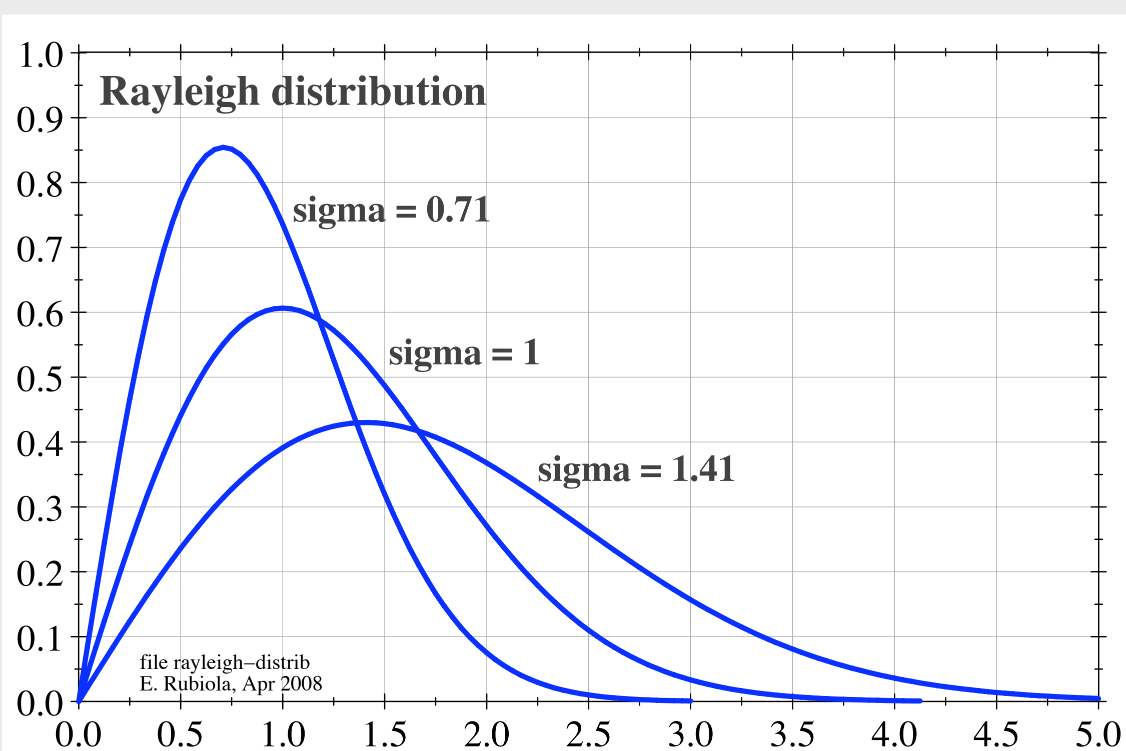
x is Rayleigh-distributed

$$f(x) = \frac{x}{\sigma^2}\,\exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{\pi}{2}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = 2\sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \frac{4-\pi}{2}\sigma^2$$



Rayleigh distribution with $\sigma^2 = 1/2$	
quantity with $\sigma^2 = 1/2$	value $[10\log(\ ),\ \mathrm{dB}]$
average $= \sqrt{\frac{\pi}{4}}$	0.886 $[-0.525]$
deviation $= \sqrt{1 - \frac{\pi}{4}}$	0.463 $[-3.34]$
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{4}{\pi} - 1}$	0.523 $[-2.82]$
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{4}{\pi} - 1}$	1.523 $[+1.83]$
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{4}{\pi} - 1}$	0.477 $[-3.21]$
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{4/\pi - 1}}{1 - \sqrt{4/\pi - 1}}$	3.19 $[5.04]$

Bessel K_0 distribution

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

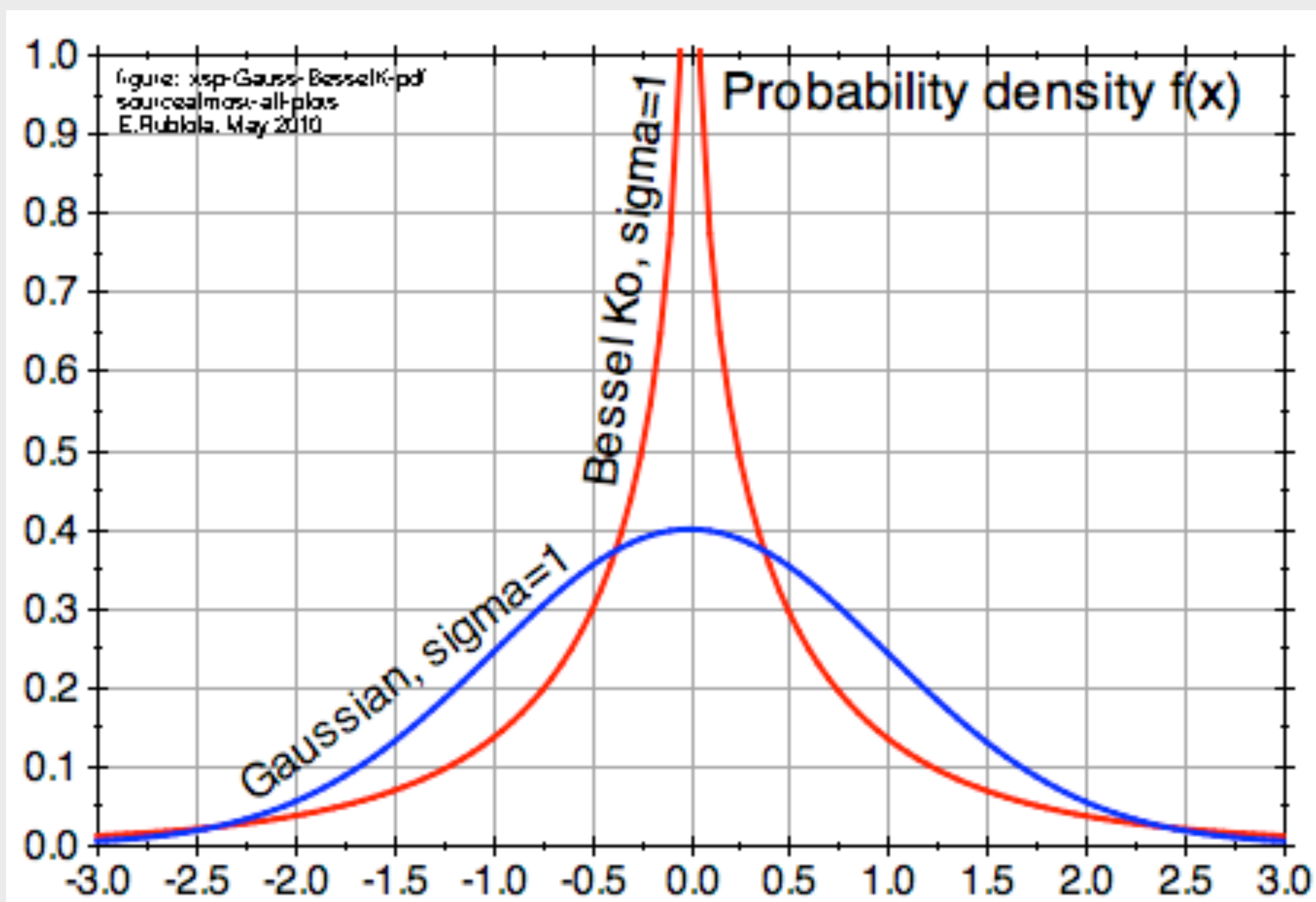
$$x = x_1\,x_2$$

x has Bessel K_0 distribution with variance $\sigma = \sigma_1^2\,\sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma}\,K_0\left(-\frac{|x|}{\sigma}\right)$$

$$\mathbb{E}\{f(x)\} = 0$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



A relevant property of random noise

A theorem states that

there is no a-priori relation between PDF[1] and spectral measure

For example, white noise can originate from

- Poisson process (emission of a particle at random time)
- Random telegraph (random switch between two level)
- Thermal noise (Gaussian)

(1) PDF = Probability Density Function

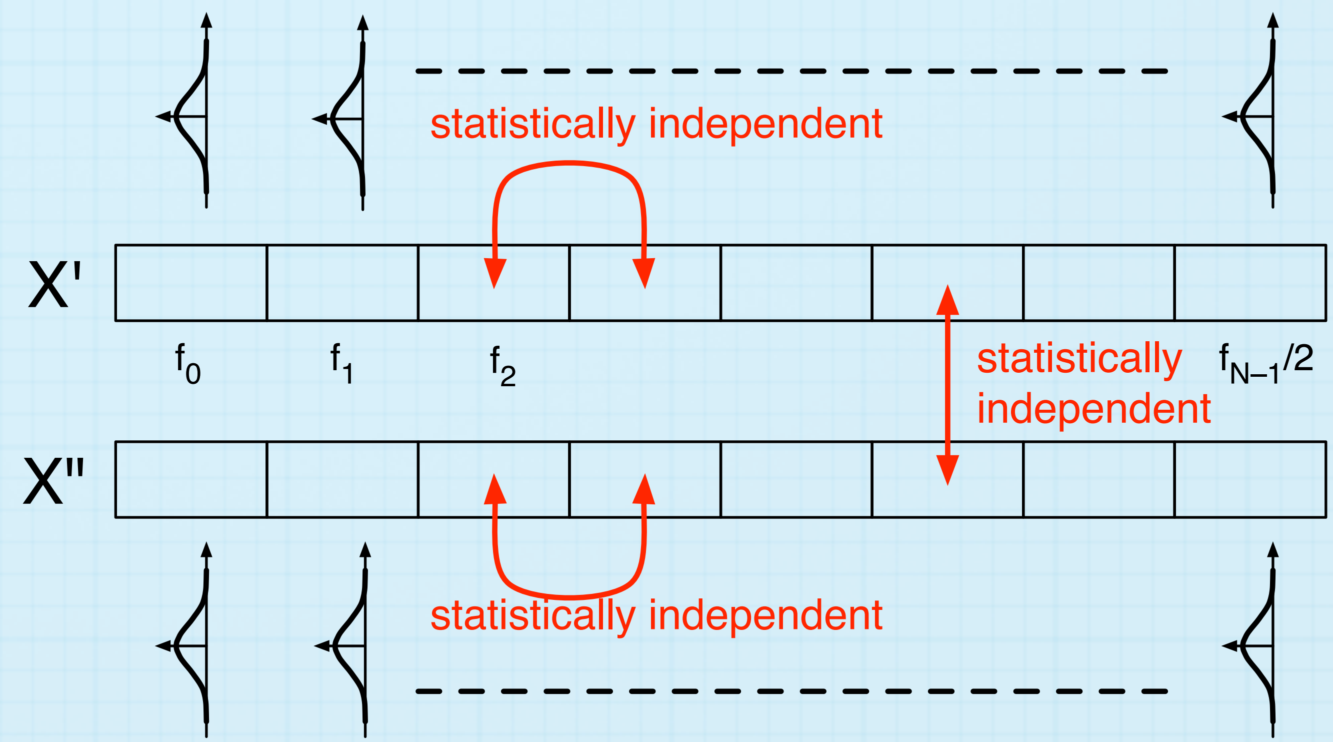
Why Gaussian White Noise?

- Whenever randomness occurs at microscopic level, noise tends to be Gaussian (central-limit theorem)
- Most environmental effects are not "noise" in strict sense (often, they are more *disturbing* than noise)
- Colored noise types ($1/f$, $1/f^2$, etc) can be whitened, analyzed, and un-whitened
- Of course, GW noise is easy to understand

Properties of Gaussian White noise with zero mean

$$x(t) \iff X(\imath f) = X'(\imath f) + \imath X''(\imath f)$$

1. $x(t) \iff X(\imath f)$ are Gaussian
2. $X(\imath f_1)$ and $X(\imath f_2)$, $f_1 \neq f_2$
    1. are statistically independent,
    2. $\mathrm{var}\{X(\imath f_1)\} = \mathrm{var}\{X(\imath f_2)\}$
3. real and imaginary part:
    1. X' and X'' are statistically independent
    2. $\mathrm{var}\{X'\} = \mathrm{var}\{X''\} = \mathrm{var}\{X\}/2$
4. $Y = X_1 + X_2$
    1. Y is Gaussian
    2. $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\} + \mathrm{var}\{X_2\}$
5. $Y = X_1 \times X_2$
    1. is Gaussian
    2. $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\}\,\mathrm{var}\{X_2\}$

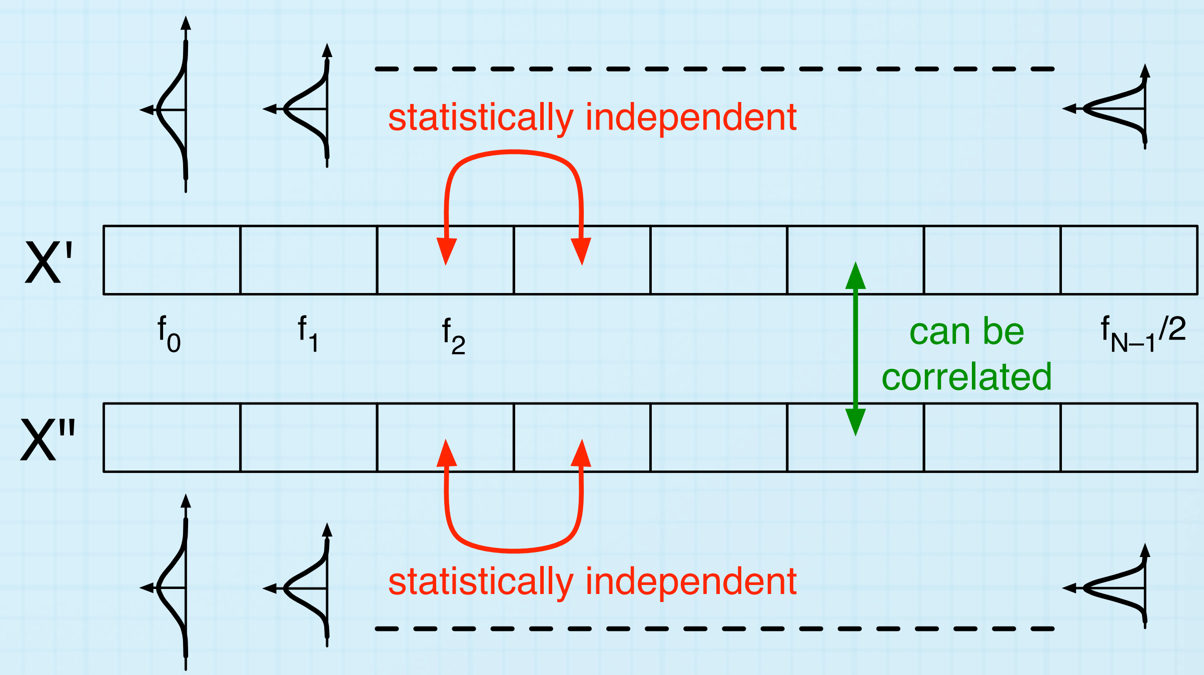


2N degrees of freedom

Properties of parametric noise

$x(t) \;\Leftrightarrow\; X(\imath f) \;=\; X'(\imath f) + \imath X''(\imath f)$

1. Pair $x(t) \Leftrightarrow X(\imath f)$
   1. there is no a-priori relation between the distribution of $x(t)$ and $X(\imath f)$ (theorem)
   2. Central limit theorem: $x(t)$ and $X(\imath f)$ end up to be Gaussian
2. $X(\imath f_1)$ and $X(\imath f_2)$
   1. generally, statistically independent
   2. $\mathrm{var}\{X(\imath f_1)\} \neq \mathrm{var}\{X(\imath f_2)\}$ in general
3. Real and imaginary part, same frequency
   1. X' and X'' can be correlated
   2. $\mathrm{var}\{X'\} \neq \mathrm{var}\{X''\} \neq \mathrm{var}\{X\}/2$
4. $Y = X_1 + X_2$, zero-mean independent Gaussian r.v.
   $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\} + \mathrm{var}\{X_2\}$
5. If X_1 and X_2 are zero-mean independent Gaussian r.v.
   1. $Y = X_1 \times X_2$ is zero-mean Gaussian
   2. $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\}\,\mathrm{var}\{X_2\}$



The process has N … 2N degrees of freedom, depending on correlation between X' and X"

Children of the Gaussian distribution

Chi-square

$\chi^2 = \sum_i x_i^2$

Bessel K_0

$x = x_1\, x_2$

Rayleigh

$x = \sqrt{(x_1^2+x_2^2)}$

Spectral measure[1] and estimation

Exercise set no 2 — Useful stuff

(1) Engineers call it Power Spectral Density (PSD)

The Spectral Measure

for stationary random process x(t)

$$C(\tau) = \mathbb{E}\{[\mathrm{x}(t+\tau) - \mu][\mathrm{x}^*(t) - \mu]\}$$

Autocovariance

Improperly referred to as the correlation and denoted with $R_{xx}(\tau)$

$$\mu = \mathbb{E}\{\mathrm{x}\}$$

$$S(\omega) = \mathcal{F}\{C(\tau)\} = \int_{-\infty}^{\infty} C(\tau)\, e^{-\imath\omega\tau} d\tau$$

Spectral measure (two-sided)

$$C(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} [x(t+\tau) - \mu][x^*(t) - \mu]\, dt$$

For ergodic process, interchange ensemble and time average

process x(*t*) –> realization *x*(*t*)

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T} X_T(\omega)\, X_T^*(\omega) = \lim_{T\to\infty} \frac{1}{T} \left|X_T(\omega)\right|^2$$

Wiener Khinchin theorem

for stationary ergodic processes

$$S^I(f) = 2S^{II}(\omega/2\pi)\,, \quad f > 0$$

In experiments we use the single-sided PSD

autocorrelation function

$$R_{\mathrm{xx}}(\tau) = \frac{1}{\sigma^2}\mathbb{E}\Big\{[\mathrm{x}(t) - \mu][\mathrm{x}(t-\tau) - \mu]\Big\}$$

Fourier transform

$$\mathcal{F}\{\xi\} = \int_{-\infty}^{\infty} \xi(t)\, e^{-\imath\omega t} dt$$

Sum of random variables

1. **The sum of Gaussian distributed random variables has Gaussian PDF**

2. **The central limit theorem states that**

   For large *m*, the PDF of the the sum of *m* statistically independent processes tends to a Gaussian distribution

   Let $X = X_1+X_2+\ldots+X_m$ be the sum of m processes of mean μ_1, μ_2, … μ_m and variance σ_1^2, σ_2^2, … σ_m^2. The process X has Gaussian PDF expectation $E\{X\} = \mu_1+\mu_2+\ldots+\mu_m$, and variance $\sigma^2 = \sigma_1^2+\sigma_2^2+\ldots+\sigma_m^2$

3. **Similarly, the average $\langle X\rangle_m = (X_1+X_2+\ldots+X_m)/m$ has Gaussian PDF, $E\{X\} = (\mu_1+\mu_2+\ldots+\mu_m)/m$, and $\sigma^2 = (\sigma_1^2+\sigma_2^2+\ldots+\sigma_m^2)/m$**

4. **Since white noise and flicker noise arise from the sum of a large number of small-scale phenomena, they are Gaussian distributed**

PDF = Probability Density Function

Product of independent zero-mean Gaussian-distributed random variables

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

$$x = x_1\, x_2$$

x has Bessel K_0 distribution with variance $\sigma = \sigma_1^2\, \sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma}\, K_0\left(-\frac{|x|}{\sigma}\right)$$

$$\mathbb{E}\{f(x)\} = 0$$

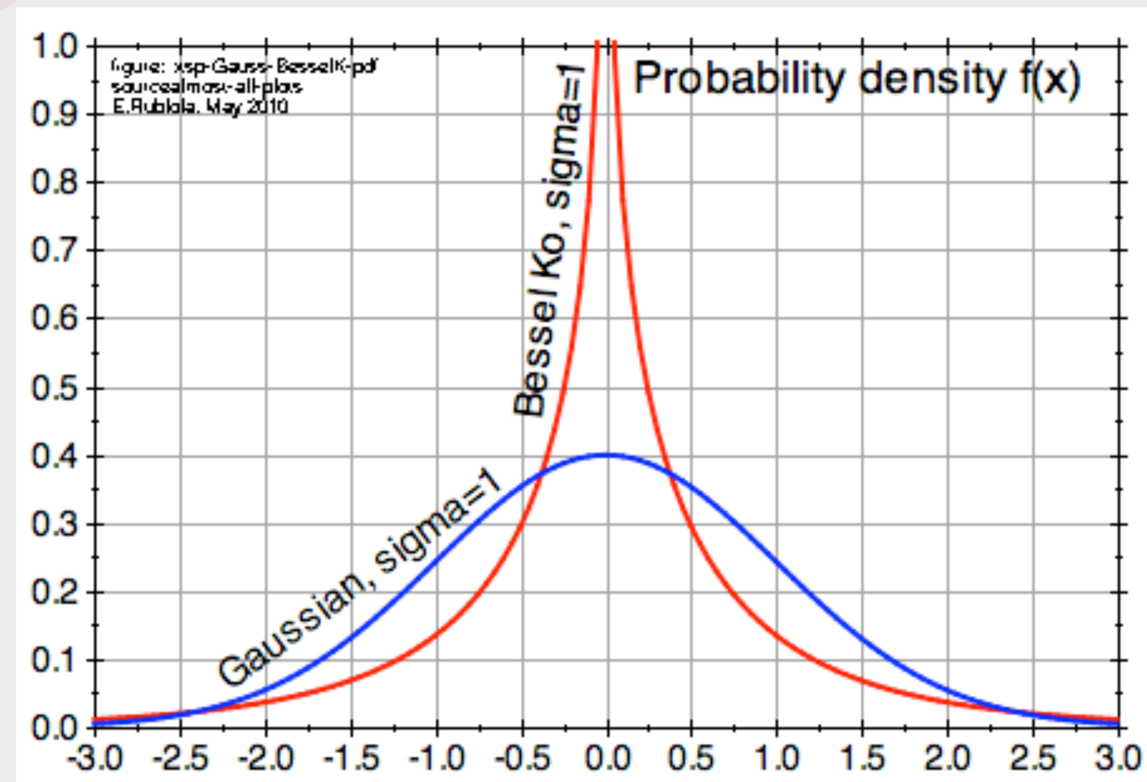
$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$

Thanks to the central limit theorem, the average

$$\langle X\rangle_m = (X_1+X_2+\ldots+X_m)/m$$

of m products has

- **Gaussian PDF,**
- **average $E\{X\} = 0$**
- **variance $V\{X\} = \sigma^2$**



Spectral Measure $S_{xx}(f)$

(Power Spectral Density)

X is white Gaussian noise
Take one frequency, S(f) –> S. Same applies to all frequencies

Spectrum

$$\begin{aligned}\langle S_{xx}\rangle_m &= \frac{1}{T}\,\langle X X^*\rangle_m \\ &= \frac{1}{T}\,\langle (X' + \imath X'') \times (X' - \imath X'')\rangle_m \\ &= \frac{1}{T}\,\boxed{\Big\langle (X')^2 + (X'')^2 \Big\rangle_m}\end{aligned}$$

white, Gaussian, avg = 0, var = 1/2

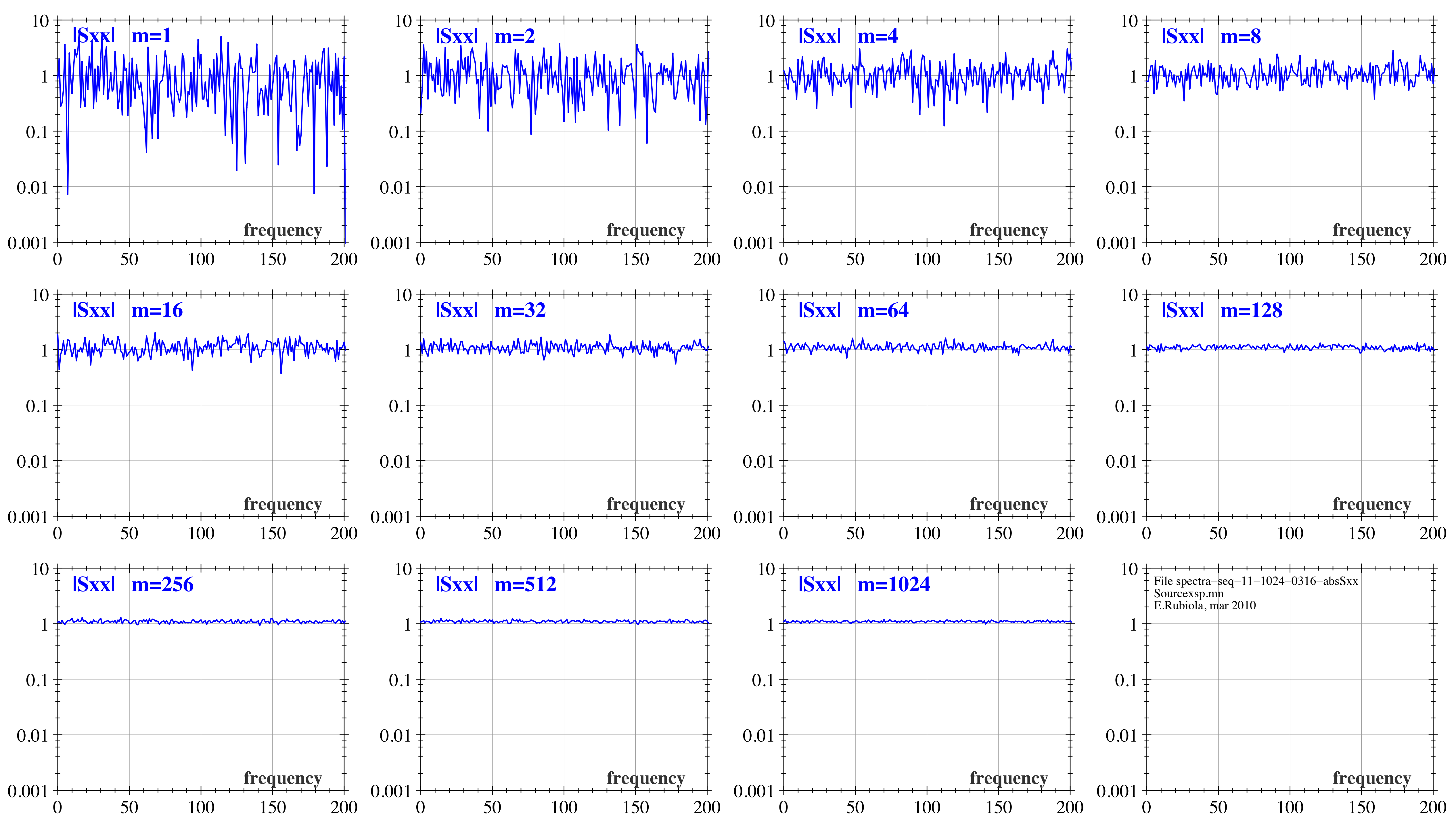
white, χ^2, with 2m degrees of freedom
avg = 1, var = 1/m

$$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{1}{m}}$$

the S_{xx} track on the FFT-SA shrinks as $1/m^{1/2}$

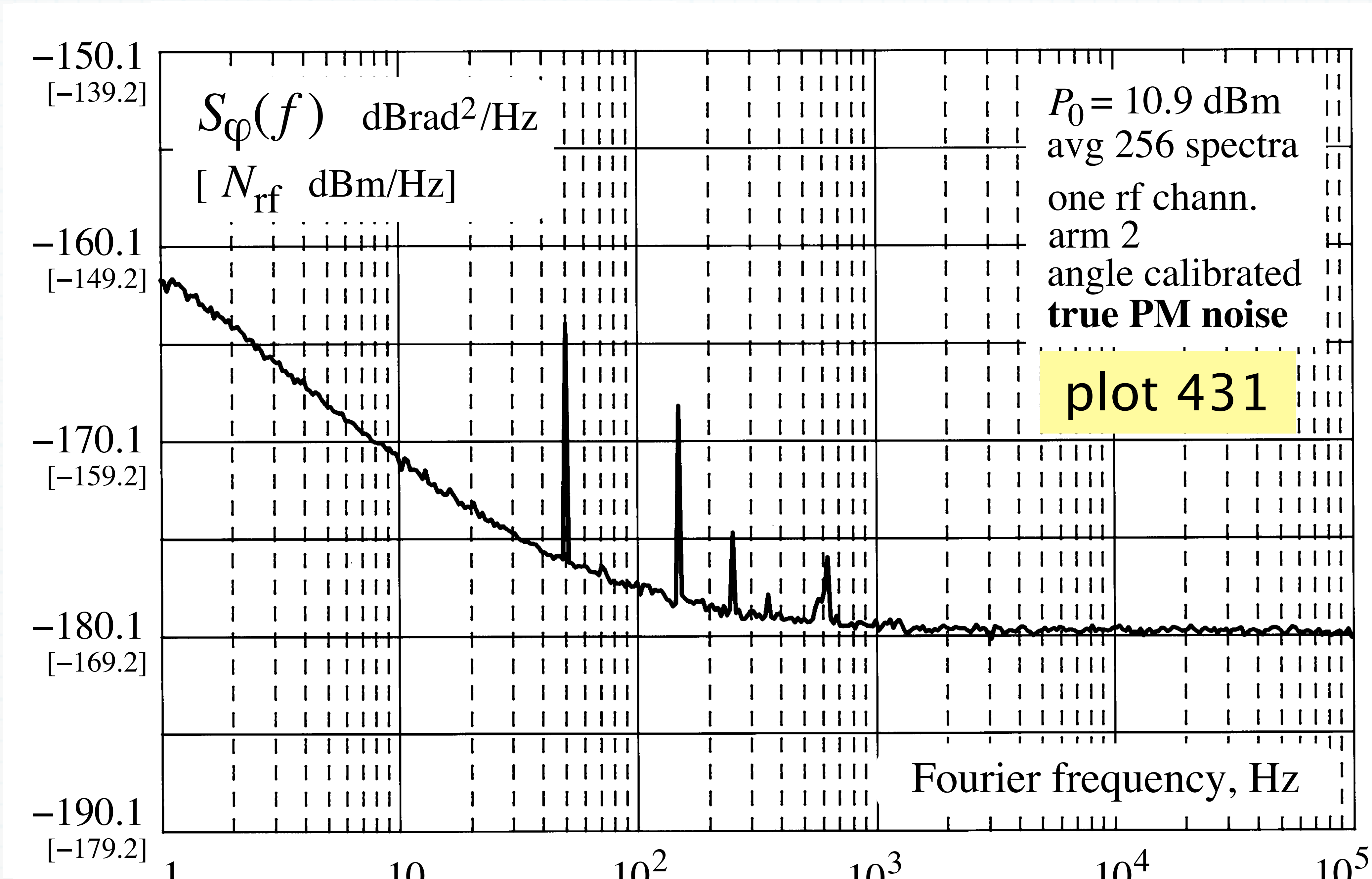
Normalization: in 1 Hz bandwidth
var{X}= 1,   and var{X'}= var{X''}= 1/2

Estimation of $|S_{xx}(f)|$

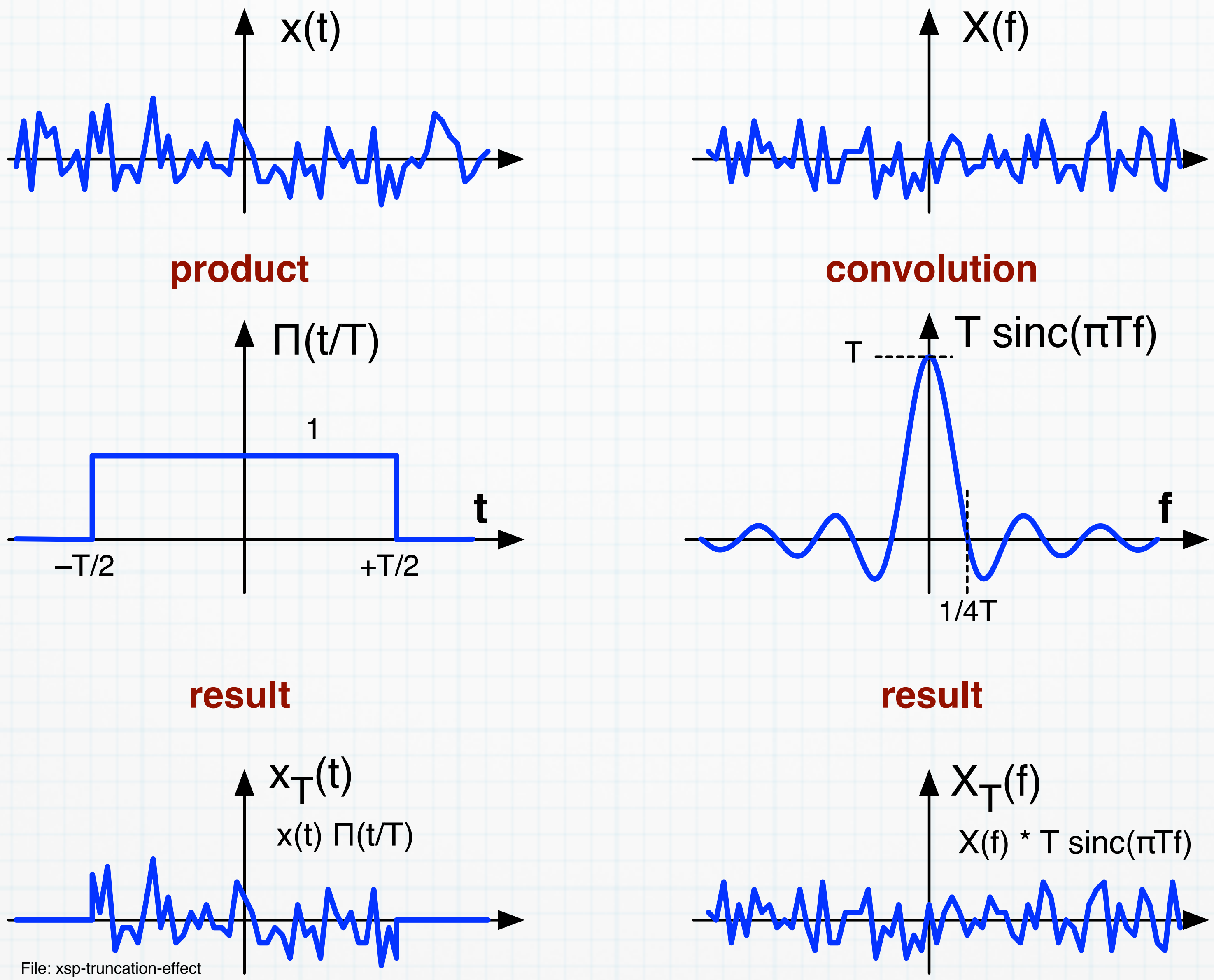


Running the measurement, m increases and S_{xx} shrinks => better confidence level

Actual spectra can be smooth like this



Statistics & finite-duration measurements



- The convolution with sinc( ) scrambles the spectrum, spreading the power of a single point all around. This introduces correlation
- In the presence of large peaks or sharp roll-off, this is disturbing
- In the measurement of smooth noise, often negligible
- Further consequences in cross-correlation measurements

Cross Spectrum Theory

Exercise set no 3

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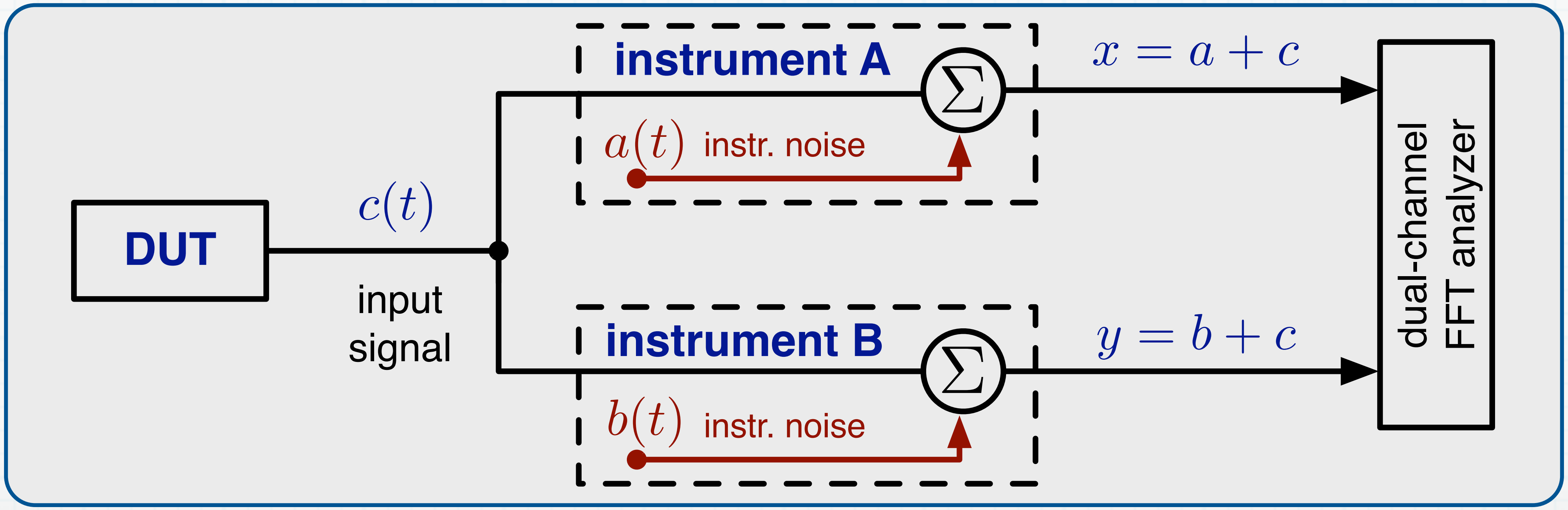
Getting close to the real game



S_{yx} with correlated term (1)

A, B = instrument background
C = DUT noise
channel 1 X = A + C
channel 2 Y = B + C
A, B, C are independent Gaussian noises
Re{ } and Im{ } are independent Gaussian noises

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Cross-spectrum

$$\langle S_{yx}\rangle_m = \frac{1}{T}\langle YX^*\rangle_m = \frac{1}{T}\langle (Y' + \imath Y'') \times (X' - \imath X'')\rangle_m$$

Expand using

$$X = (A' + \imath A'') + (C' + \imath C'') \qquad \text{and} \qquad Y = (B' + \imath B'') + (C' + \imath C'')$$

Split S_{yx} into three sets

$$\langle S_{yx}\rangle_m = \langle S_{yx}\rangle_m\Big|_{\text{instr}} + \langle S_{yx}\rangle_m\Big|_{\text{mixed}} + \langle S_{yx}\rangle_m\Big|_{\text{DUT}}$$

background only | background and DUT noise | DUT noise only

... and work it out !!!

S_{yx} with correlated term κ≠0 (2)

All the DUT signal goes in Re{S_{yx}}, Im{S_{yx}} contains only noise

Real

$$\Re\left\{\langle S_{yx}\rangle_m\right\} = \frac{1}{T}\left\{\langle B'A' + B''A''\rangle_m + \langle B'C' + B''C''\rangle_m + \langle A'C' + A''C''\rangle_m + \langle (C')^2 + (C'')^2\rangle_m\right\}$$

var=1/2 var=1/2 var= κ²/2 var=1/2 var= κ²/2 var= κ²/2

Bessel K_0, avg=0, var=1/4

Bessel K_0, avg=0, var=κ^2/4

white, χ^2, 2 DF avg = κ^2, var = κ^4

Gaussian, avg = 0, var = 1/2m

Gaussian, avg = 0, var = κ^2/2m

Gaussian, avg = 0, var = κ^2/2m

white, χ^2 2m deg. of freedom avg = κ^2, var = κ^4/m Set $\mathscr{C}$

Set $\mathscr{A}$ Gaussian, avg = 0, var = (1+2κ^2)/2m

Note: DF < 2m
See vol.XVI p.56

Imaginary

$$\Im\left\{\langle S_{yx}\rangle_m\right\} = \frac{1}{T}\left\{\langle B''A' + B'A''\rangle_m + \langle B''C' - B'C''\rangle_m + \langle A'C'' - A''C'\rangle_m\right\}$$

Bessel K_0, avg = 0, var = 1/4

Bessel K_0, avg = 0, var = κ^2/4

Gaussian, avg = 0, var = 1/2m

Gaussian, avg = 0, var = κ^2/2m

Gaussian, avg = 0, var = κ^2/2m

Set $\mathscr{B}$ Gaussian, avg = 0, var = (1+2κ^2)/2m

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = κ^2/2

A, B, C are independent Gaussian noises
Re{ } and Im{ } are independent Gaussian noises

Expand S_{yx}

$$S_{yx} = \frac{1}{T}\,\mathbb{E}\left\{\mathscr{A} + \imath\mathscr{B} + \mathscr{C}\right\}$$

Bessel K_0,
avg=0, var=1/4

$$\mathscr{A} = B'A' + B''A'' + B'C' + B''C'' + C'A' + C''A''$$
$$\mathscr{B} = B''A' + B'A'' + B''C' - B'C'' + C''A' - C'A''$$
$$\mathscr{C} = C'^2 + C''^2$$

Bessel K_0,
avg=0, var=$\kappa^2/4$

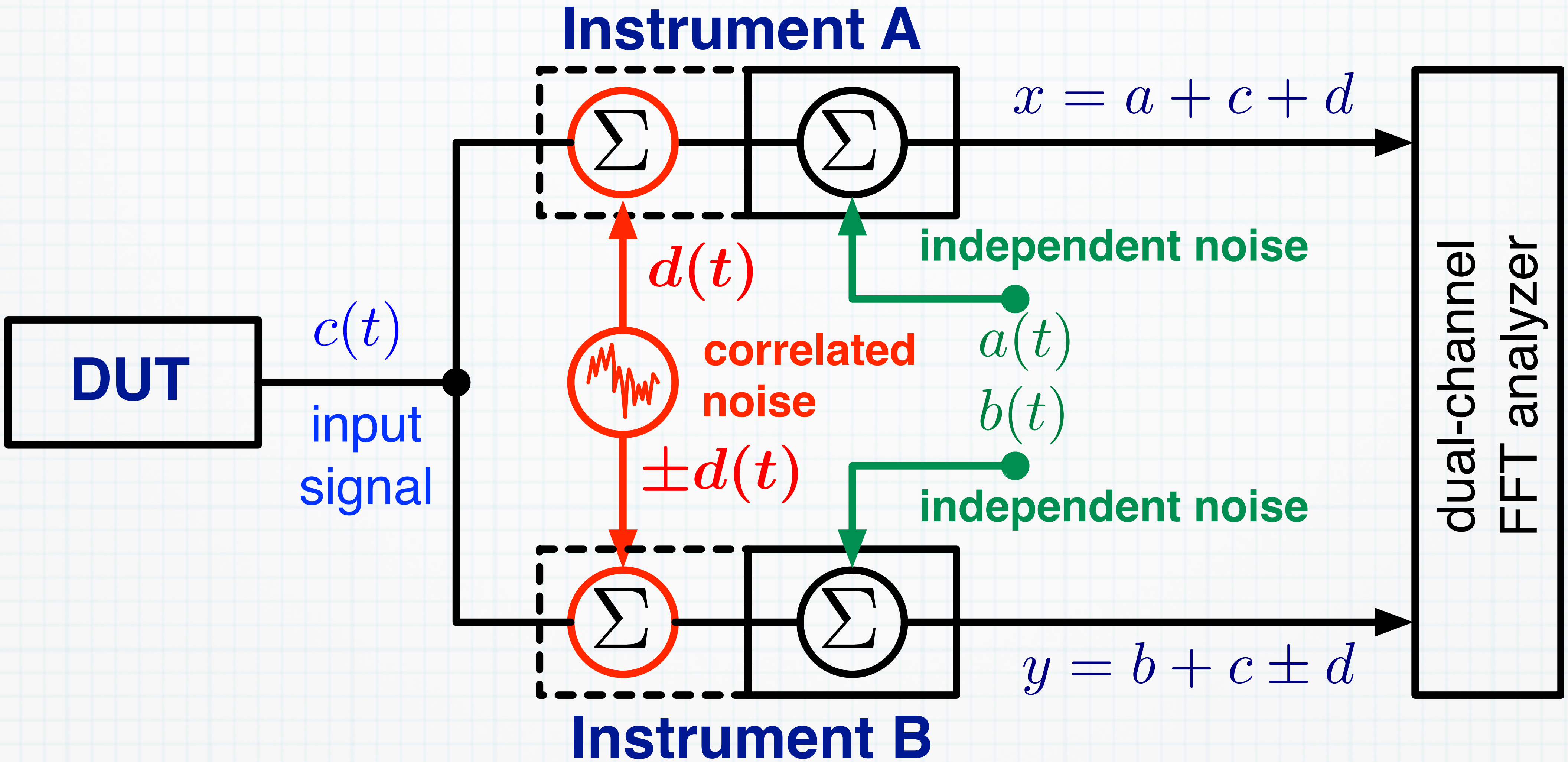
white, χ^2, 2 DF
avg = κ^2, var = κ^4

After averaging, the Bessel K_0 distribution turns into a Gaussian distribution (central limit theorem)

term	$\mathbb{E}$	$\mathbb{V}$	PDF	comment
$\langle\mathscr{A}\rangle_m$	0	$\dfrac{1+2\kappa^2}{2m}$	Gauss	average (sum) of zero-mean
$\langle\mathscr{B}\rangle_m$	0	$\dfrac{1+2\kappa^2}{2m}$	Gauss	Gaussian processes
$\langle\mathscr{C}\rangle_m$	κ^2	κ^4/m	χ^2 $\nu = 2m$	average (sum) of chi-square processes
$\langle\tilde{\mathscr{C}}\rangle_m$	κ^2	κ^4/m	Gauss	approximates $\langle\mathscr{C}\rangle_m$ for large m

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Stray correlated effects



The cross spectrum $S_{yx}(f)$ converges to $S_c(f) \pm S_d(f)$

- **Common pollution (DUT AM noise…) is not rejected**
- **Overestimation or underestimation of noise is possible**

C. Nelson & al., Rev. Sci. Instr. 85(3)
See also E. Rubiola, R. Boudot, IEEE TUFFC 54(5) 2007 / arXiv:physics/0609147
and E. Rubiola, F. Vernotte, arXiv:1003.0113v1, 2010.

Estimator $\hat{S} = |\langle S_{yx} \rangle_m|$

Simplest estimator – the instrument default

$$
\begin{aligned}
\left| \langle S_{yx} \rangle_m \right| &= \frac{1}{T} \sqrt{\left[\Re\left\{\langle Y X^* \rangle_m\right\}\right]^2 + \left[\Im\left\{\langle Y X^* \rangle_m\right\}\right]^2} \\
&= \frac{1}{T} \sqrt{\left[\langle \mathscr{A} \rangle_m + \langle \tilde{\mathscr{C}} \rangle_m\right]^2 + \left[\langle \mathscr{B} \rangle_m\right]^2} \,.
\end{aligned}
$$

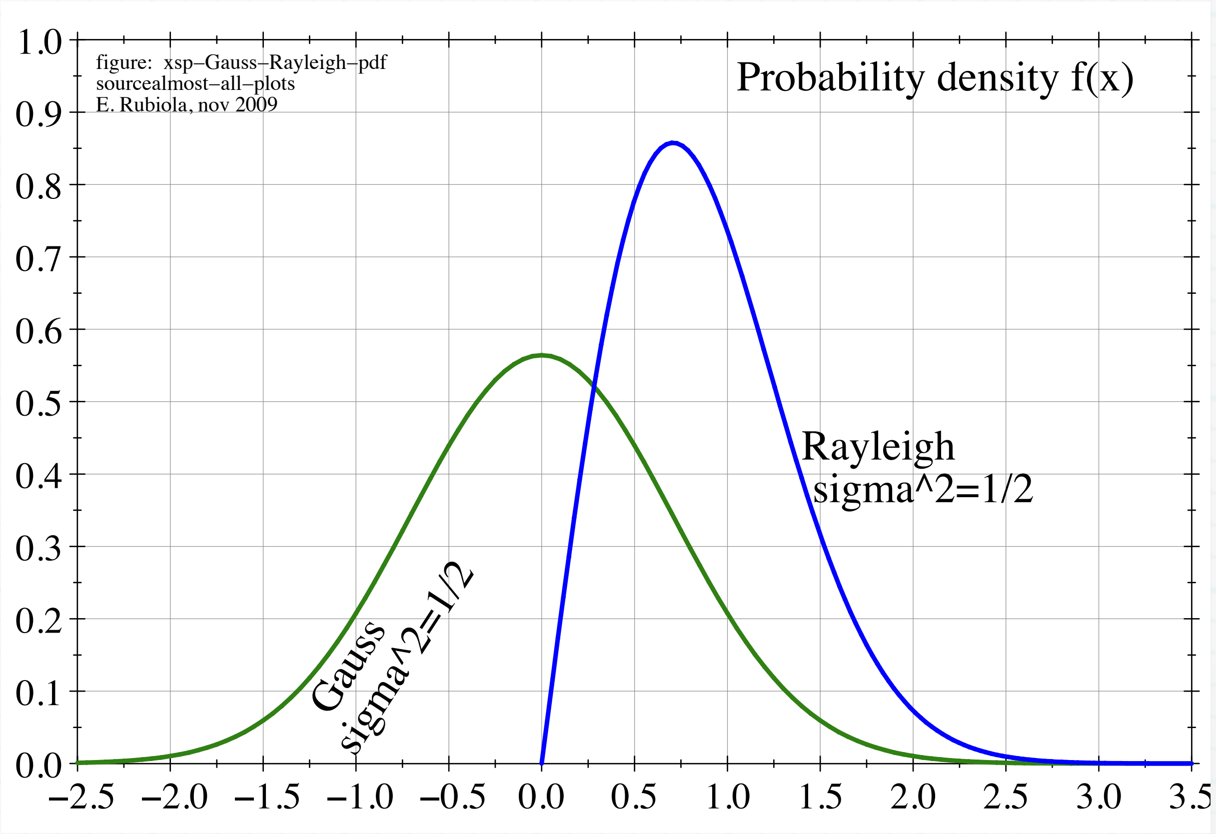
κ → 0 Rayleigh distribution

$$
\langle \mathscr{Z} \rangle_m = \sqrt{\left[\langle \mathscr{A} \rangle_m\right]^2 + \left[\langle \mathscr{B} \rangle_m\right]^2} \,.
$$

$$
\mathbb{E}\{\langle \mathscr{Z} \rangle_m\} = \sqrt{\frac{\pi}{4m}} = \frac{0.886}{\sqrt{m}}
$$

$$
\mathbb{V}\{\langle \mathscr{Z} \rangle_m\} = \frac{1}{m}\left(1 - \frac{\pi}{4}\right) = \frac{0.215}{m}
$$

$$
\frac{\mathrm{dev}\{\left| \langle S_{yx} \rangle_m \right|\}}{\mathbb{E}\{\left| \langle S_{yx} \rangle_m \right|\}} = \sqrt{\frac{4}{\pi} - 1} = 0.523
$$

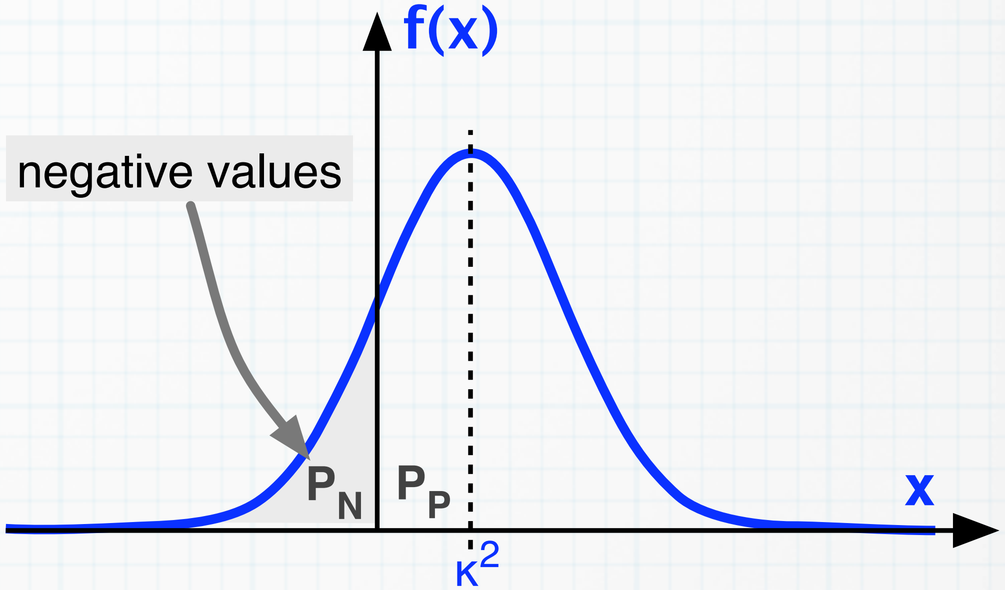


Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Estimator $\hat{S} = \mathrm{Re}\{\langle S_{yx}\rangle_m\}$

Best (unbiased) estimator

$$\langle \mathscr{Z} \rangle_m = \langle \mathscr{A} \rangle_m + \langle \tilde{\mathscr{C}} \rangle_m$$



$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\kappa^2}{\sqrt{2}\,\sigma}\right)$$

$$\mathbb{E}\left\{\langle \mathscr{Z} \rangle_m\right\} = \kappa^2$$

$$\mathbb{V}\left\{\langle \mathscr{Z} \rangle_m\right\} = \frac{1 + 2\kappa^2 + 2\kappa^4}{2m}$$

$$\mathrm{dev}\left\{\langle \mathscr{Z} \rangle_m\right\} = \sqrt{\frac{1 + 2\kappa^2 + 2\kappa^4}{2m}} \approx \frac{1+\kappa^2}{\sqrt{2m}}$$

$$\frac{\mathrm{dev}\left\{\langle \mathscr{Z} \rangle_m\right\}}{\mathbb{E}\left\{\langle \mathscr{Z} \rangle_m\right\}} = \frac{\sqrt{1 + 2\kappa^2 + 2\kappa^4}}{\kappa^2\,\sqrt{2m}} \approx \frac{1+\kappa^2}{\kappa^2\,\sqrt{2m}}$$

0 dB SNR requires that $m = 1/2\kappa^4$.

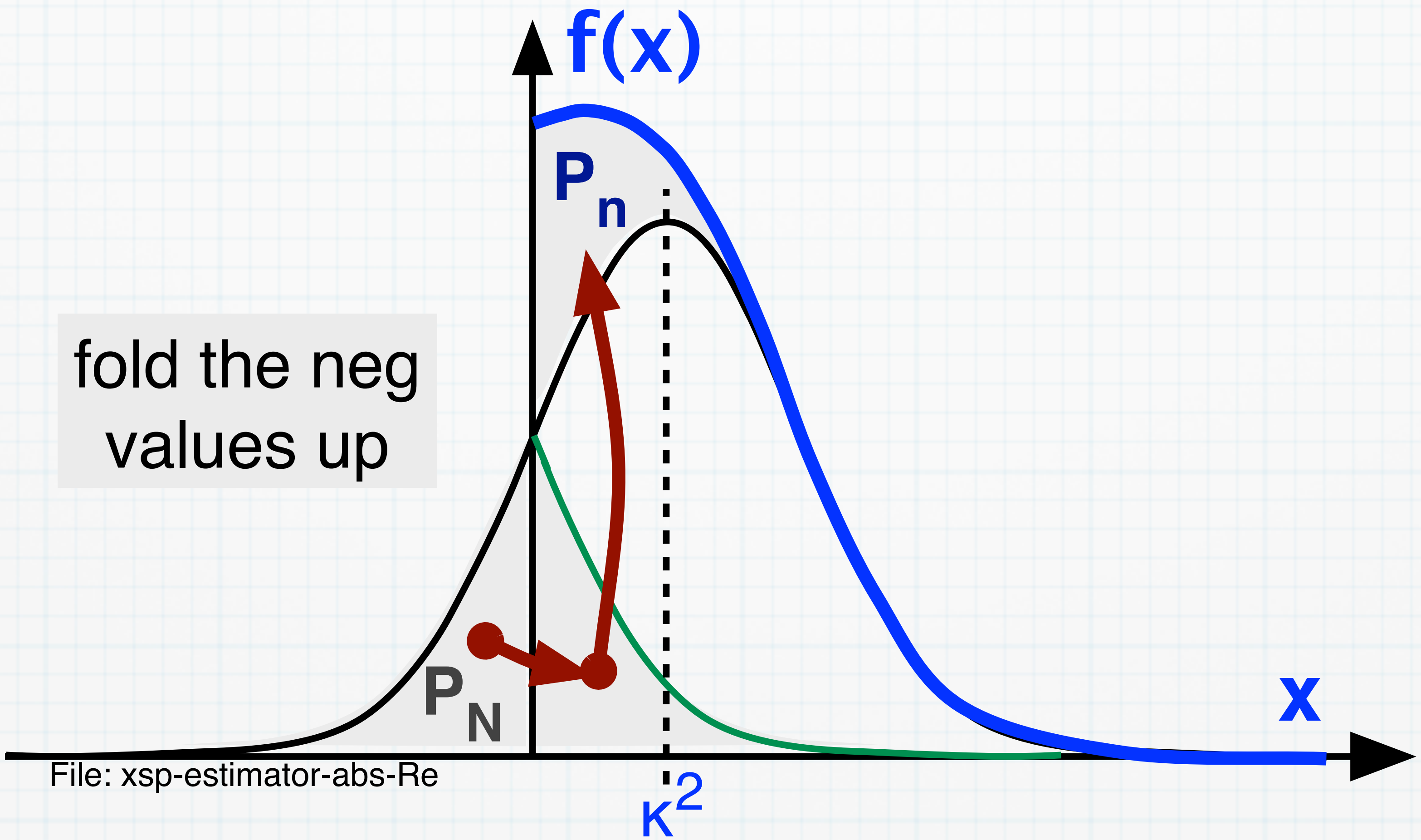
Example $\kappa = 0.1$ (DUT noise 20 dB lower than single-channel background) averaging on 5×10^3 spectra is necessary to get SNR = 0 dB.

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Estimator $\hat{S} = |Re\{<S_{yx}>_m\}|$

Good (yet biased) estimator: fold the negative values

$$\left|\Re\left\{\langle S_{yx}\rangle_m\right\}\right| = \frac{1}{T}\left|\langle \mathscr{A}\rangle_m + \langle \tilde{\mathscr{C}}\rangle_m\right|$$

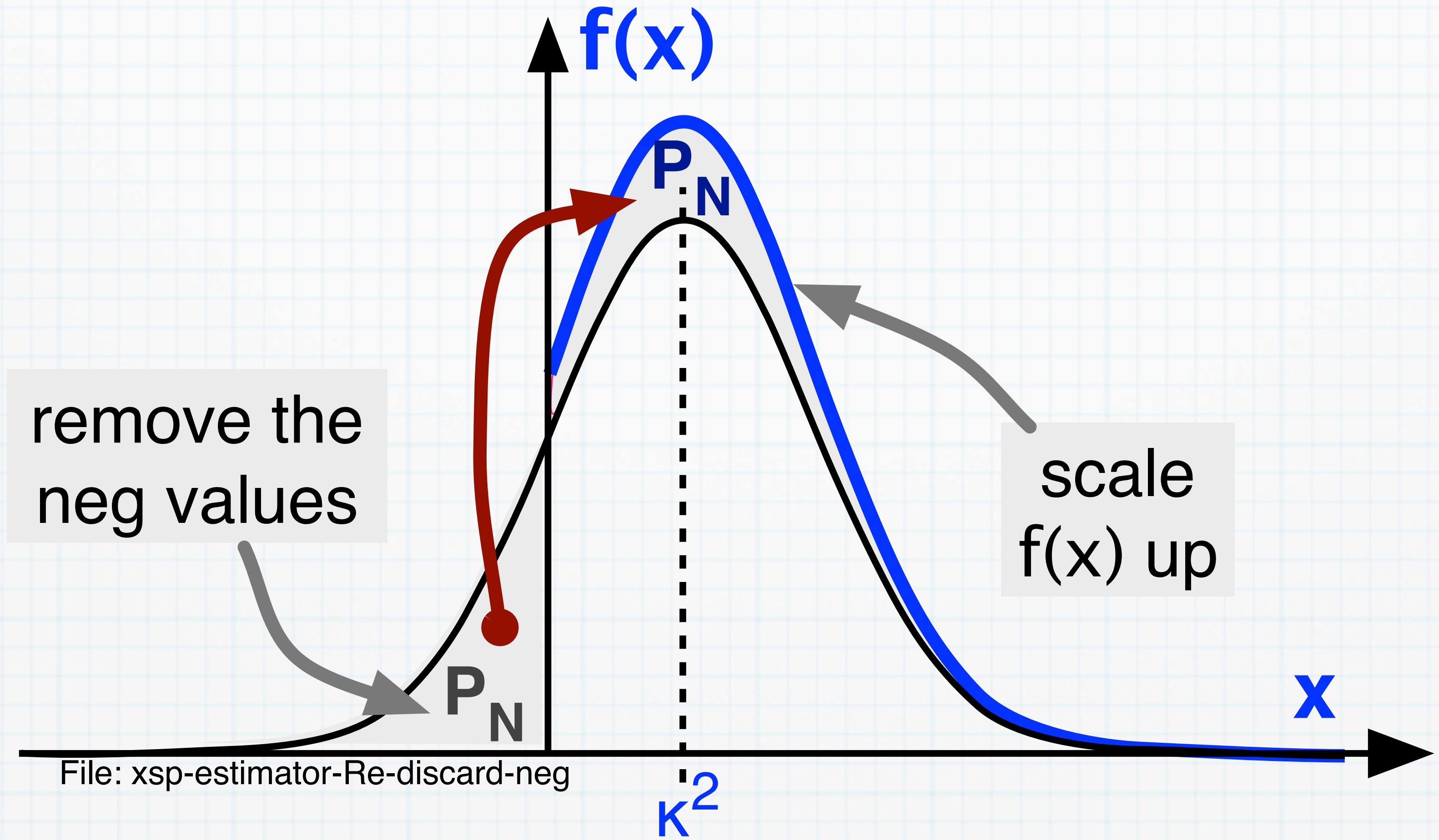


$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\kappa^2}{\sqrt{2}\,\sigma}\right)$$

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Estimator $\hat{S} = \mathrm{Re}\{<S_{yx}>_{m'<m}\}$ averaged on positive values only

Naive (poor) solution: discard the negative values



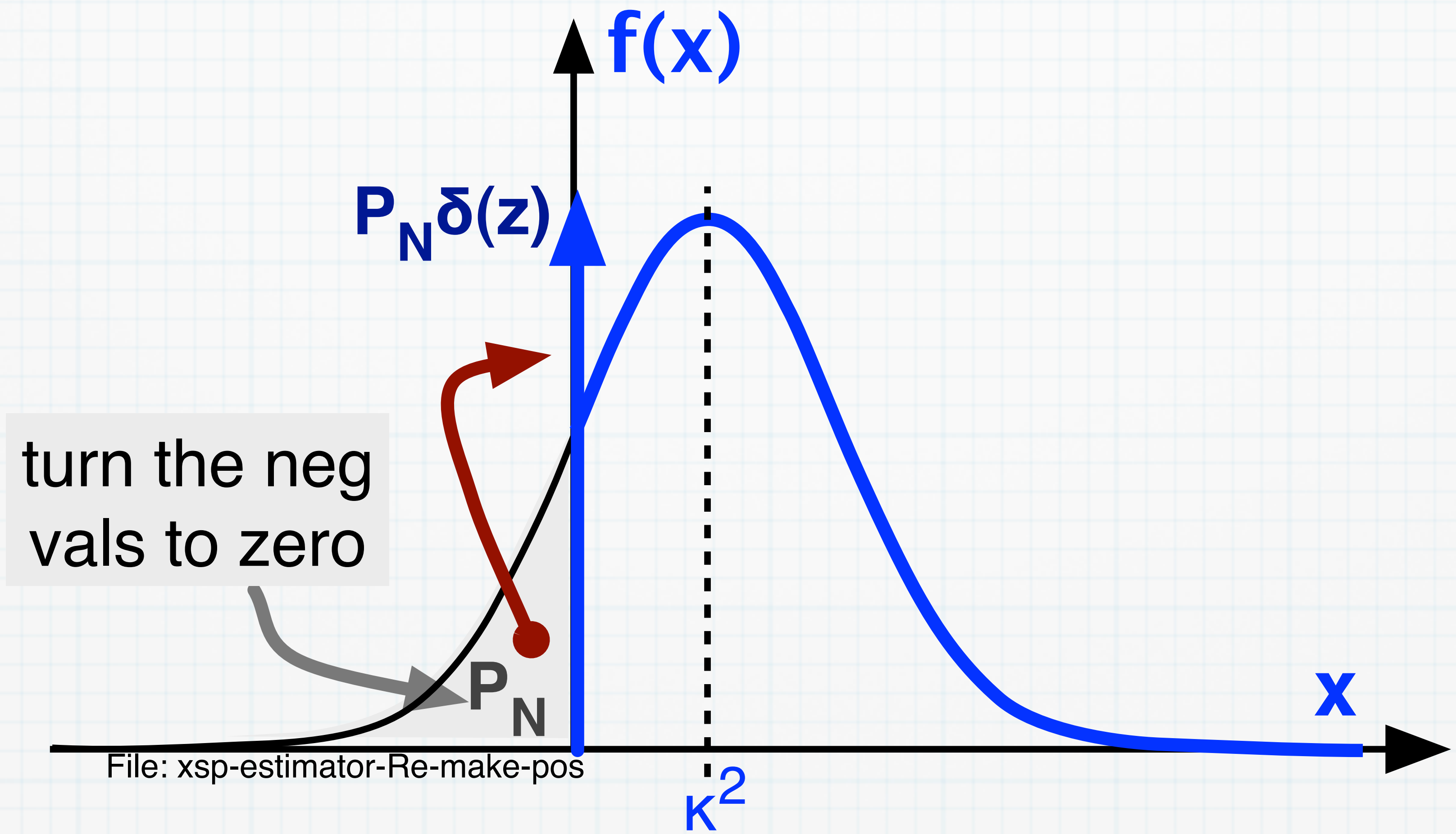
$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\kappa^2}{\sqrt{2}\,\sigma}\right)$$

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Estimator $\hat{S} = \langle \max(\mathrm{Re}\{S_{yx}\}, 0_+) \rangle_m$

Replace the negative values with 0+

Smart – suitable to log scale



Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Noise rejection, $|S_{yx}(f)|$

Independent X and Y, var{X} = var{Y}= 1/2

$|S_{yx}|$ => Rayleigh distribution

average

$$\mathbb{E}\{S\} = \sqrt{\frac{\pi}{4m}}$$

$= 0.886/\sqrt{m}$

deviation

$$\sqrt{\mathbb{E}\left\{|S - \mathbb{E}\{S\}|^2\right\}} = \sqrt{\left(1 - \frac{\pi}{4}\right)\frac{1}{m}}$$

$= \sqrt{(0.215/m)}$

$|\langle S_{yx}\rangle_m|$ ~ $-5\log_{10}(m) - 0.53$ dB

dev / avg ratio is independent of m

$$\frac{\sqrt{\mathbb{E}\{|S - \mathbb{E}\{S\}|^2\}}}{\mathbb{E}\{S\}} = \sqrt{\frac{4}{\pi} - 1}$$

$= 0.523$

The track thickness on the analyzer logarithmic scale is constant because the dev / avg ratio is independent of m

Noise rejection, $|\mathrm{Re}\{S_{yx}(f)\}|$

Independent X and Y, var{X} = var{Y}= 1/2

$|\mathrm{Re}\{S_{yx}\}|$ => one-sided Gaussian distribution

average

$$\mathbb{E}\{S\} = \sqrt{\frac{1}{\pi m}}$$

= 0.564/√m

deviation

$$\sqrt{\mathbb{E}\left\{|S - \mathbb{E}\{S\}|^2\right\}} = \sqrt{\left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{m}}$$

= √(0.182/m)

$|\langle \mathrm{Re}\{S_{yx}\rangle_m\}|$ ~ – 5 $\log_{10}$(m) – 2.49 dB

dev / avg ratio is independent of m

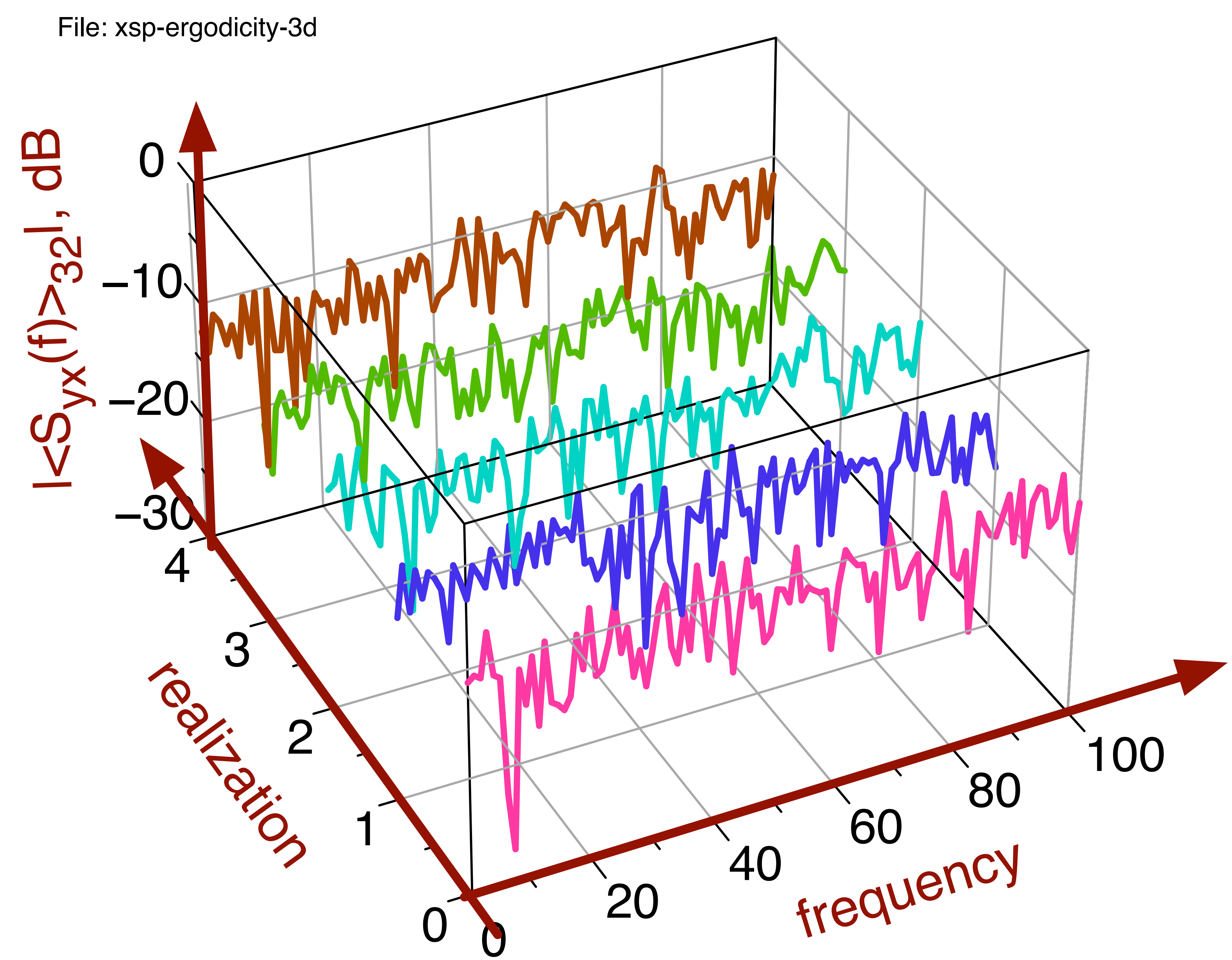
$$\frac{\sqrt{\mathbb{E}\{|S - \mathbb{E}\{S\}|^2\}}}{\mathbb{E}\{S\}} = \sqrt{\frac{\pi}{2} - 1}$$

= 0.756

The track thickness on the analyzer logarithmic scale is constant because the dev / avg ratio is independent of m

Ergodicity

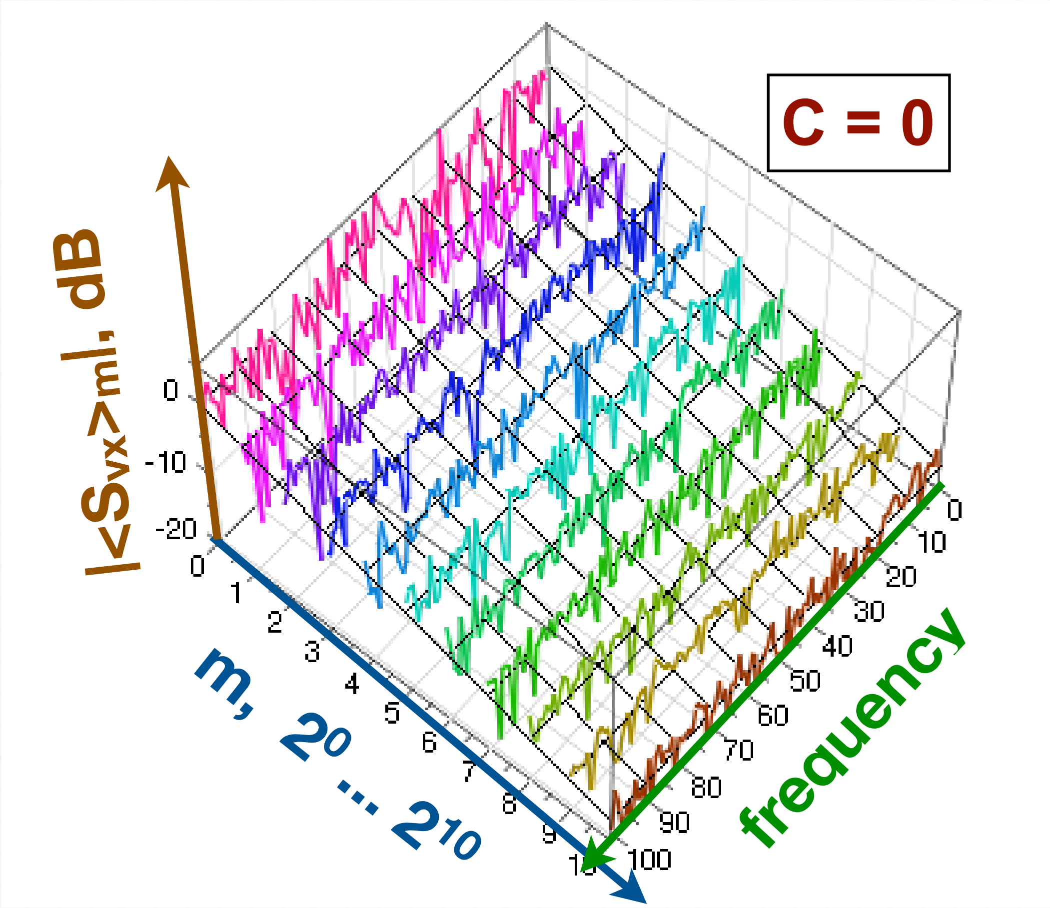
Let's collect a sequence of spectra

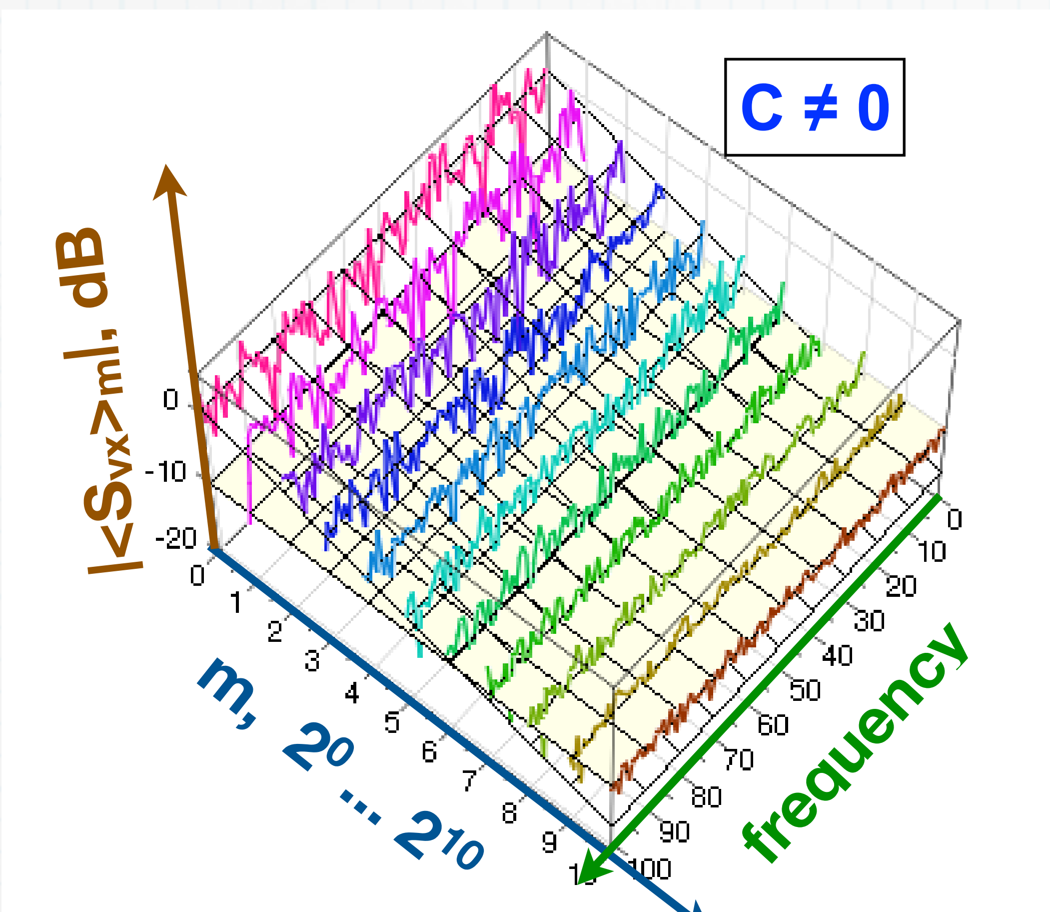


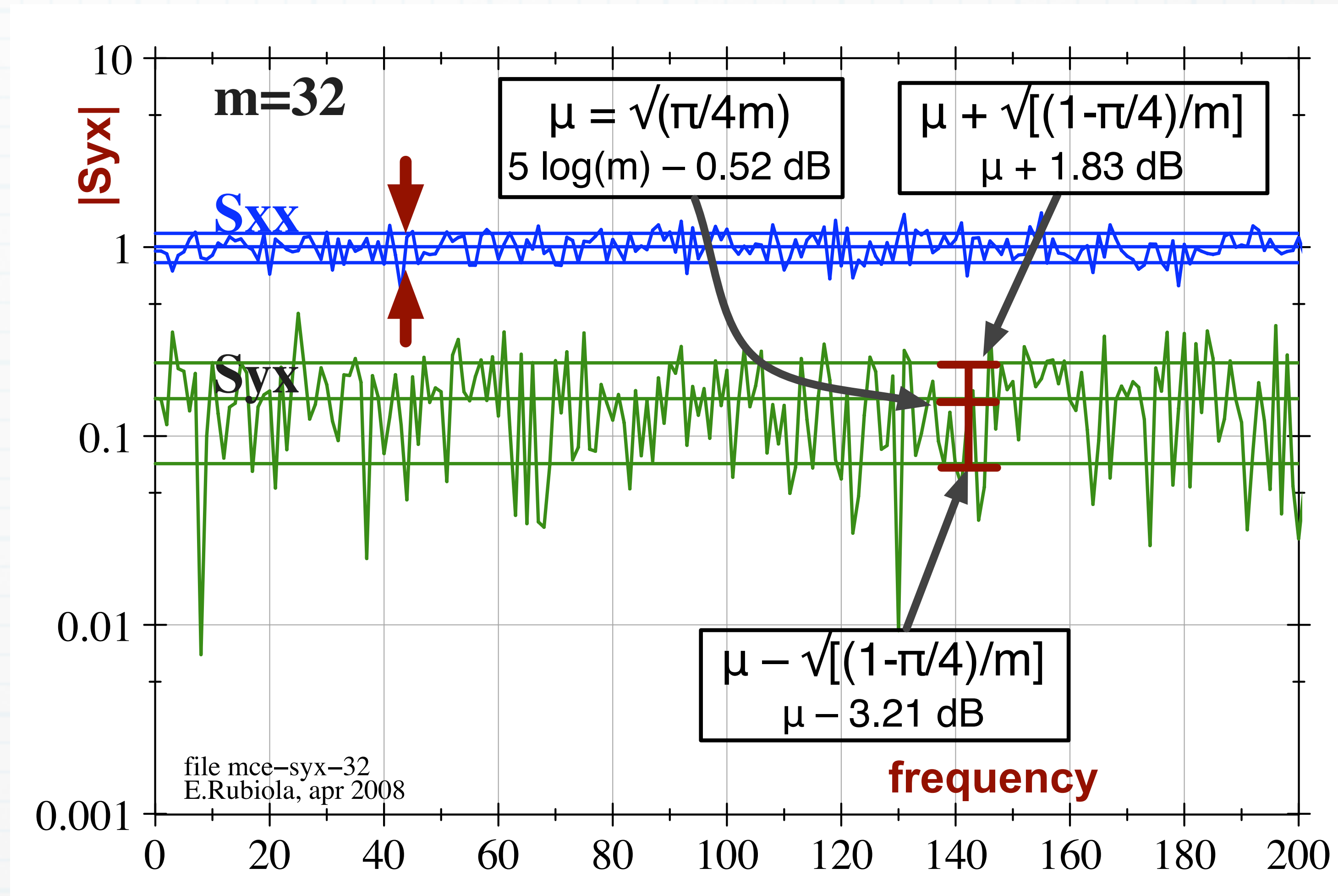
Ergodicity allows to interchange time statistics and ensemble statistics, thus the running index i of the sequence and the frequency f.

The average and the deviation calculated on the frequency axis are the same as the average and the deviation of the sequence of spectra.

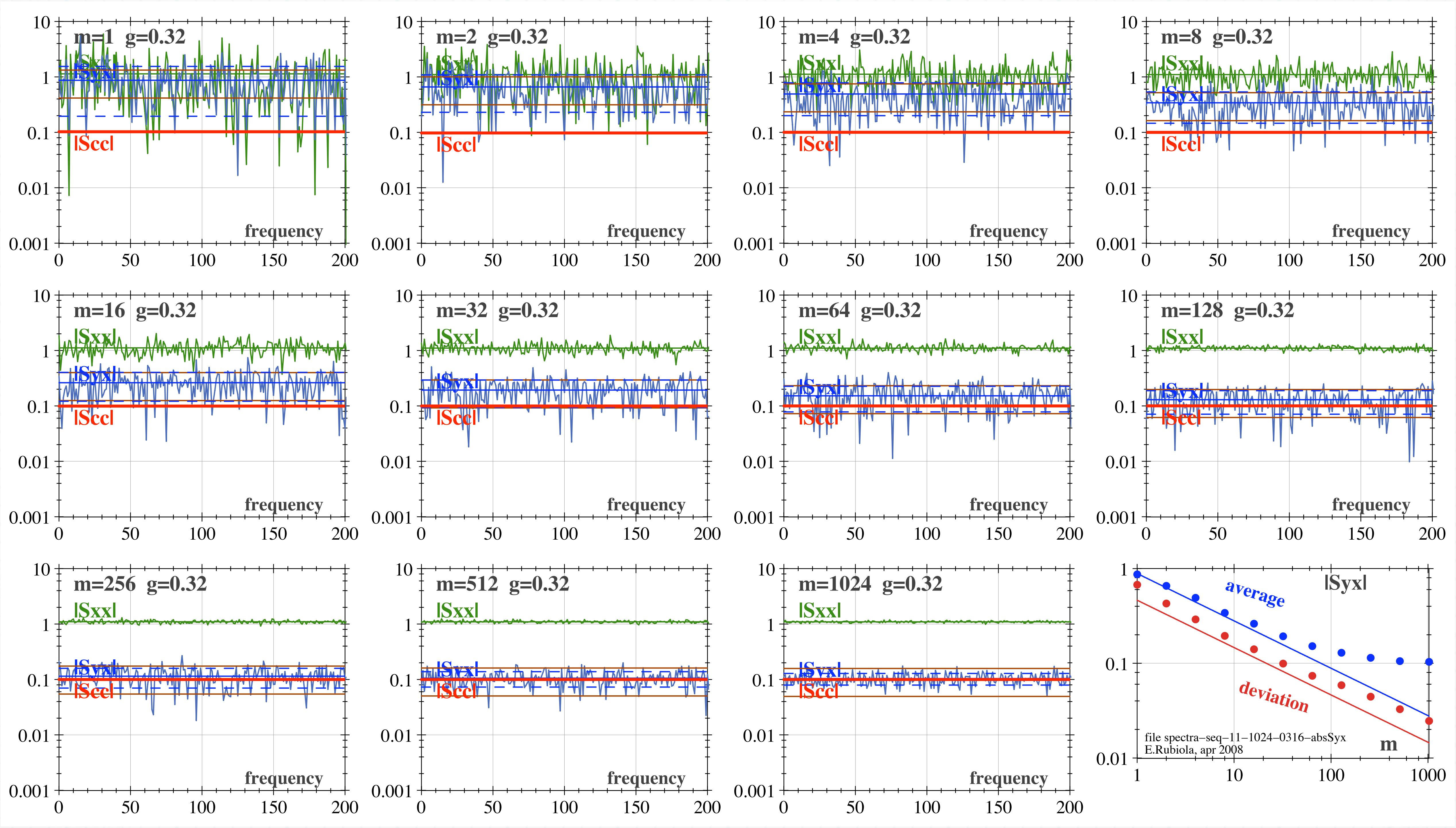
Example: Measurement of |Syx|





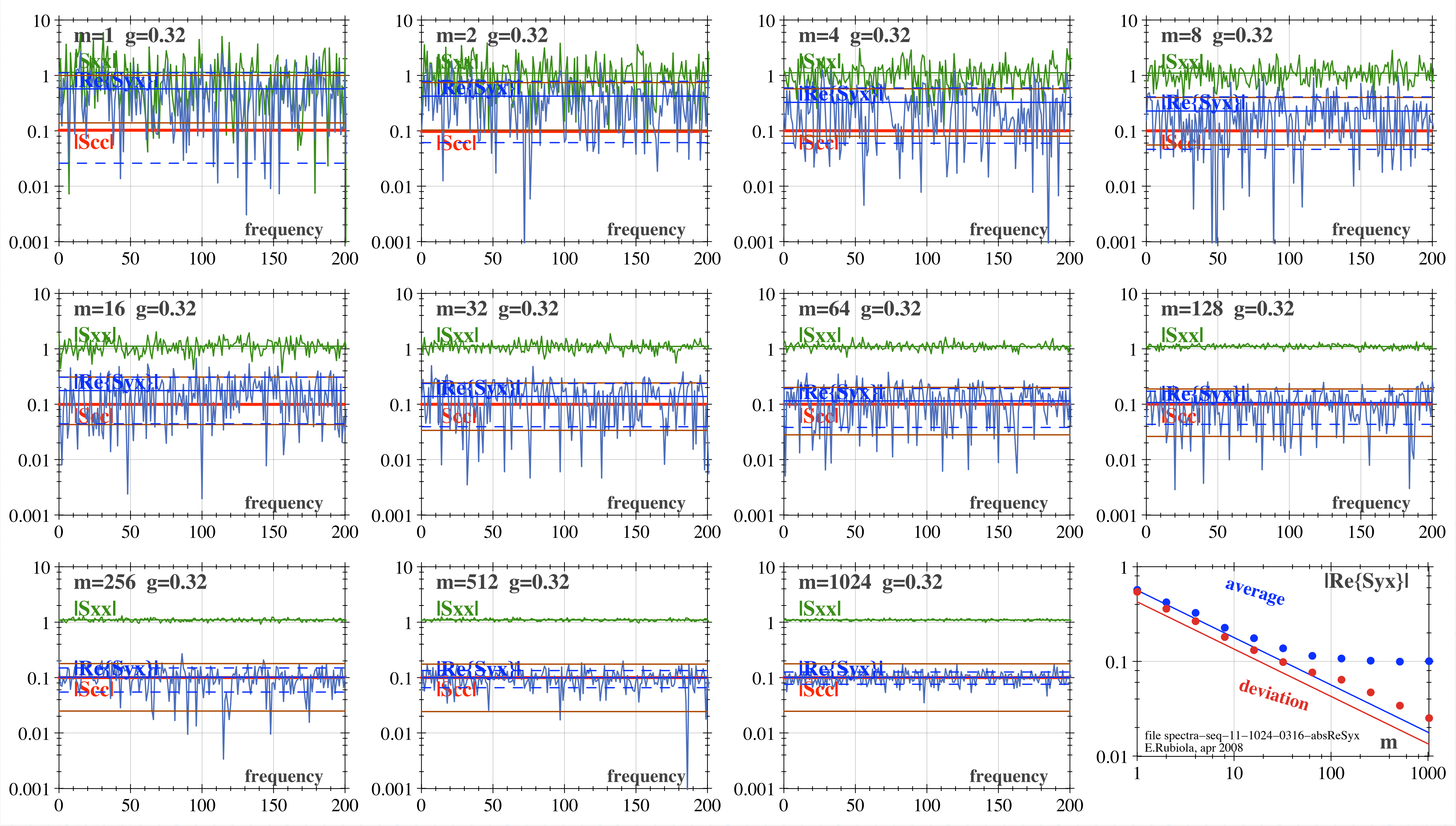


Measurement (C≠0), |Syx|



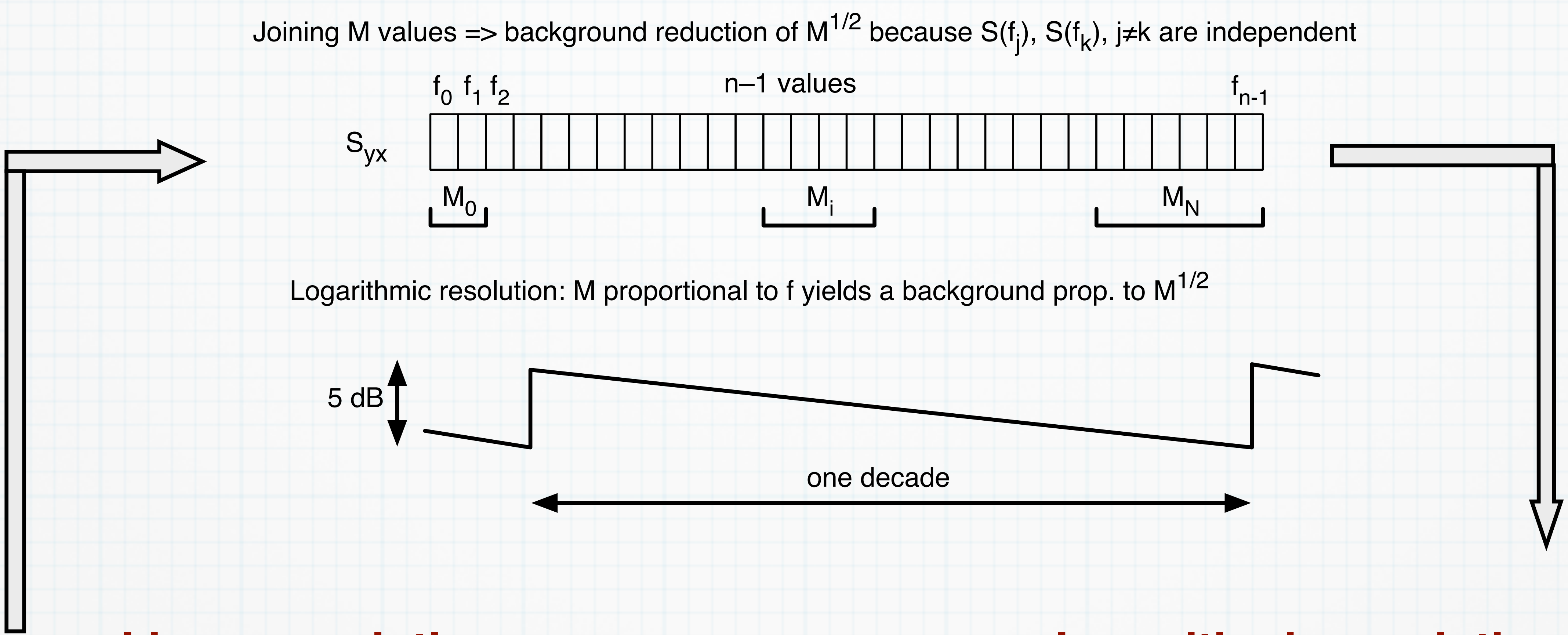
Running the measurement, m increases
S_{xx} shrinks => better confidence level
S_{yx} decreases => higher single-channel noise rejection

Measurement (C≠0), |Re{Syx}|



Running the measurement, m increases
S_{xx} shrinks => better confidence level
S_{yx} decreases => higher single-channel noise rejection

Linear vs. logarithmic resolution



Linear resolution

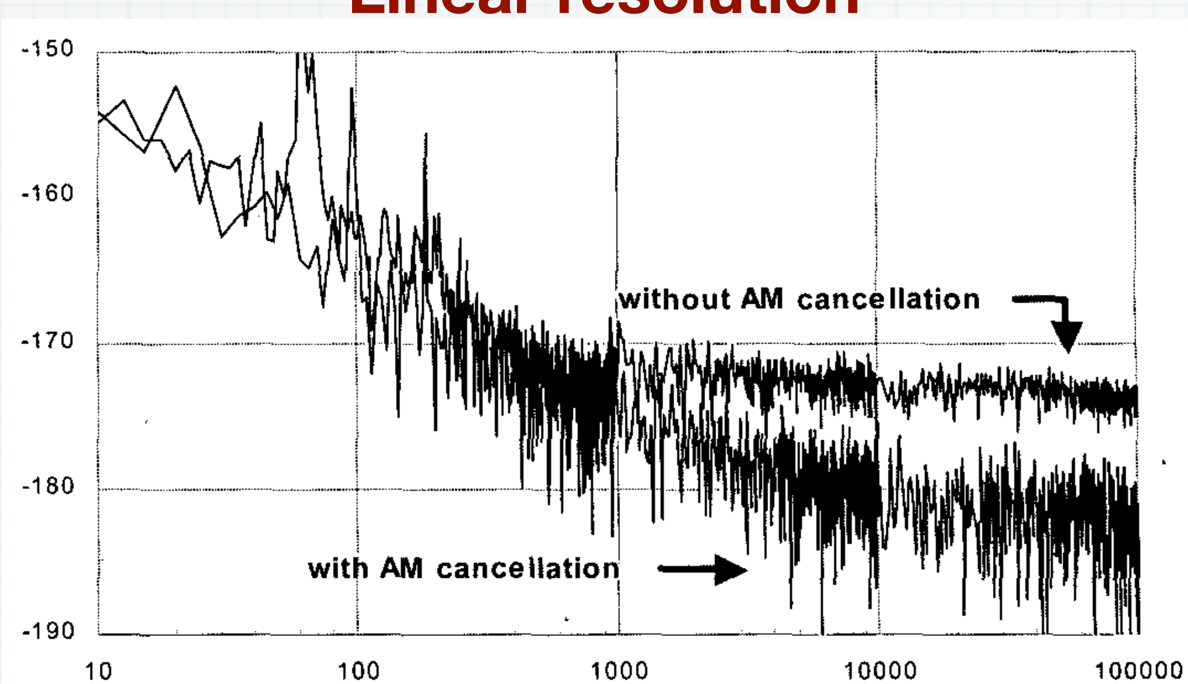


Fig.5, G. Cibiel, TUFFC 49(6) jun 2002

Logarithmic resolution

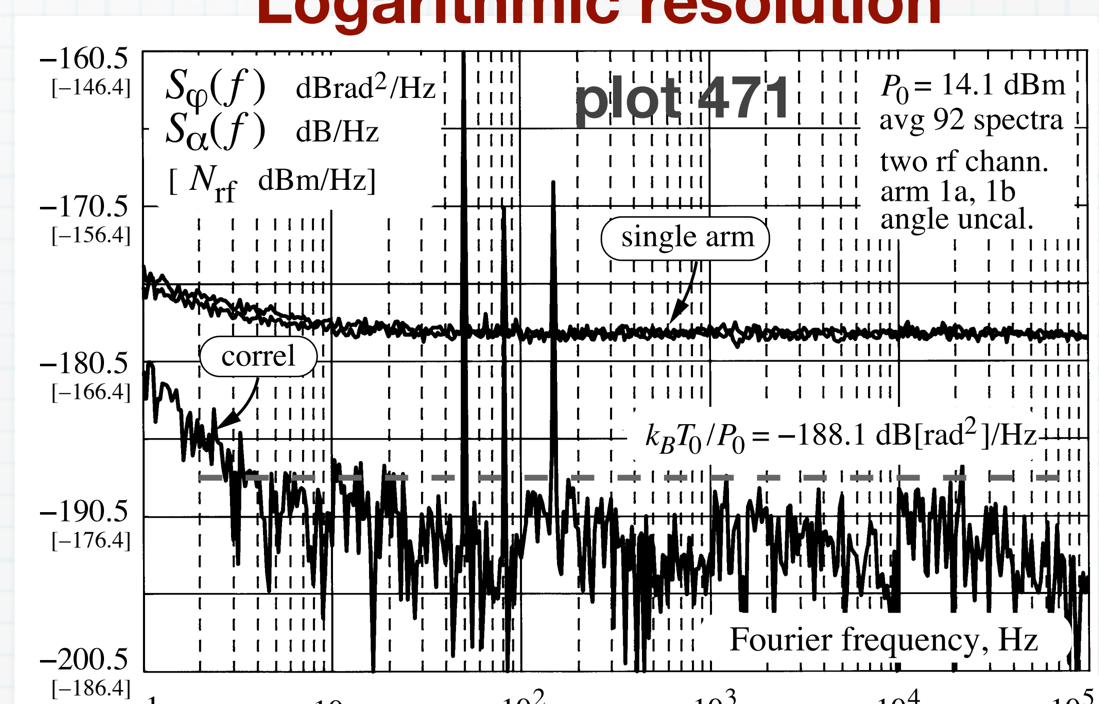


Fig.7, E. Rubiola, V. Giordano, RSI 73(6) jun 2002

Applications

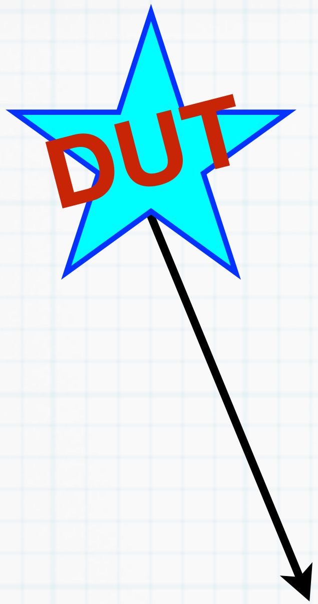
The real fun starts here

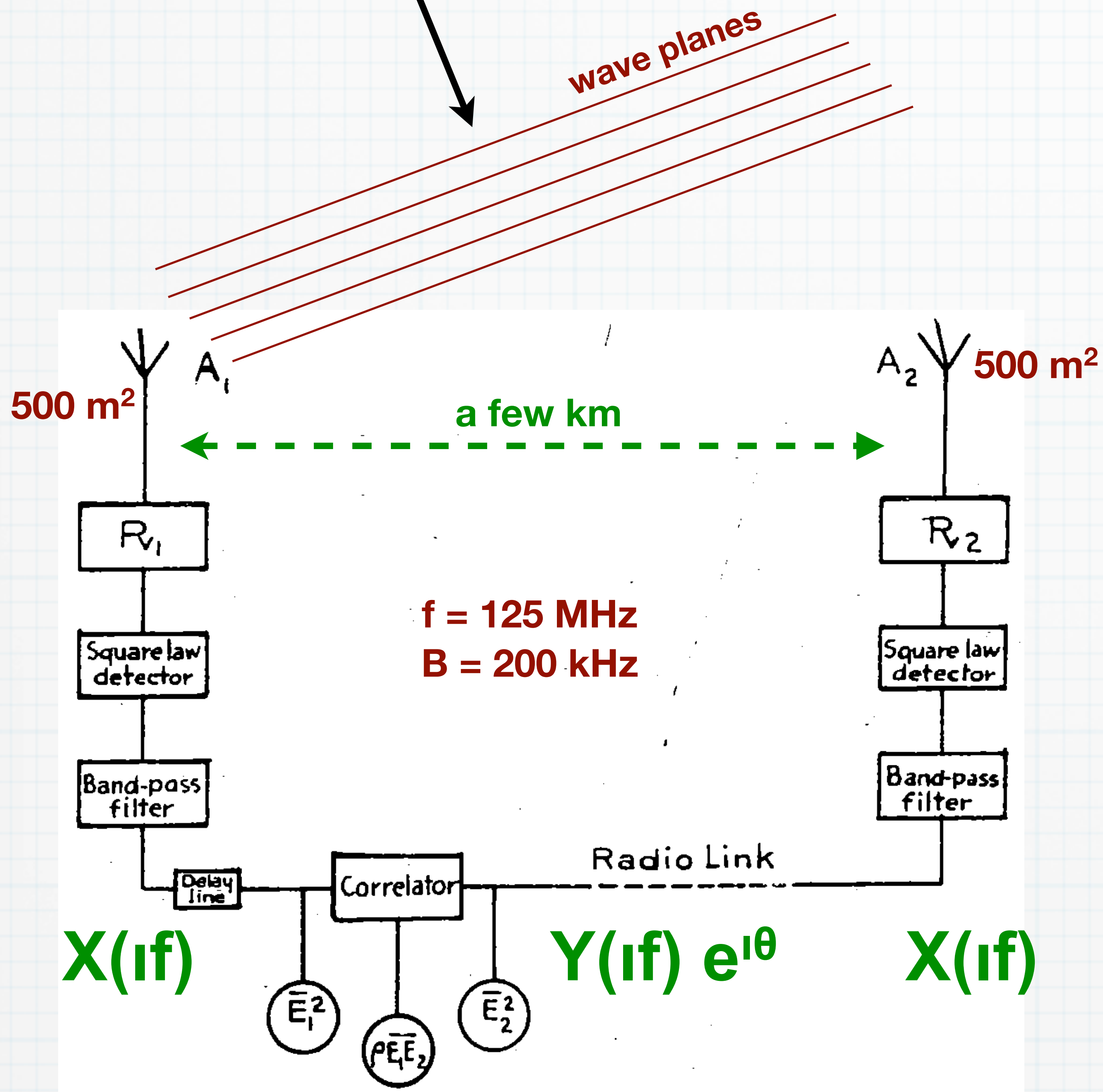
Applications

- Radio-astronomy (Hanbury-Brown, 1952)
- Early implementations
- Radiometry (Allred, 1962)
- Noise calibration (Spietz, 2003)
- Frequency noise (Vessot 1964)
- Phase noise (Walls 1976)
- Dual delay line system (Lance, 1982)
- Phase noise (Rubiola 2000 & 2002)
- Effect of amplitude noise (Rubiola, 2007)
- Frequency stability of a resonator (Rubiola)
- Dual-mixer time-domain instrument (Allan 1975, Stein 1983)
- Amplitude noise & laser RIN (Rubiola 2006)
- Noise of a power detector (Grop & Rubiola, in progress)
- Noise in chemical batteries (Walls 195)
- Semiconductors (Sampietro RSI 1999)
- Electromigration in thin films (Stoll 1989)
- Fundamental definition of temperature
- Hanbury Brown - Twiss effect (Hanbury-Brown & Twiss 1956, Glattli 2004)

Radio-astronomy

Cassiopeia A (or Cygnus A) radio source

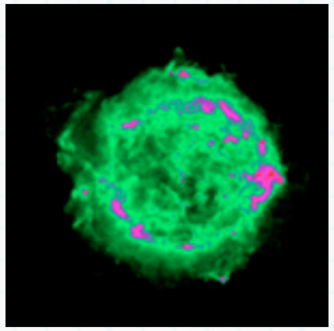




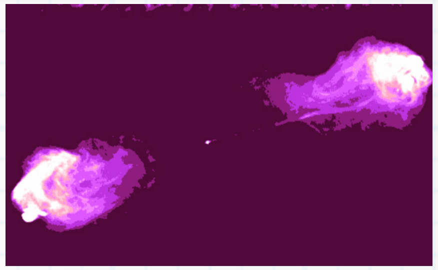
Measurement of the apparent angular size of stellar radio sources

Jodrell Bank, Manchester

Cassiopeia A (Harvard)



Cygnus A (Harvard)

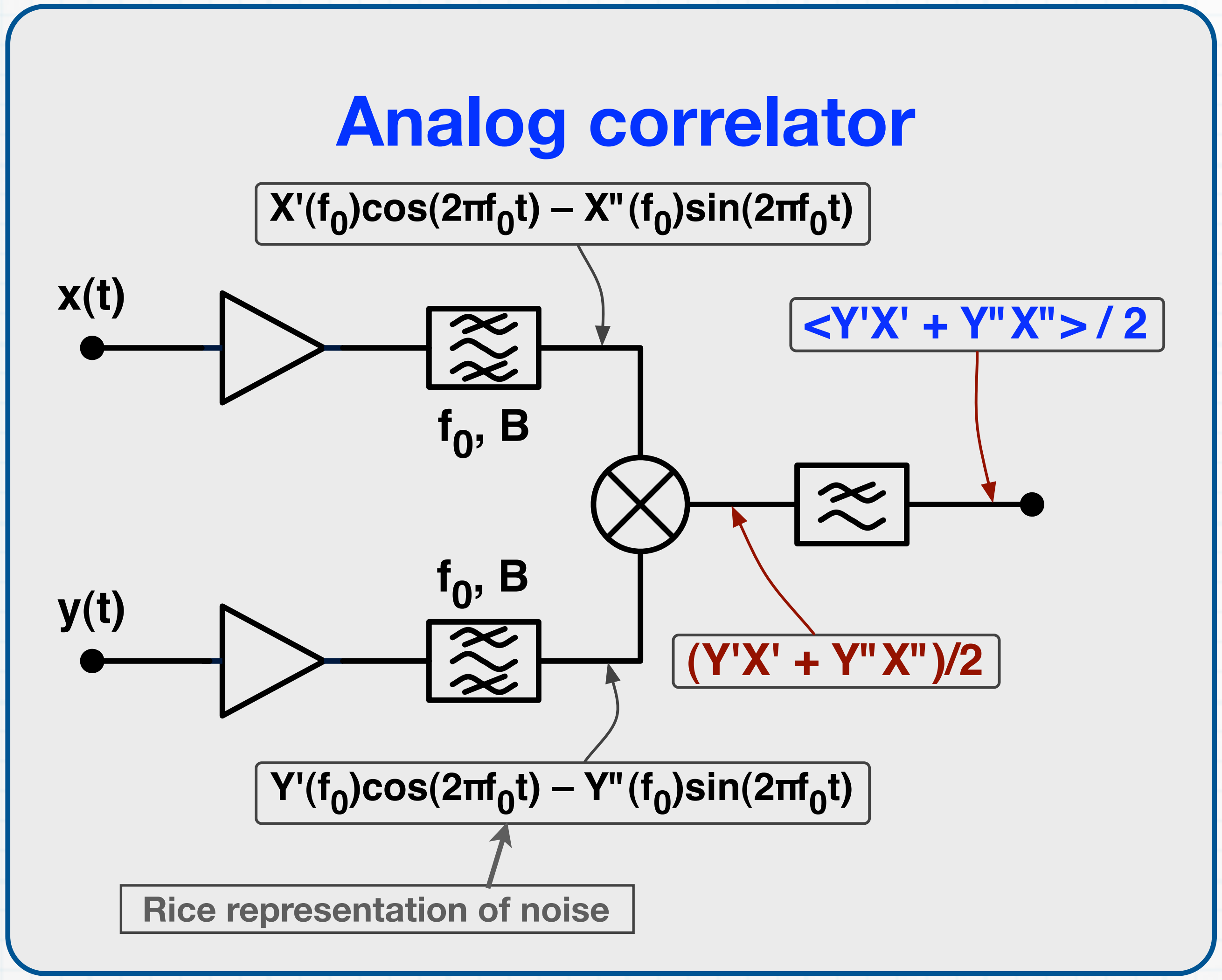


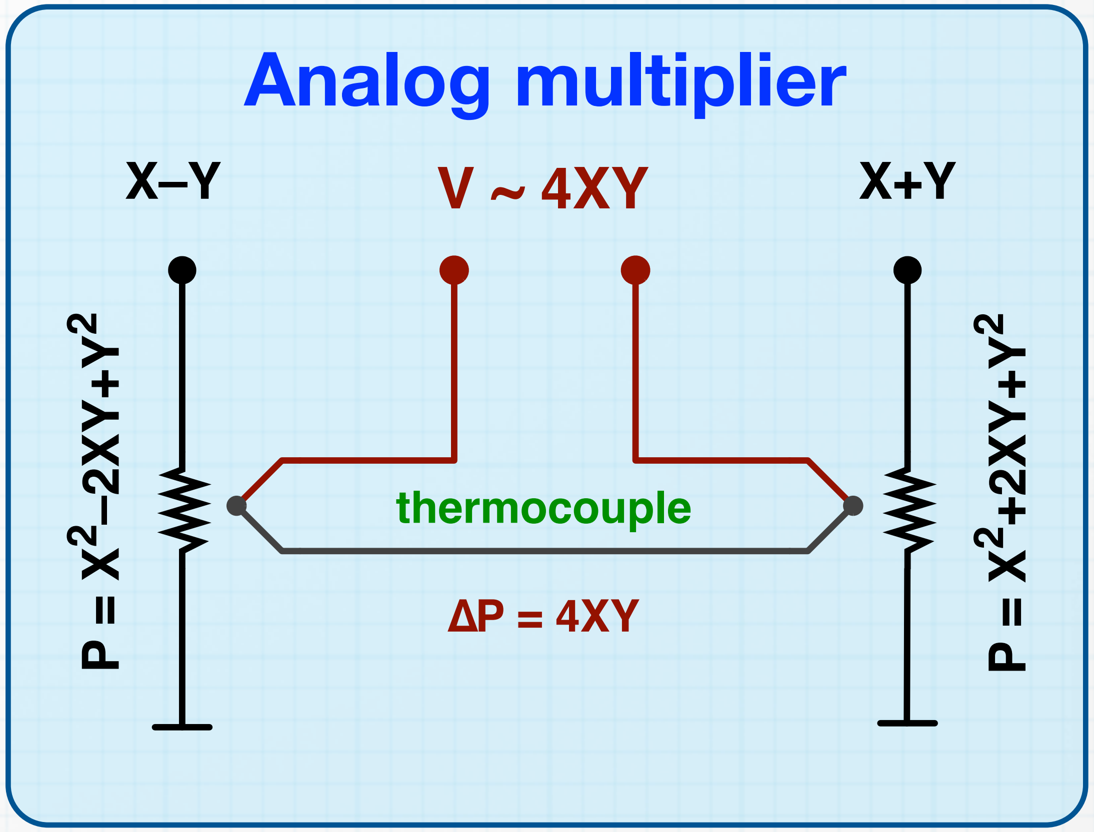
- The radio link breaks the hypothesis of symmetry of the two channels, introducing a phase θ
- The cross spectrum is complex
- The the antenna directivity results from the phase relationships
- The phase of the cross spectrum indicates the direction of the radio source

R. Hanbury Brown & al., Nature 170(4338) p.1061-1063, 20 Dec 1952
R. Hanbury Brown, R. Q. Twiss, Phyl. Mag. ser.7 no.366 p.663-682

Early implementations

1940-1950 technology

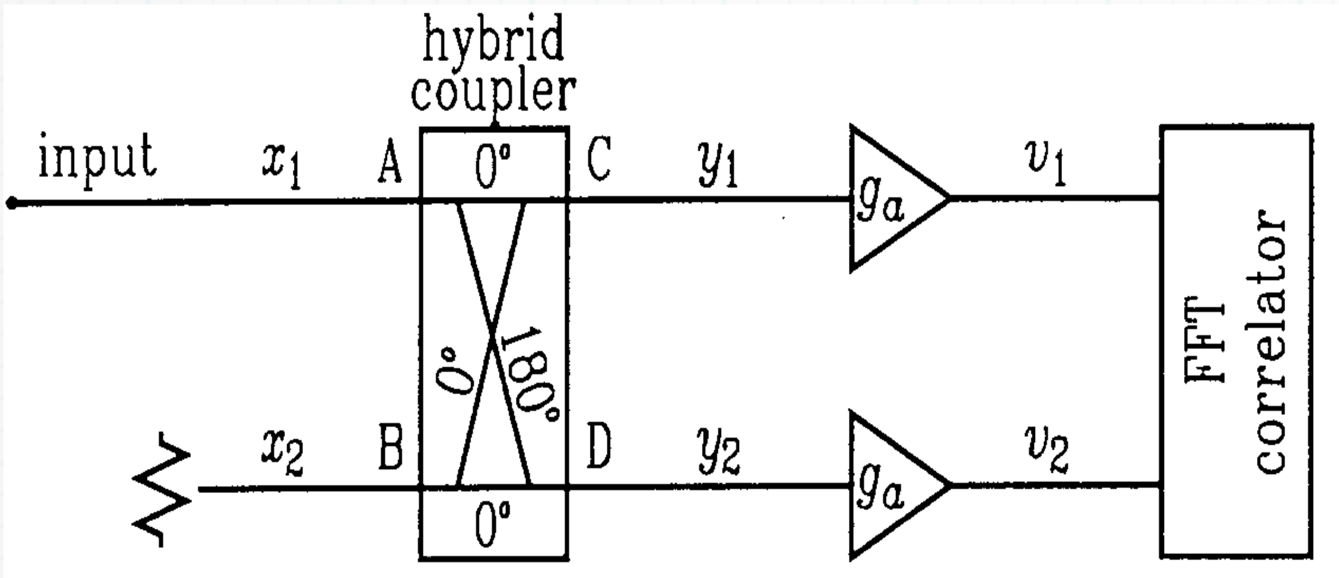


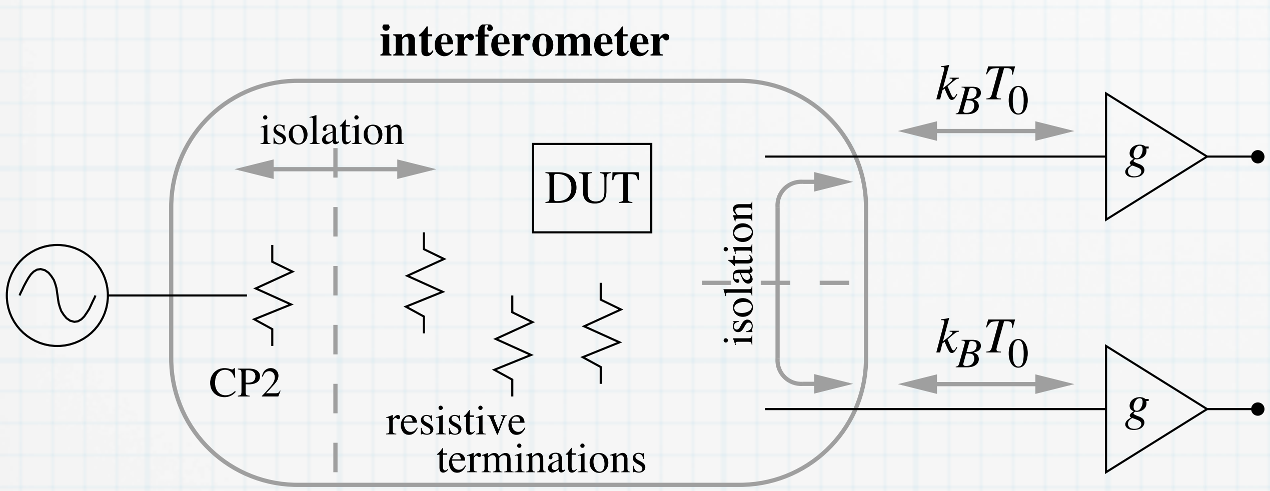


Spectral analysis at the single frequency f_0, in the bandwidth B

Need a filter pair for each Fourier frequency

Thermal noise compensation





Correlation-and-averaging rejects the thermal noise

hybrid output

$$
\begin{aligned}
y_1(t) &= \frac{1}{\sqrt{2}}\, x_2(t) + \frac{1}{\sqrt{2}}\, x_1(t) \\
y_2(t) &= \frac{1}{\sqrt{2}}\, x_2(t) - \frac{1}{\sqrt{2}}\, x_1(t)
\end{aligned}
$$

correlation

$$
\begin{aligned}
\mathcal{R}_{y_1 y_2}(\tau) &= \lim_{\theta\to\infty} \frac{1}{\theta} \int_\theta y_1(t)\, y_2^*(t-\tau)\, \mathrm{d}t \\
&= \frac{1}{2}\, \mathcal{R}_{x_2 x_2}(\tau) - \frac{1}{2}\, \mathcal{R}_{x_1 x_1}(\tau)
\end{aligned}
$$

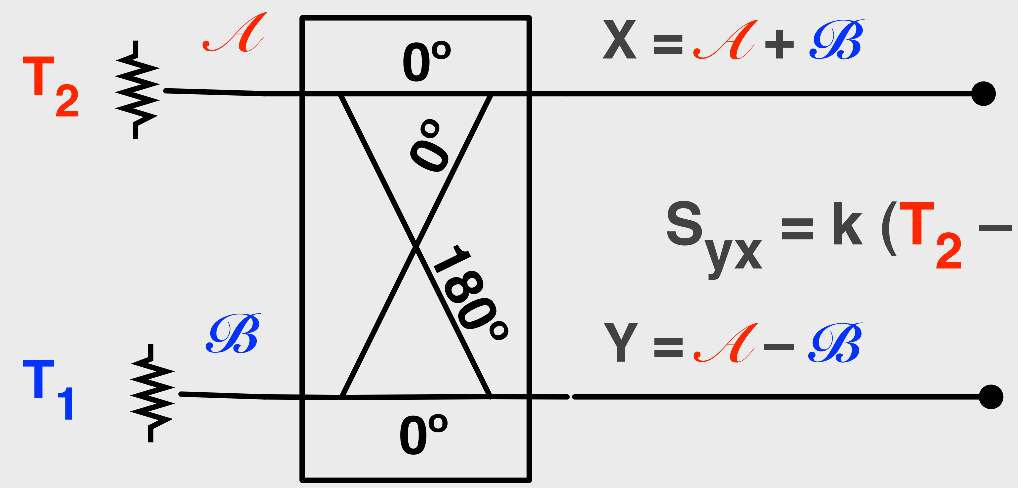
Fourier transform and thermal noise

$$
S_{y_1 y_2}(f) = \frac{1}{2}\, S_{x_2}(f) - \frac{1}{2}\, S_{x_1}(f)
$$

$$
S_{y_1 y_2}(f) = \frac{k_B (T_2 - T_1)}{2}
$$

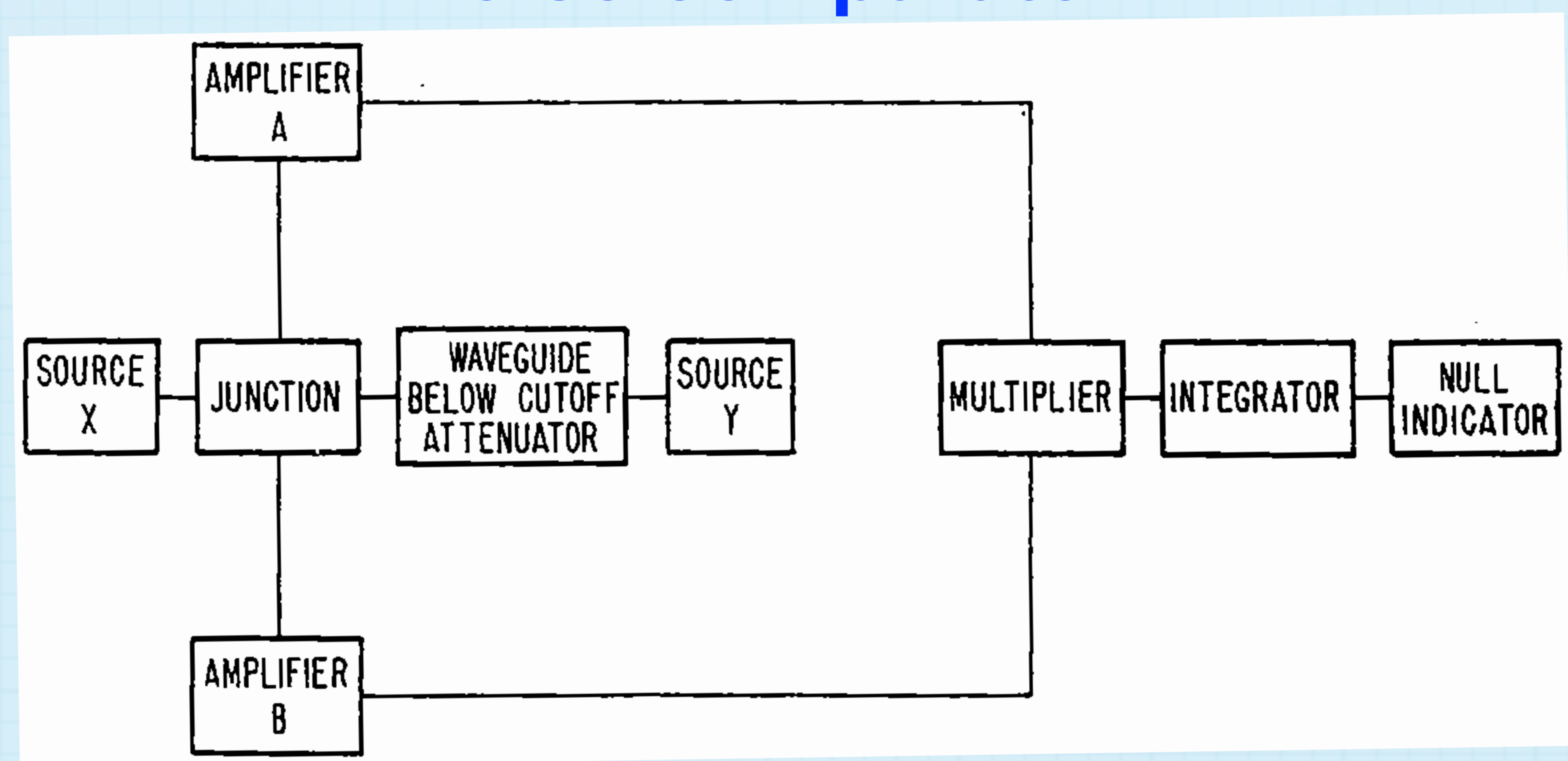
Radiometry & Johnson thermometry

correlation and anti-correlation



$$S_{yx} = k\,(T_2 - T_1)\,/\,2$$

noise comparator



C. M. Allred, A precision noise spectral density comparator, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962

Re-definition of the Kelvin?

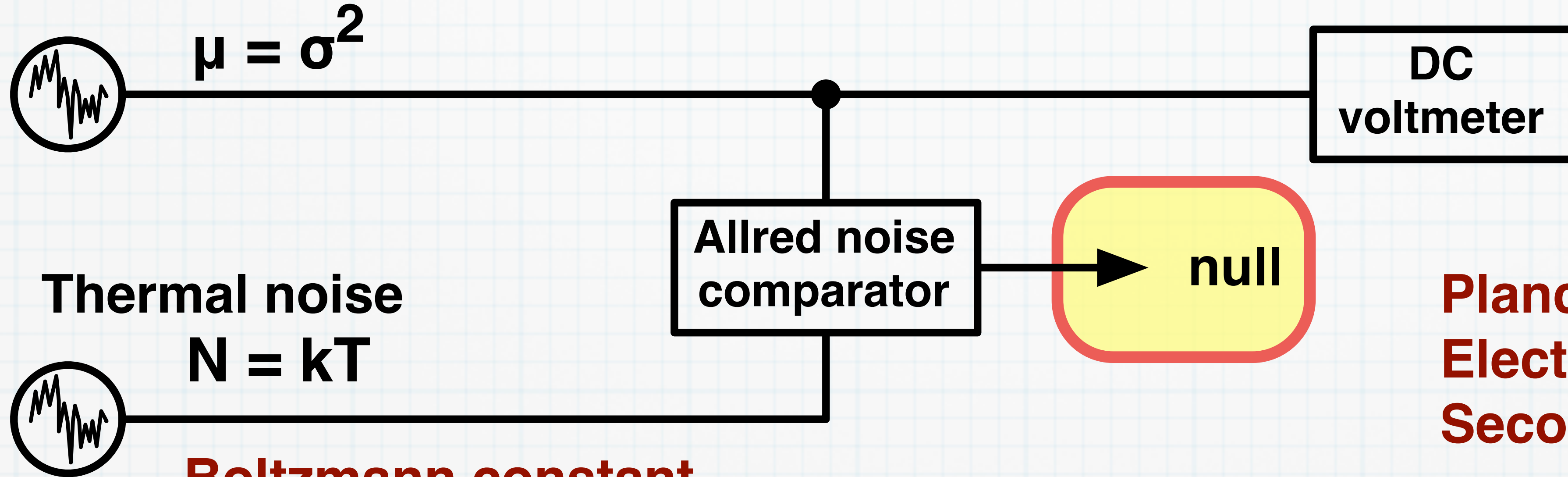
thermal noise $S = kT$

shot noise $S = 2qI_{avg}R$

high accuracy of I_{avg} with a dc instrument

Poisson process

Josephson effect



$V_{DC} = hv / 2e$

Thermal noise

Planck constant
Electron charge
Second (Cesium)

Boltzmann constant

Property of the Poisson process

$$\mu = \sigma^2$$

Noise calibration

thermal noise $S = kT$

shot noise $S = 2qI_{avg}R$

high accuracy of I_{avg} with a dc instrument

Compare shot and thermal noise with a noise bridge

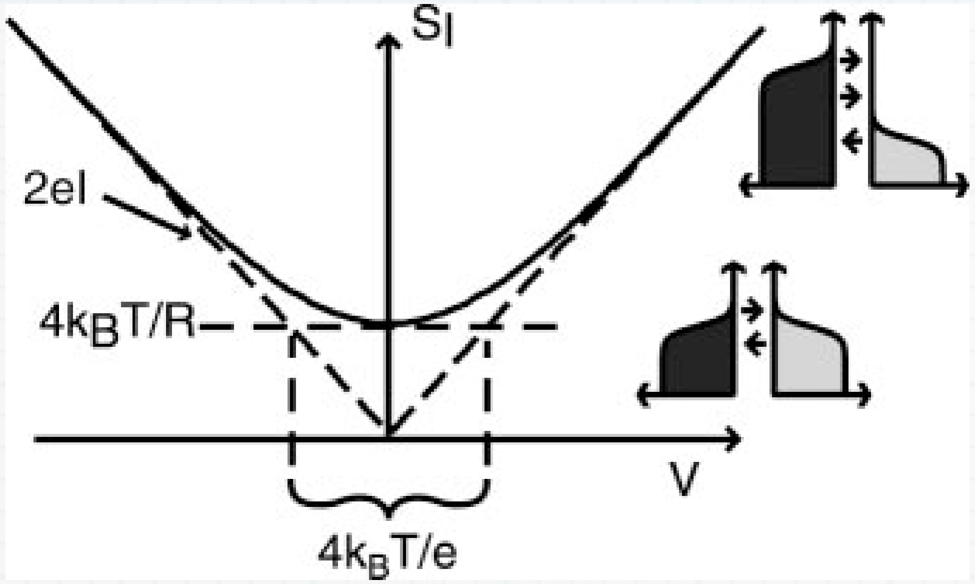


Fig. 1. Theoretical plot of current spectral density of a tunnel junction (Eq. 3) as a function of dc bias voltage. The diagonal dashed lines indicate the shot noise limit, and the horizontal dashed line indicates the Johnson noise limit. The voltage span of the intersection of these limits is $4k_BT/e$ and is indicated by vertical dashed lines. The bottom inset depicts the occupancies of the states in the electrodes in the equilibrium case, and the top inset depicts the out-of-equilibrium case where $eV \gg k_BT$.

This idea could turn into a re-definition of the temperature

In a tunnel junction, theory predicts the amount of shot and thermal noise

L. Spietz & al., Primary electronic thermometry using the shot noise of a tunnel junction, Science 300(20) p. 1929-1932, jun 2003

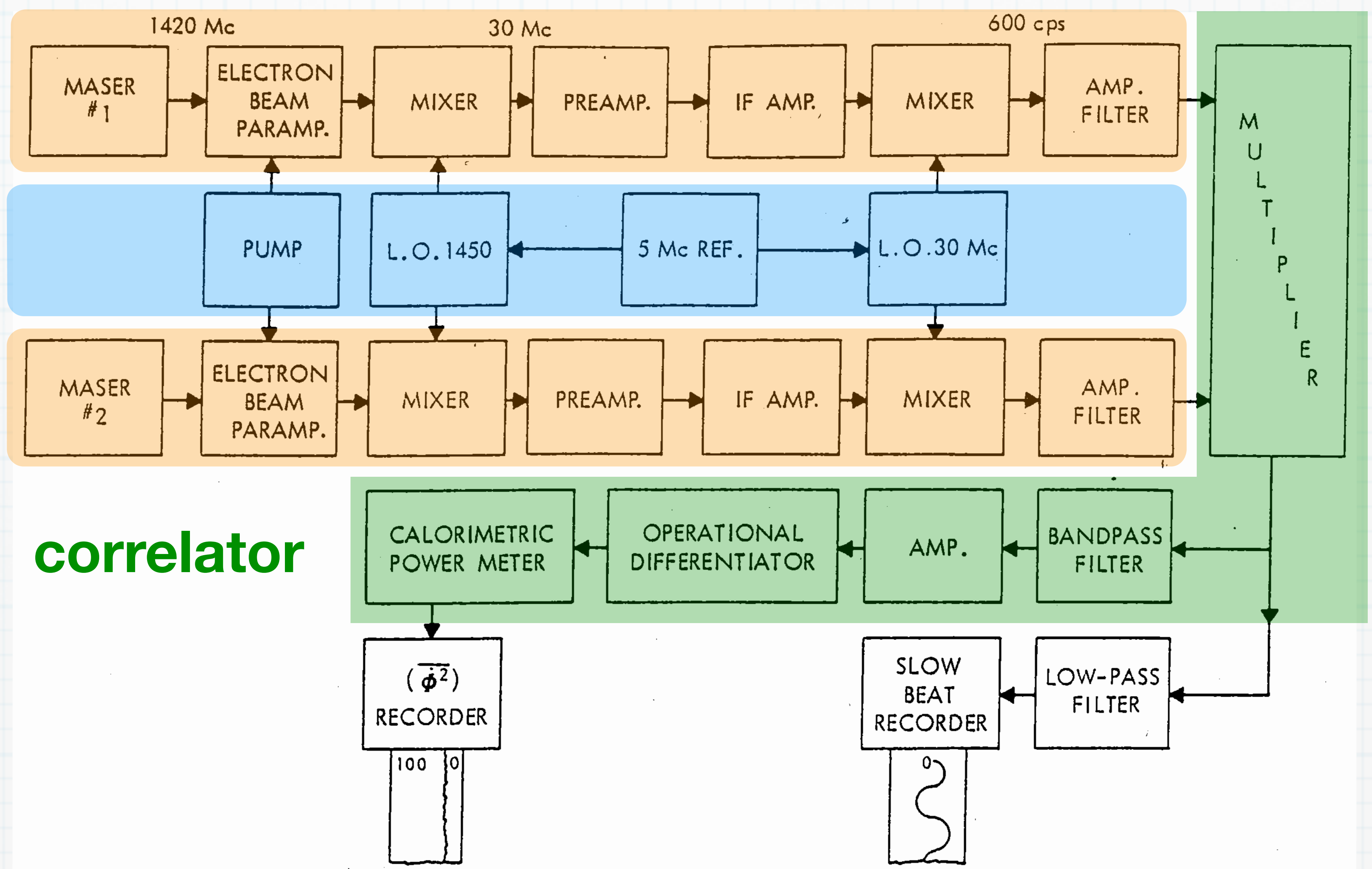
Measurement of the frequency noise of a H-maser

H maser

common synthesizer

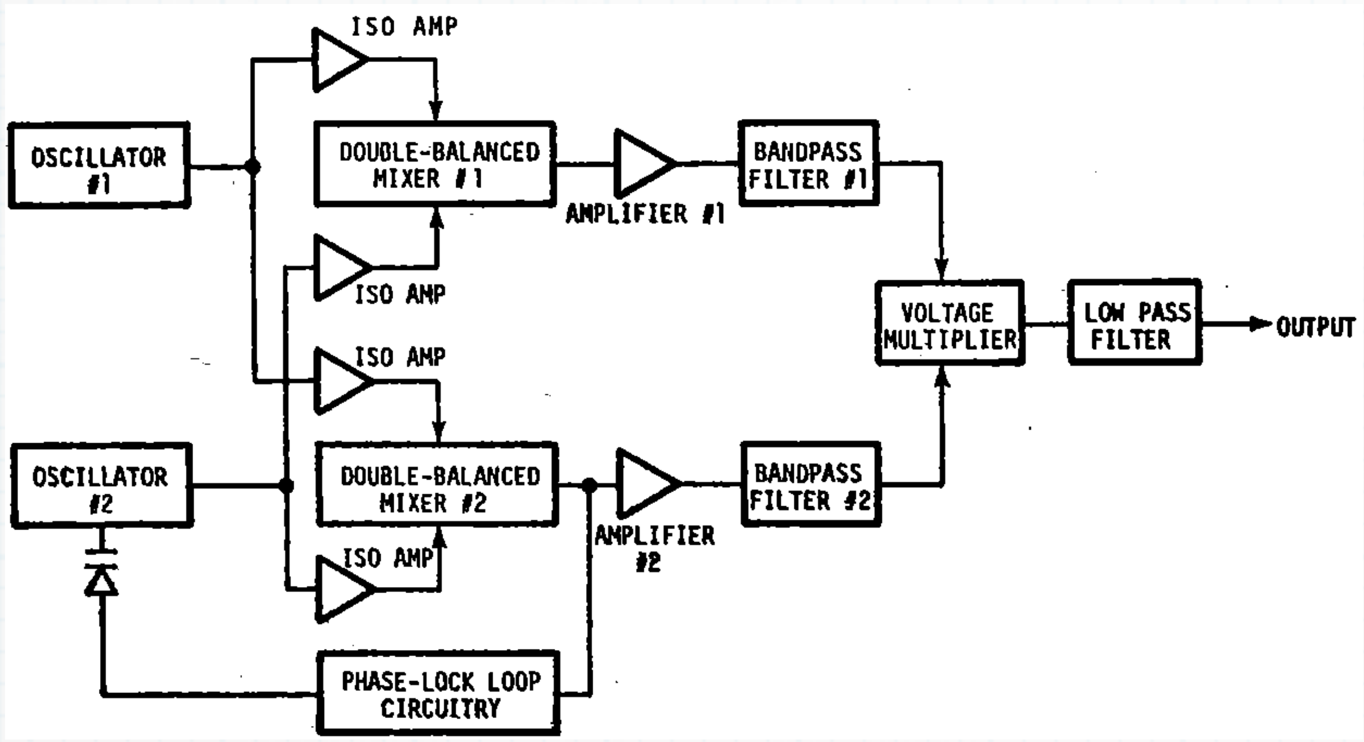
H maser

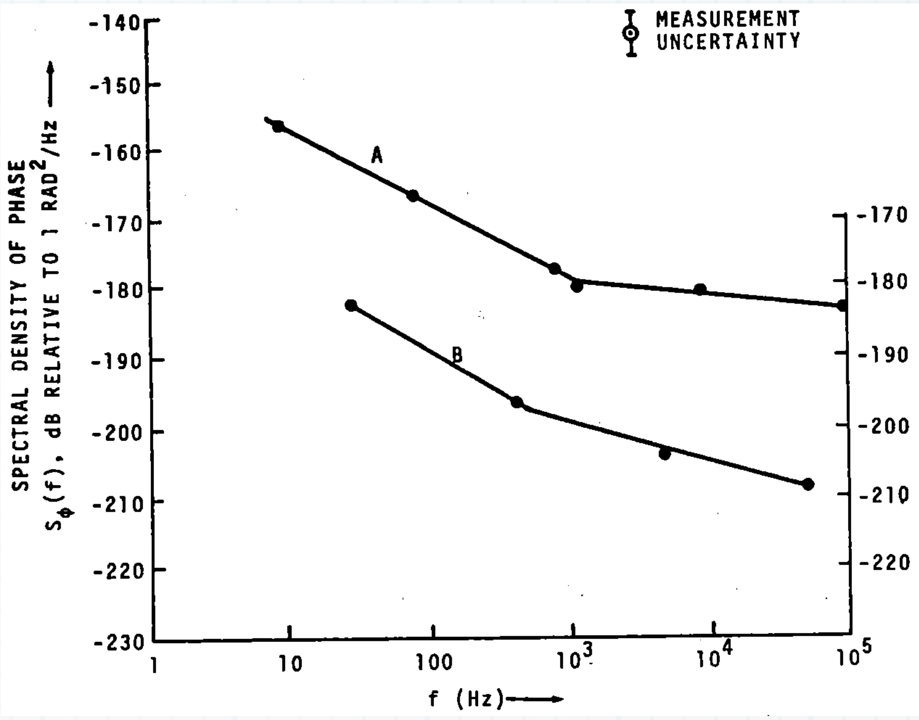
correlator



R. F. C. Vessot, Proc. Nasa Symp. on Short Term Frequency Stability p.111-118, Greenbelt, MD, 23-24 Nov 1964

Phase noise measurement

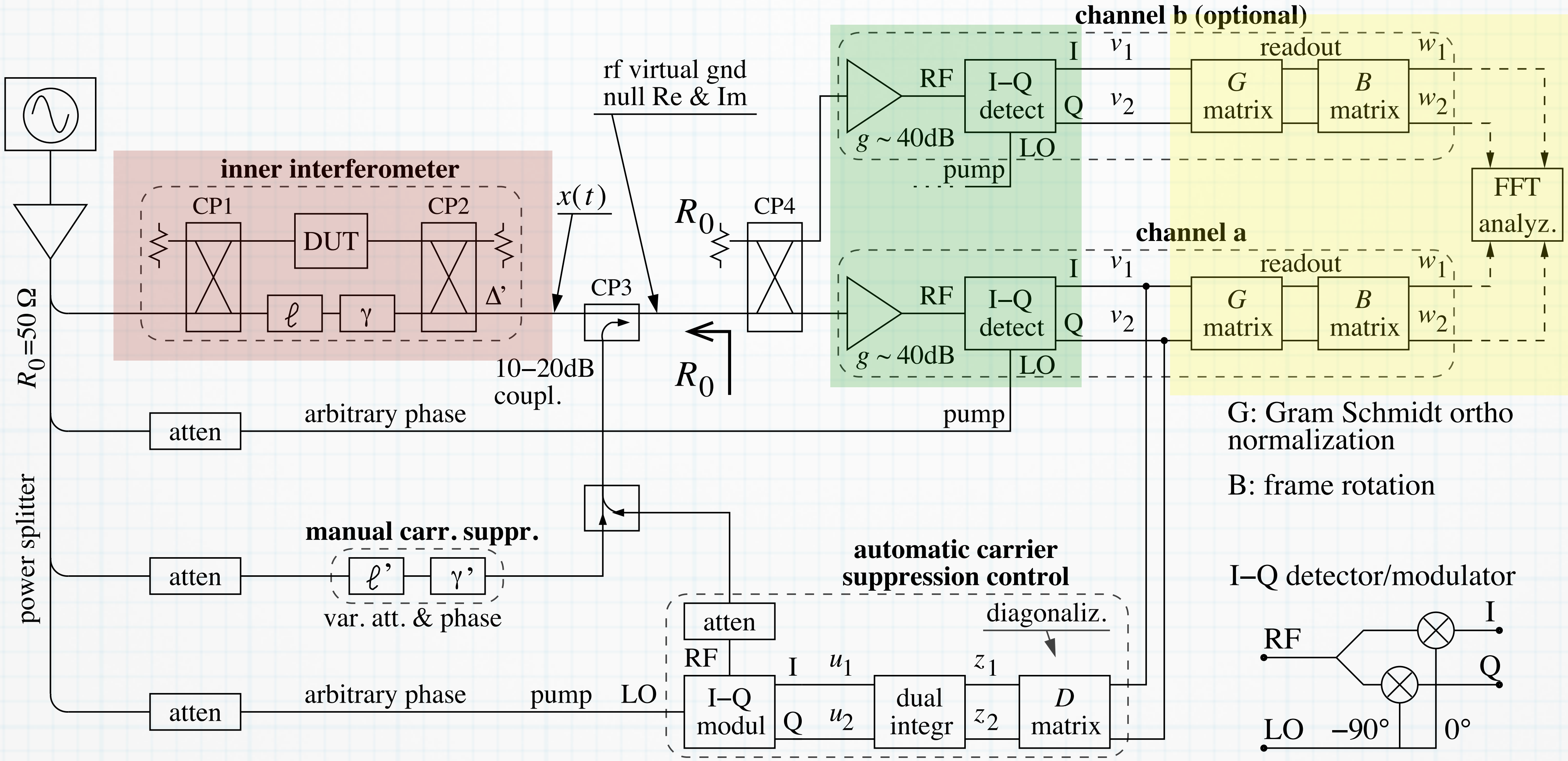




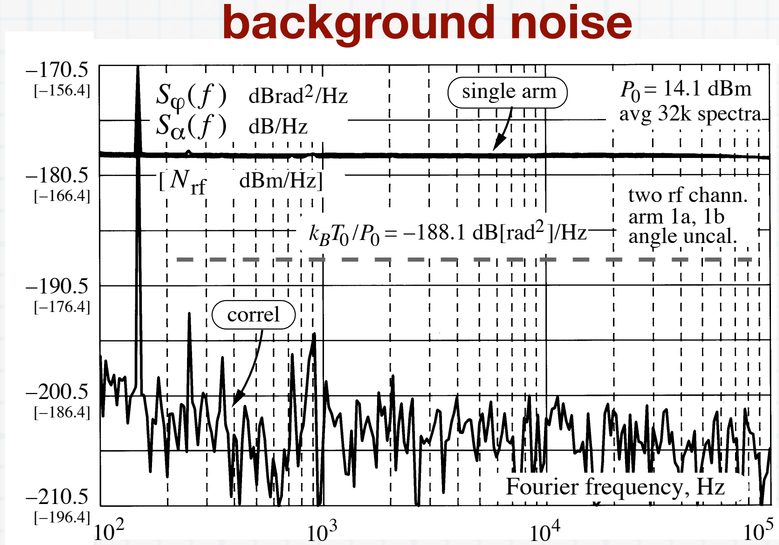
(relatively) large correlation bandwidth provides low noise floor in a reasonable time

F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976
More popular after W. Walls, Proc. 46th FCS pp.257-261, 1992

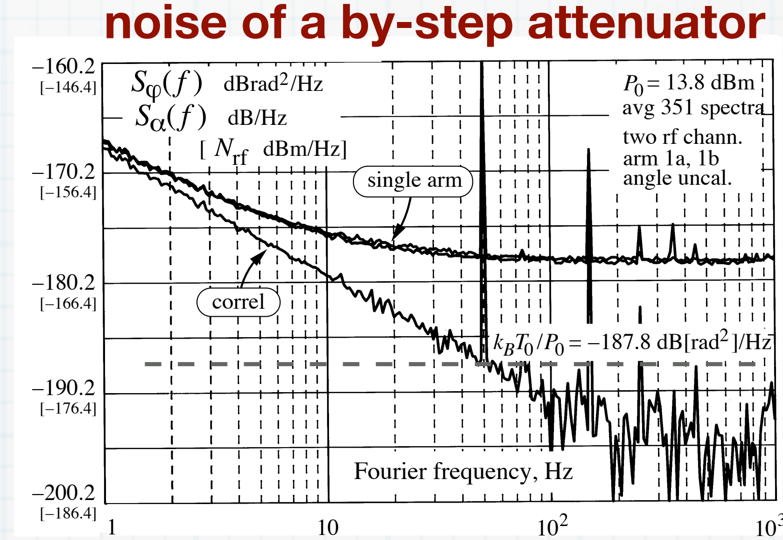
Phase noise measurement



background noise



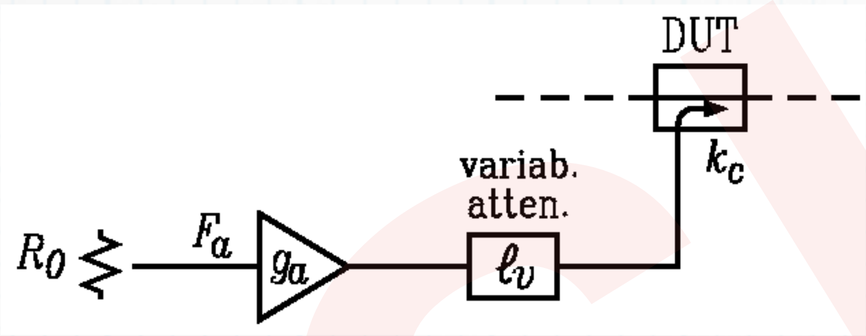
noise of a by-step attenuator

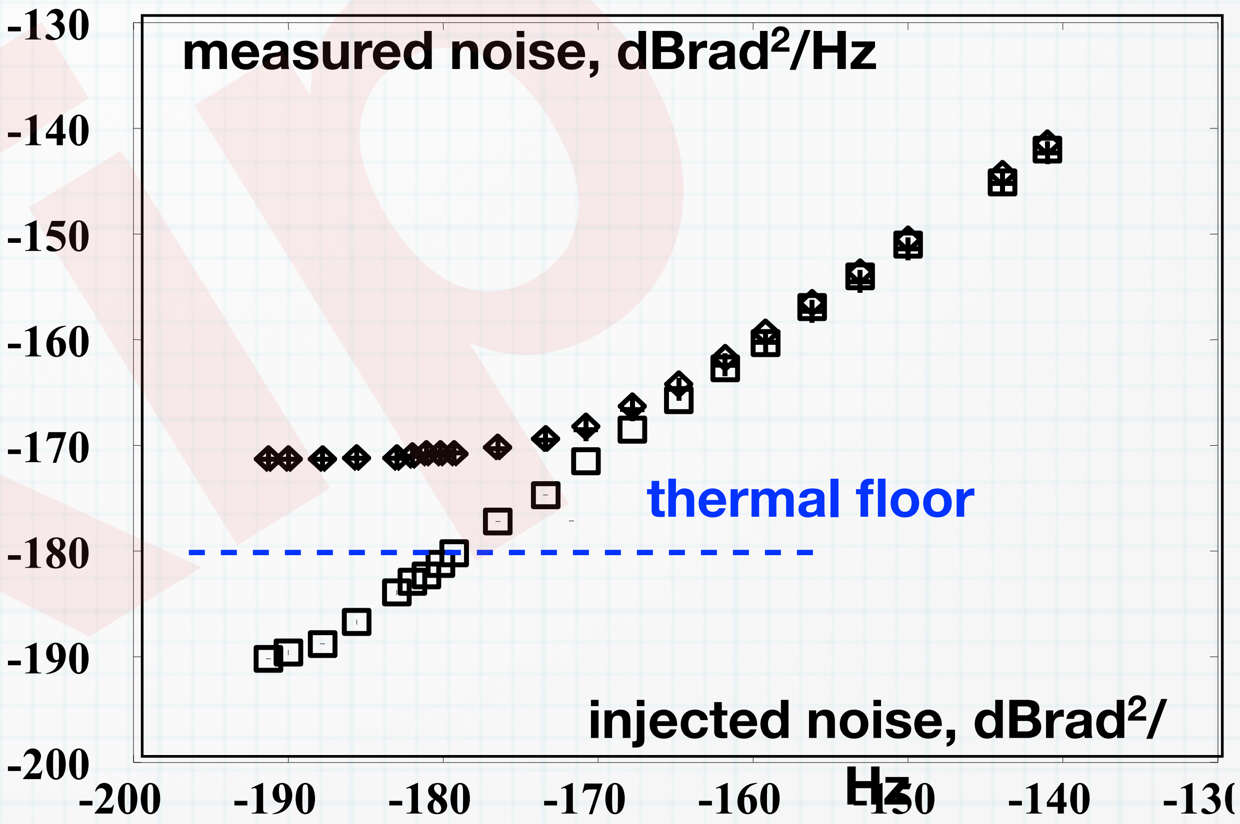


E. Rubiola, V. Giordano, Rev. Sci. Instrum. 71(8) p.3085-3091, aug 2000
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 73(6) pp.2445-2457, jun 2002

Thermal noise compensation

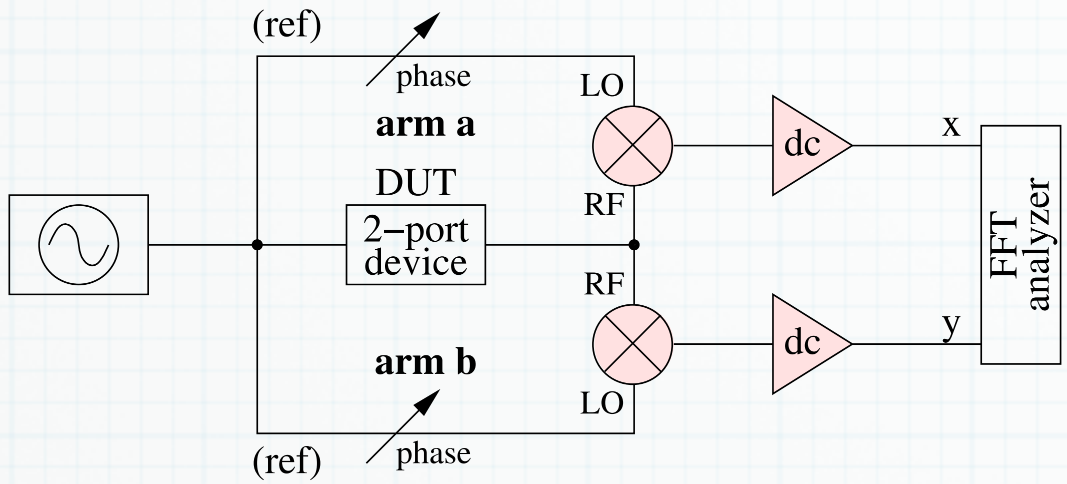
100 MHz prototype,
carrier power P_o = 8 dBm

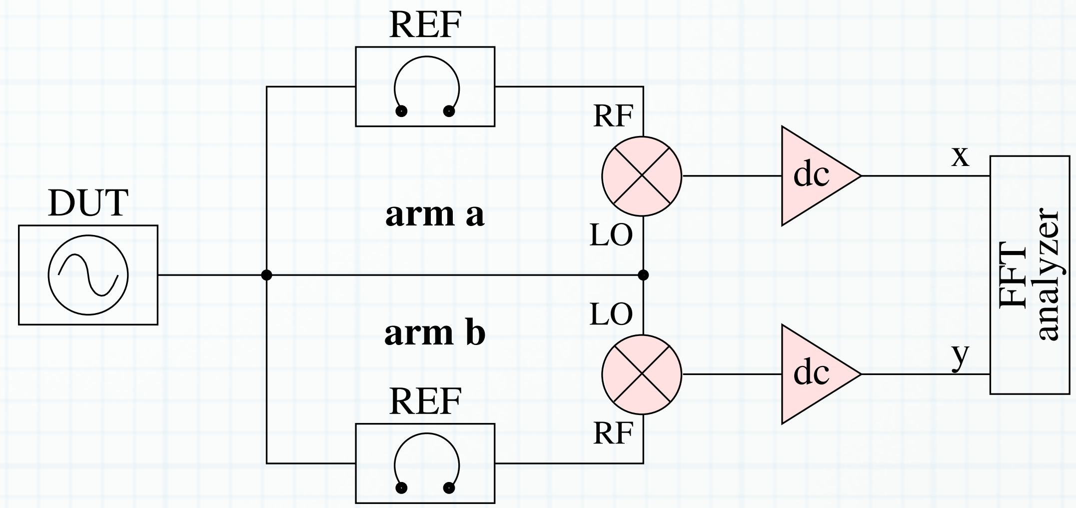


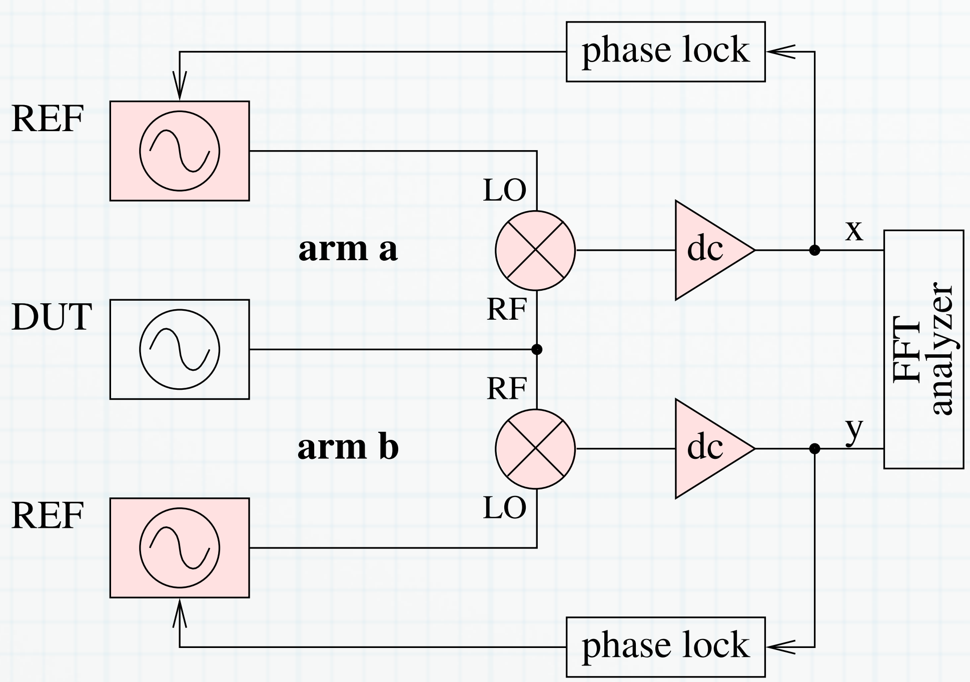


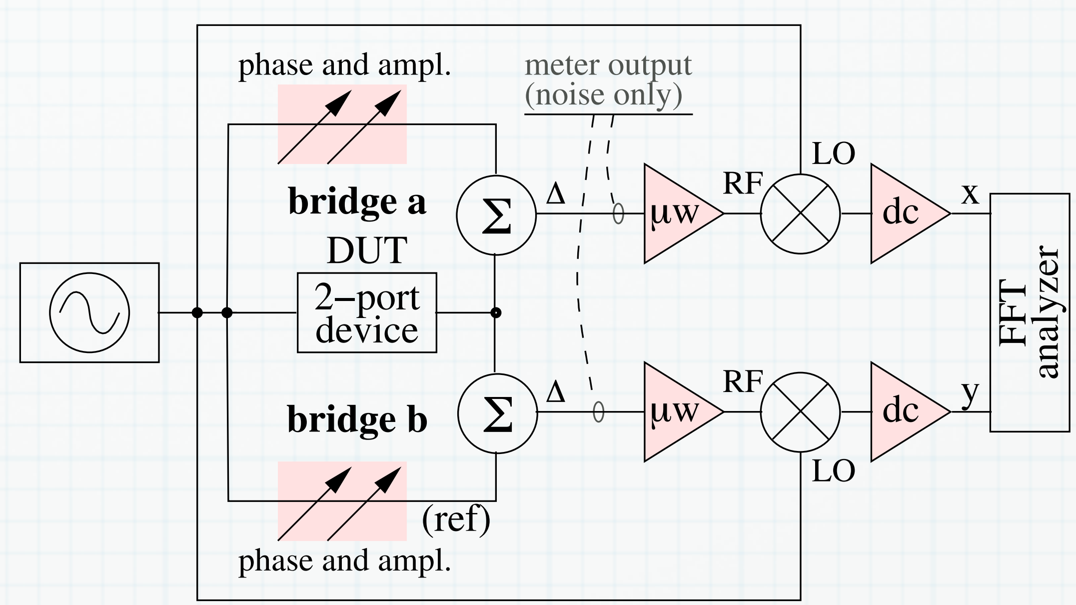
skip

Phase noise


(ref)
phase
LO
arm a
DUT
2–port device
RF
RF
arm b
LO
(ref)
phase
dc
dc
x
y
FFT analyzer

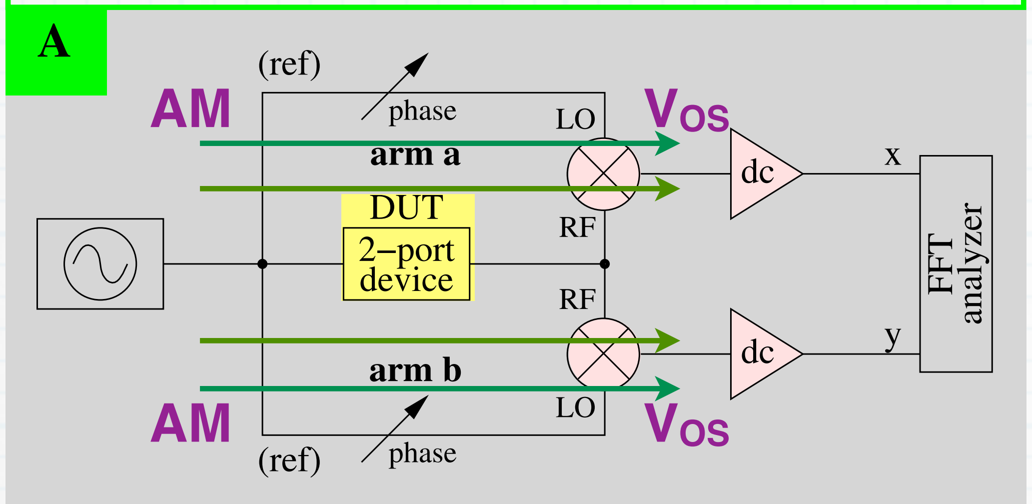

REF
RF
DUT
arm a
LO
LO
arm b
REF
RF
dc
dc
x
y
FFT analyzer


phase lock
REF
LO
arm a
DUT
RF
RF
arm b
REF
LO
dc
dc
x
y
FFT analyzer
phase lock

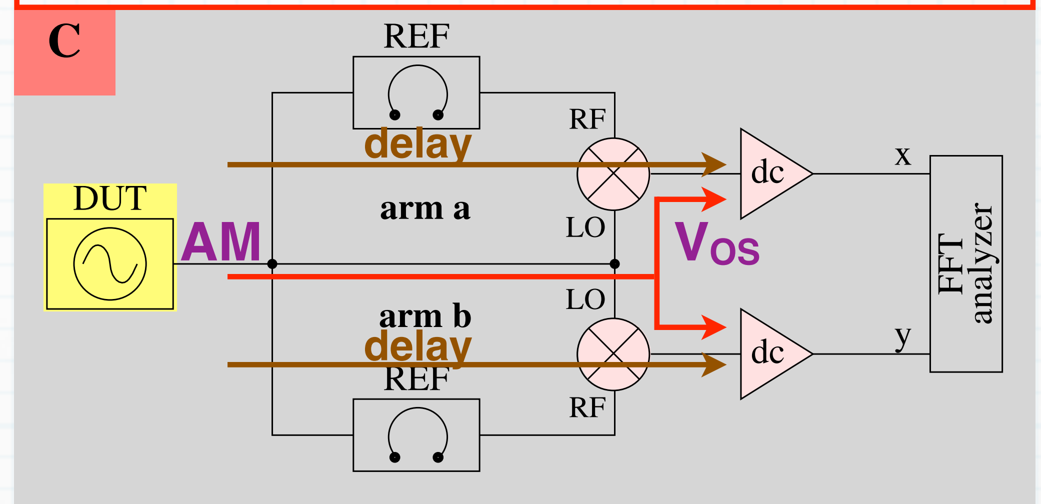

phase and ampl.
meter output
(noise only)
bridge a
DUT
2–port device
bridge b
Σ
Σ
Δ
Δ
µw
µw
RF
RF
LO
LO
dc
dc
x
y
FFT analyzer
(ref)
phase and ampl.

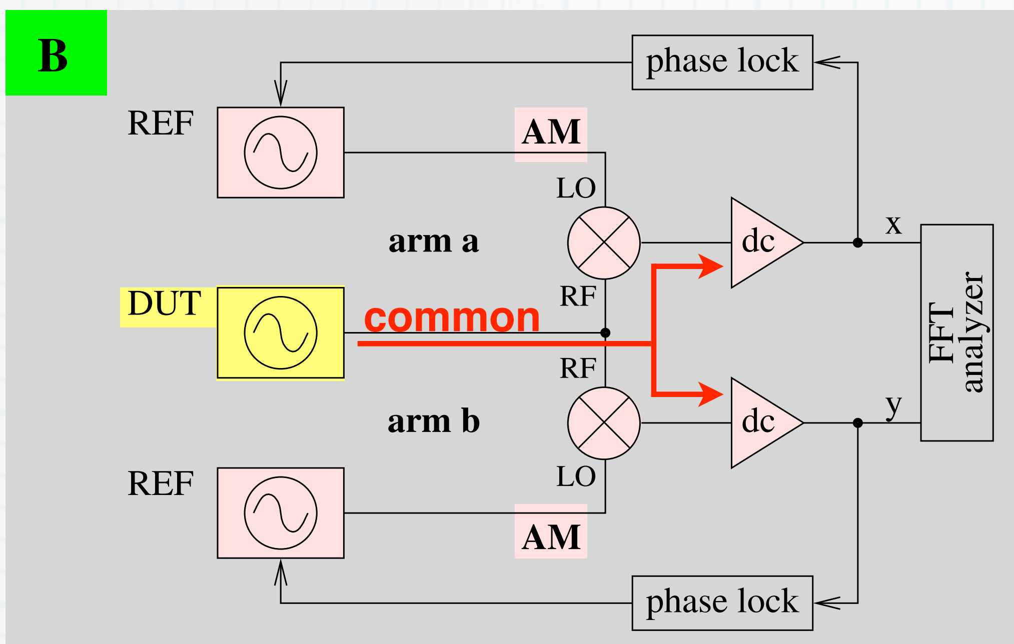
Effect of amplitude noise

Should set both channels at the sweet point, if exists

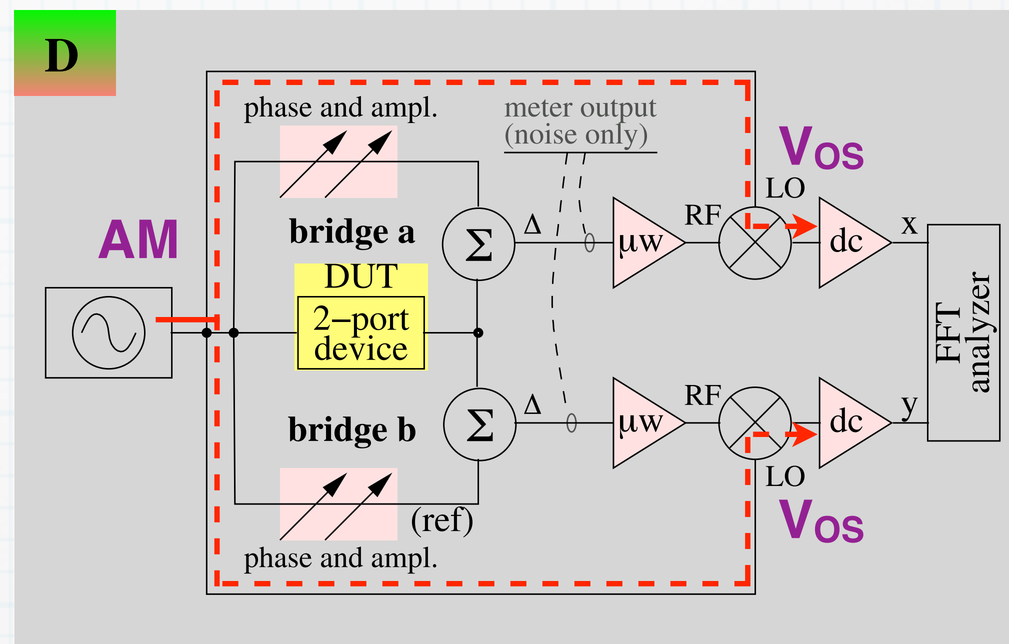


The delay de-correlates the two inputs, so there is no sweet point





Should set both channels at the sweet point of the RF input, if exists, by offsetting the PLL or by biasing the IF

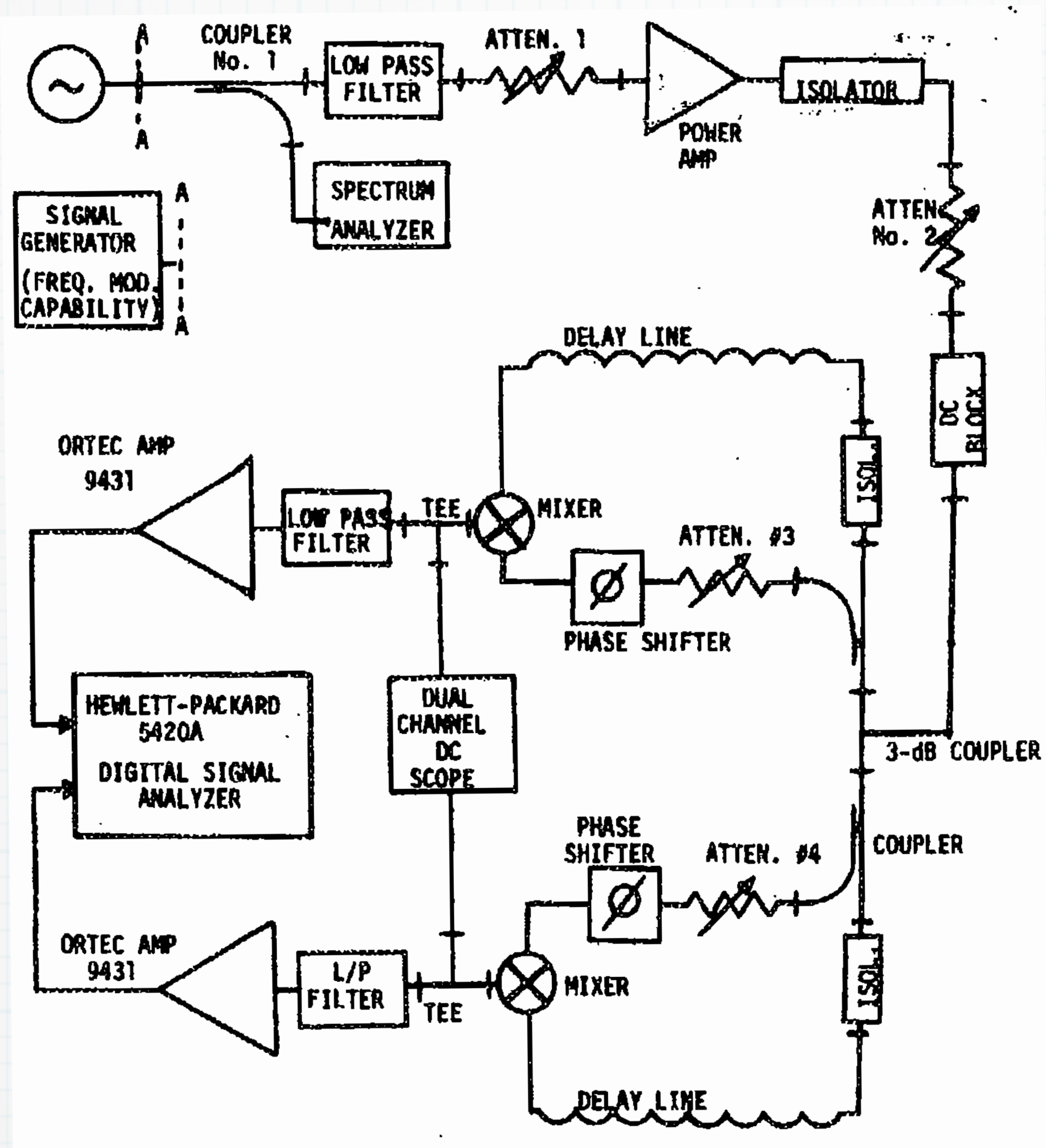


The effect of the AM noise is strongly reduced by the RF amplification

pink: noise rejected by correlation and averaging

E. Rubiola, R. Boudot, IEEE Transact. UFFC 54(5) pp.926-932, may 2007

Dual-delay-line method



Original idea:
D. Halford's NBS notebook
F10 p.19-38, apr 1975

First published: A. L. Lance & al, CPEM Digest, 1978

The delay line converts the frequency noise into phase noise

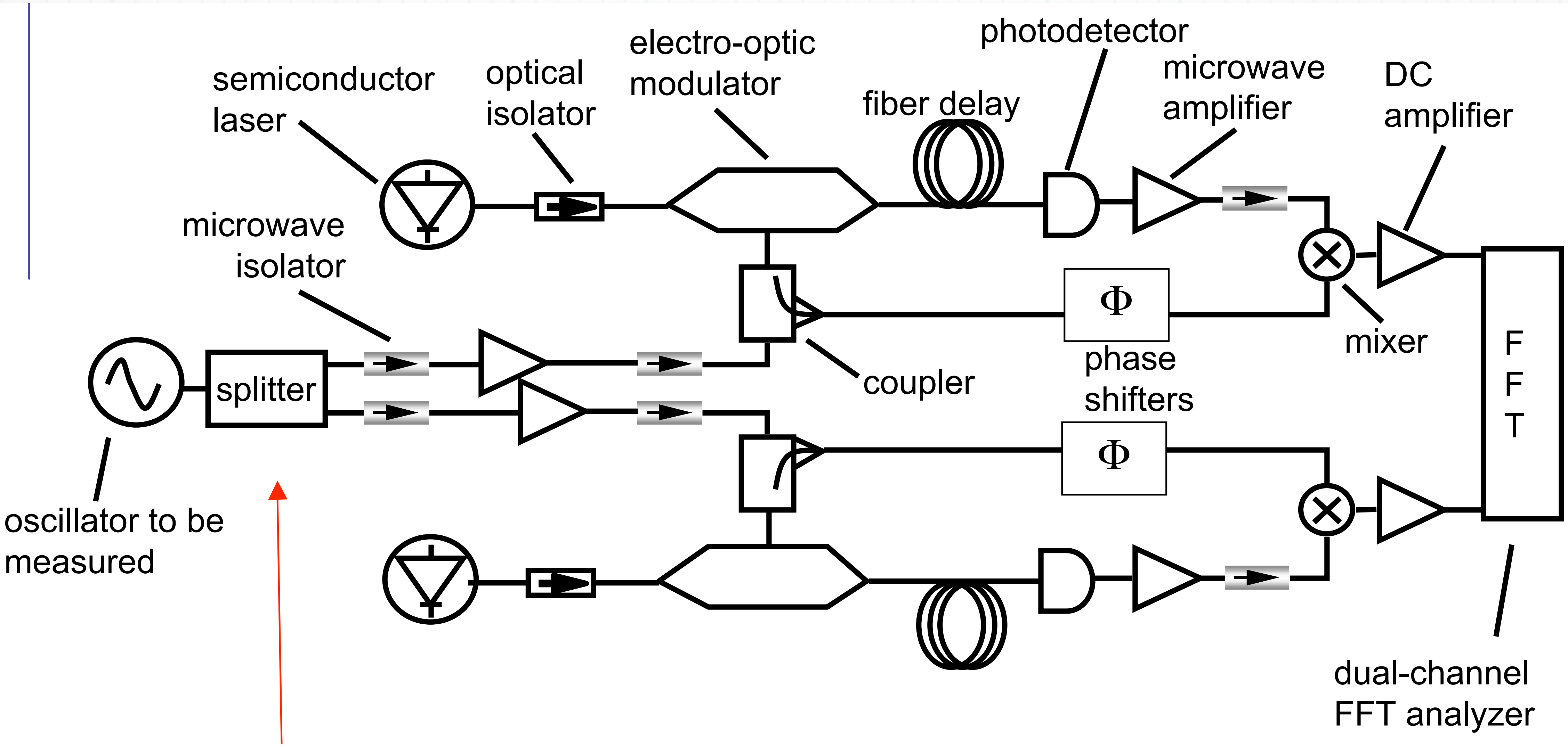
The high loss of the coaxial cable limits the maximum delay

Updated version:
The optical fiber provides long delay with low attenuation (0.2 dB/km or 0.04 dB/μs)

A.L. Lance, W.D. Seal, F. Labaar ISA Transact.21 (4) p.37-84, Apr 1982

Optical version of the dual-delay-line method

Two completely separate systems measure the same oscillator under test



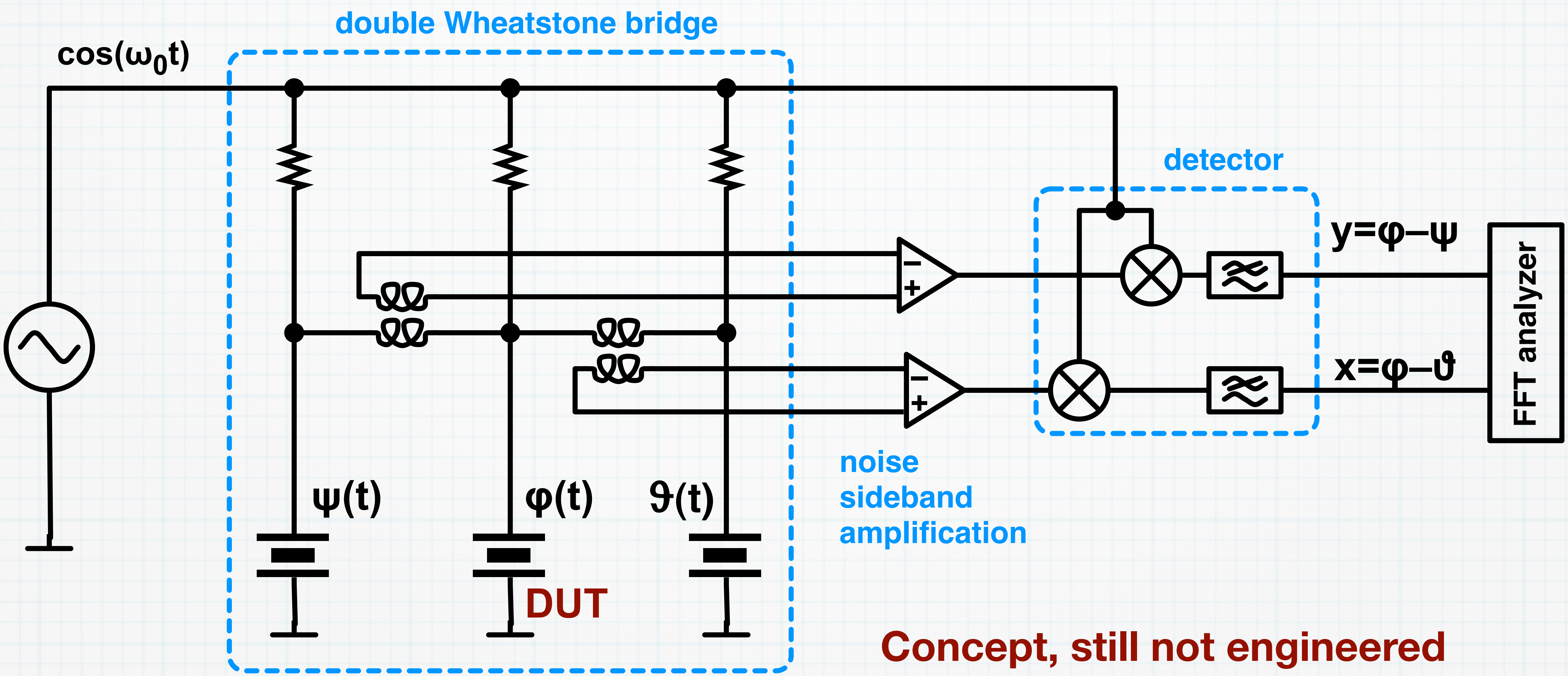
The only common part of the setup is the power splitter.

E. Salik, N. Yu, L. Maleki, E. Rubiola, Proc. Ultrasonics-FCS Joint Conf., Montreal, Aug 2004 p.303-306
Volyanskiy & al., JOSAB 25(12) 2140-2150, Dec.2008. Also arXiv:0807.3494v1 [physics.optics] July 2008

Frequency stability of a resonator

Enrico's weird brain

– however, with the cryogenic sapphire oscillators we can do way better –

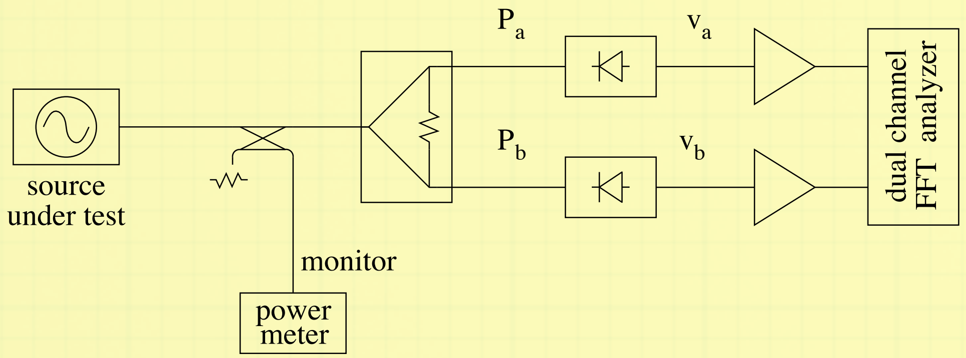


Concept, still not engineered

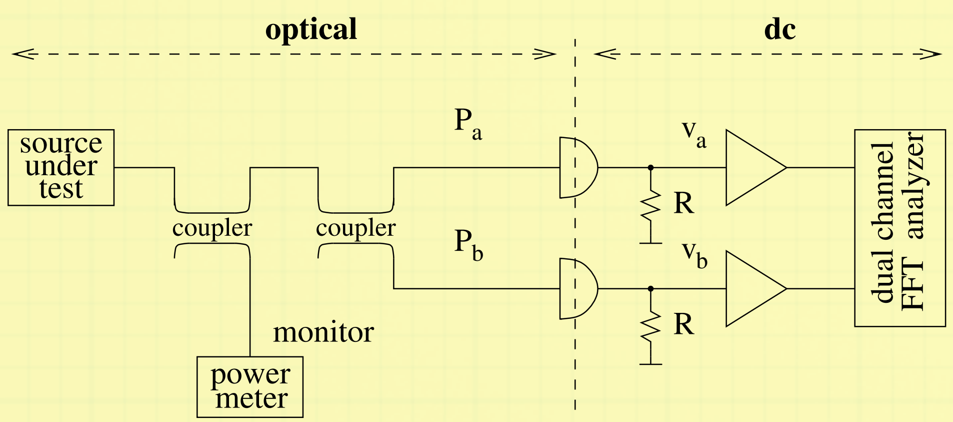
- Bridge in equilibrium
  - The amplifier cannot flicker around ω_0, which it does not know
- The fluctuation of the resonator natural frequency is estimated from phase noise
- Q matching prevents the master-oscillator noise from being taken in
- Correlation removes the noise of the instruments and the reference resonators

Amplitude noise & laser RIN

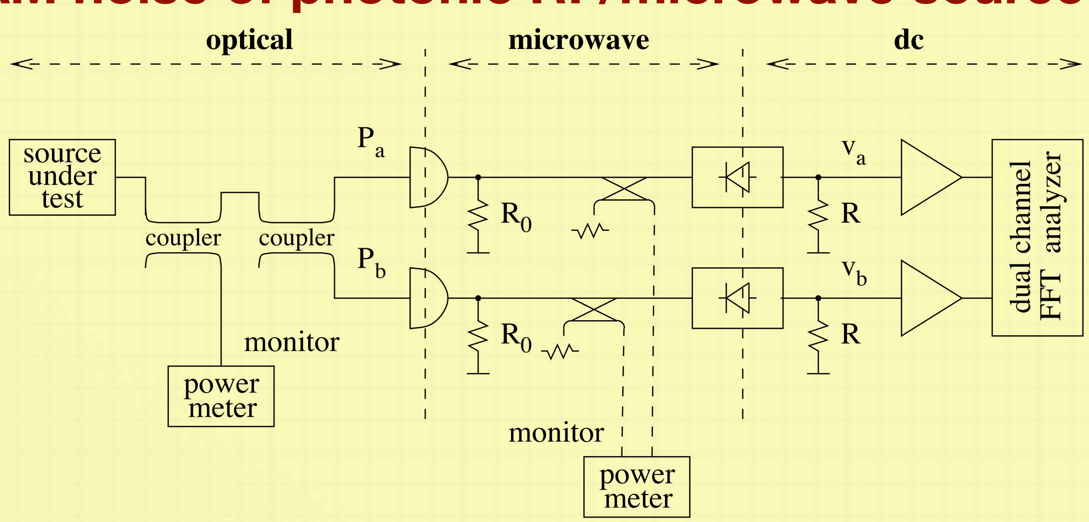
AM noise of RF/microwave sources



Laser RIN

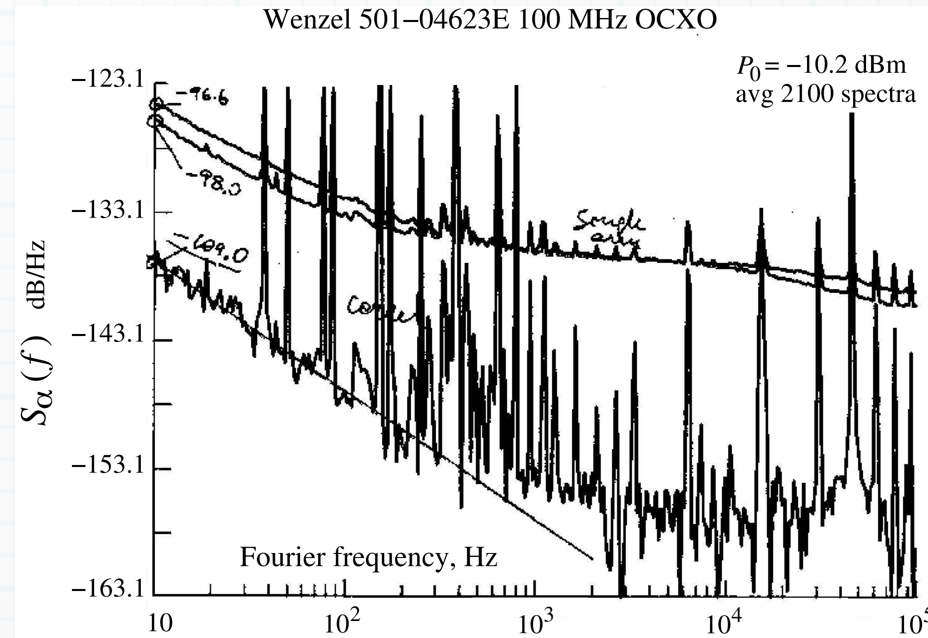


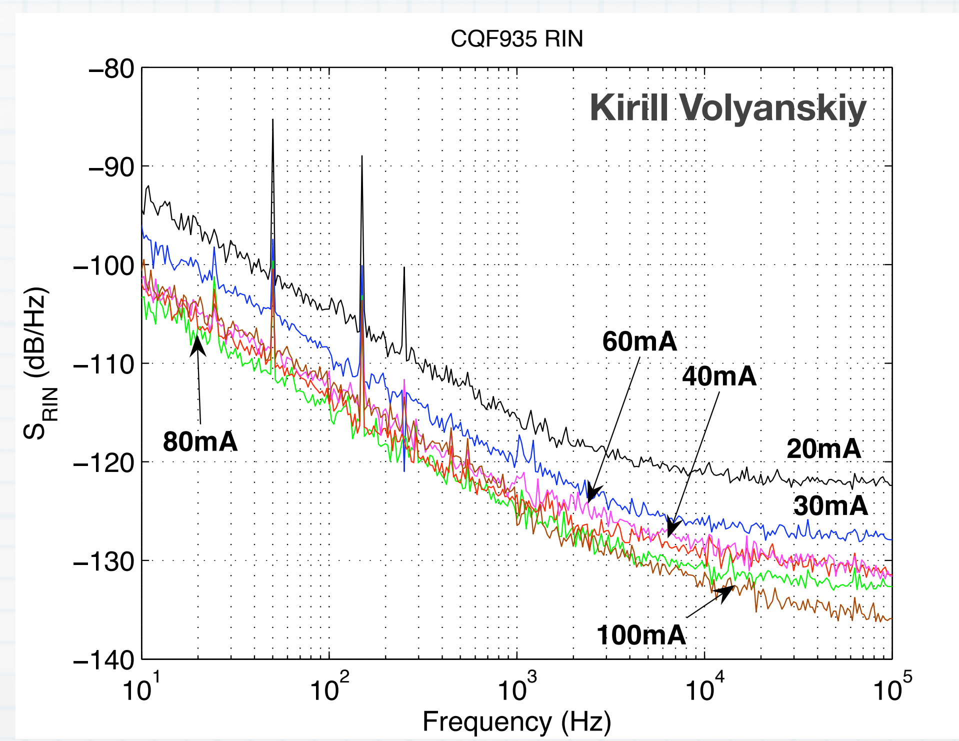
AM noise of photonic RF/microwave sources



E. Rubiola, the measurement of AM noise, dec 2005
arXiv:physics/0512082v1 [physics.ins-det]

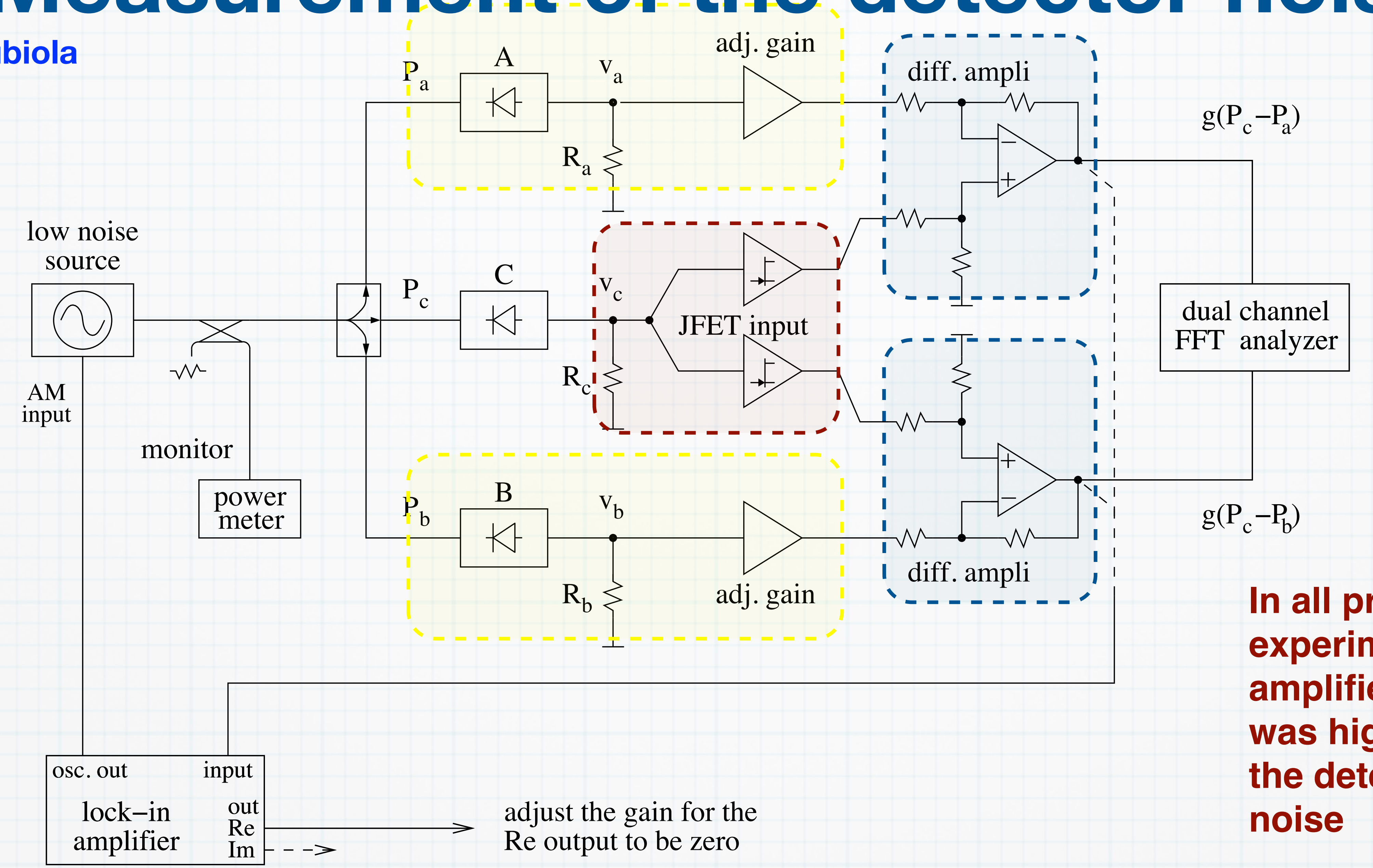
- In PM noise measurements, one can validate the instrument by feeding the same signal into the phase detector
- In AM noise this is *not possible* without a lower-noise reference
- Provided the crosstalk was measured otherwise, correlation enables to validate the instrument





Measurement of the detector noise

Grop & Rubiola

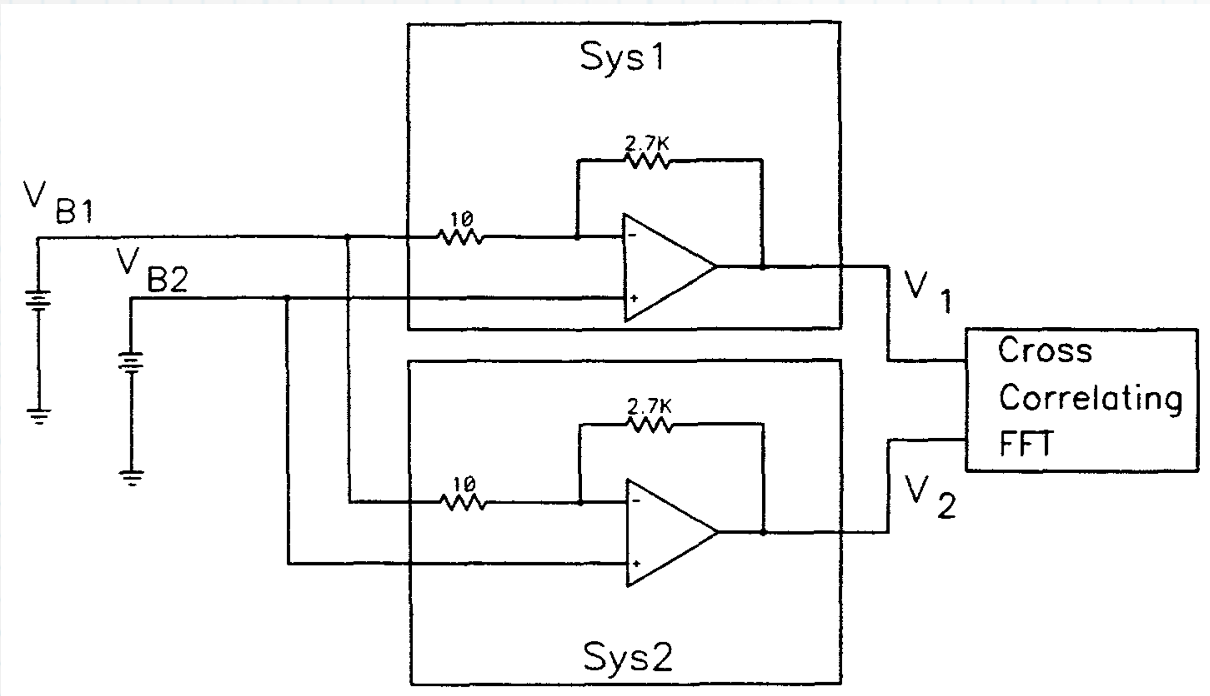


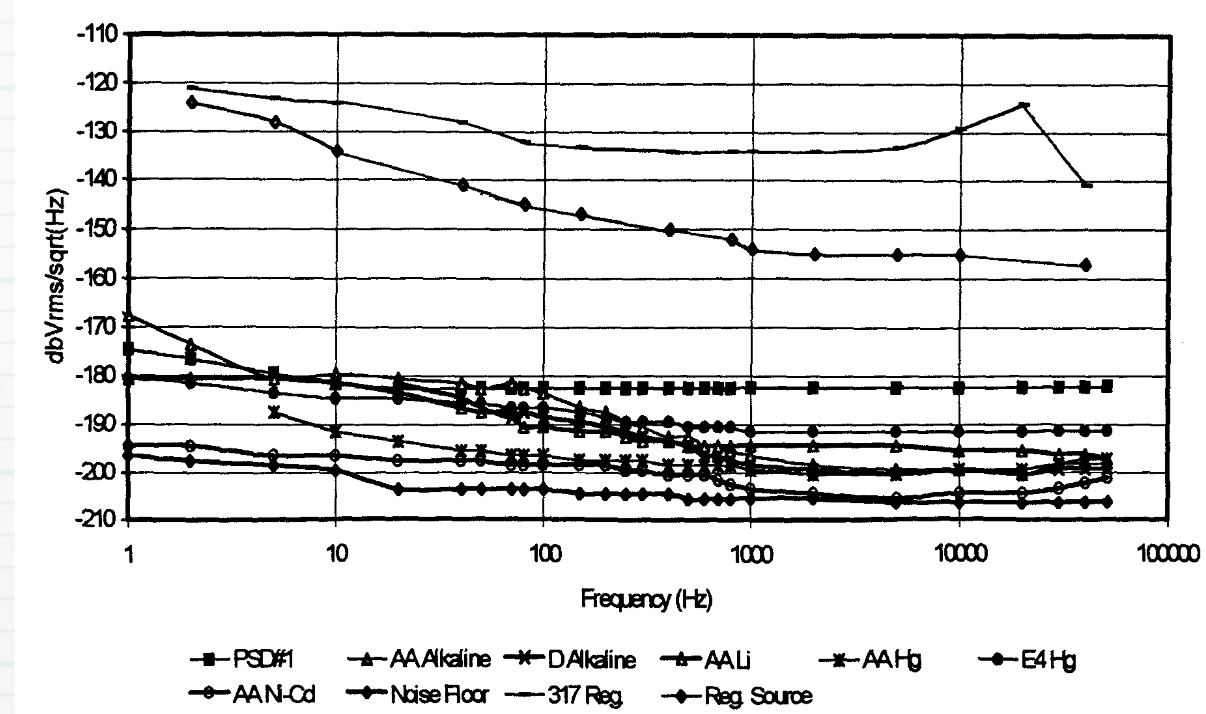
In all previous experiments, the amplifier noise was higher than the detector noise

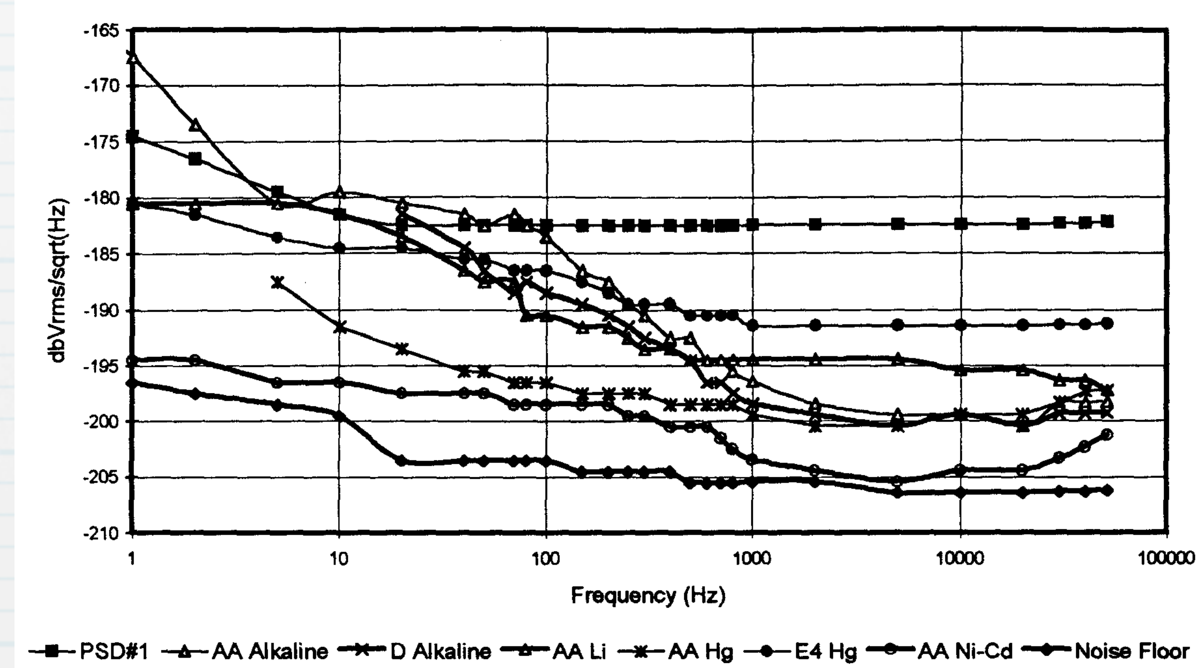
Basic ideas

- Remove the noise of the source by balancing C–A and C–B
    - Use a lock-in amplifier to get a sharp null measurement
- Channels A and B are independent –> noise is averaged out
- Two separate JFET amplifiers are needed in the C channel
    - JFETs have virtually no bias-current noise
- Only the noise of the detector C remains

Noise in chemical batteries







- **Do not waste DAC bits for a constant DC, $V = V_{B2} - V_{B1}$ has (almost) zero mean**
- **Two separate amplifiers measure the same quantity V**
- **Correlation rejects the amplifier nose, and the FFT noise as well**

C. K. Boggs, A. D. Doak, F. L. Walls, Proc. IFCS p.367-373 1995

Noise in semiconductors

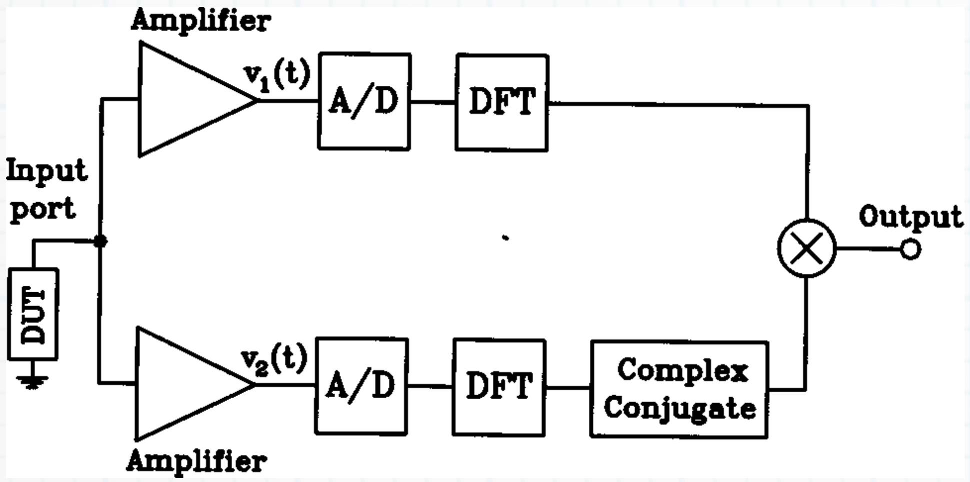


FIG. 2. Schematics of the building blocks of our correlation spectrum analyzer performing the suppression of the uncorrelated input noises by a digital processing of sampled data.

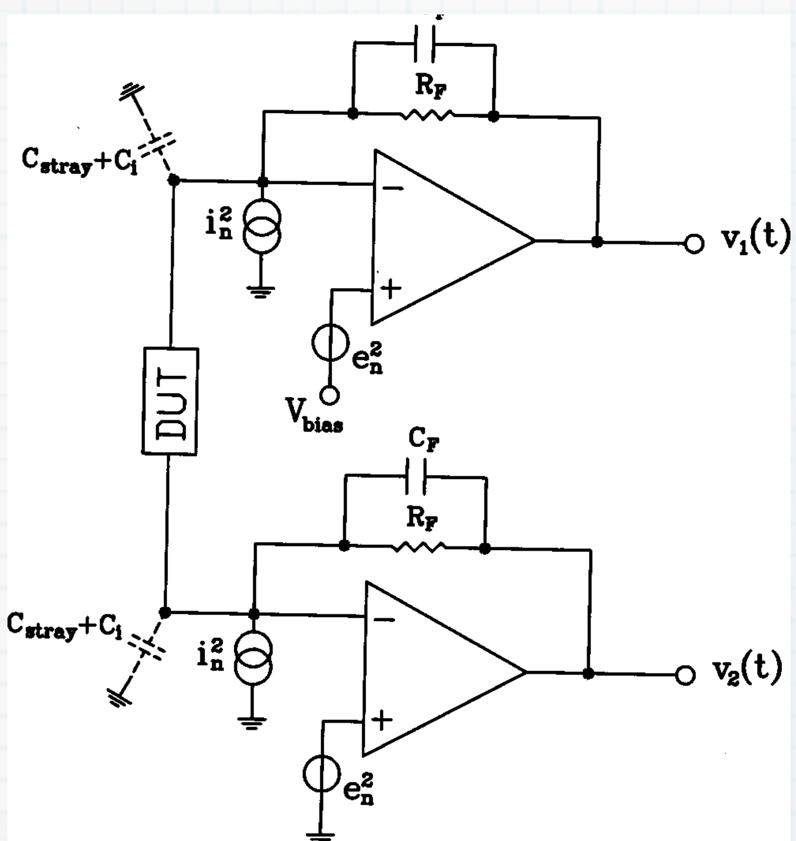


FIG. 3. Schematics of the active test fixture for current noise measurements.

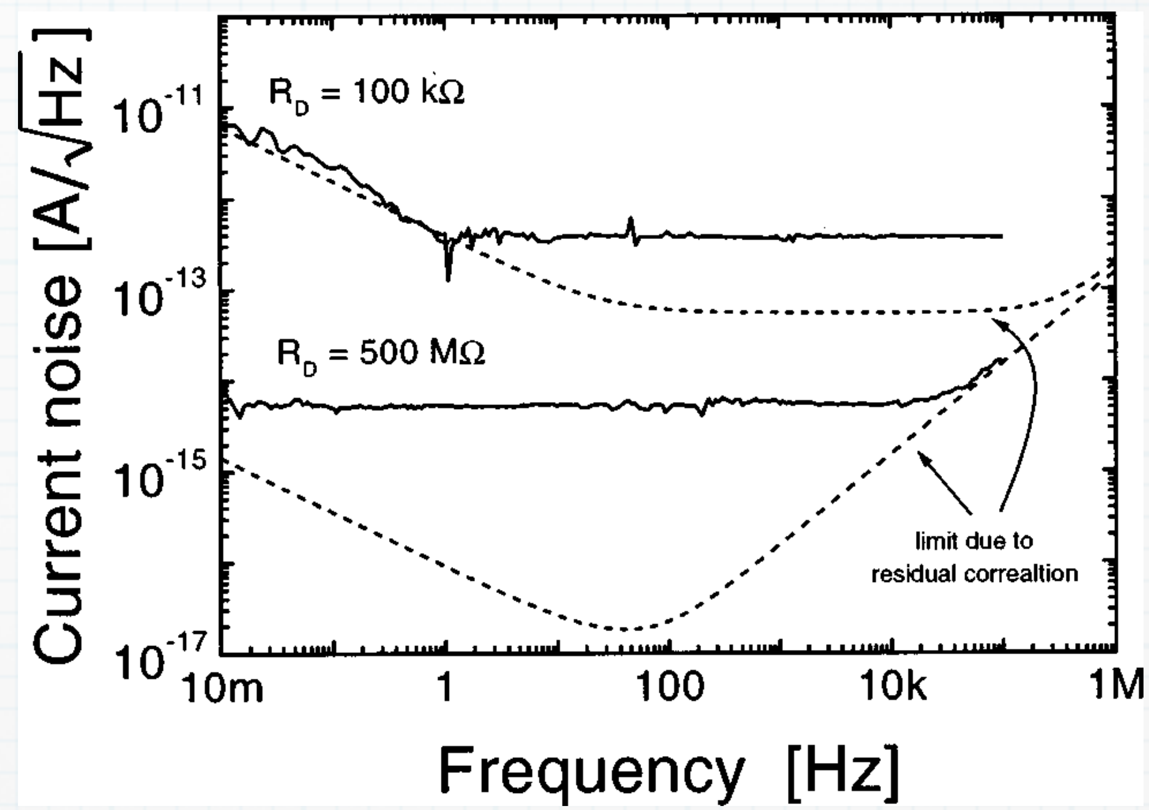
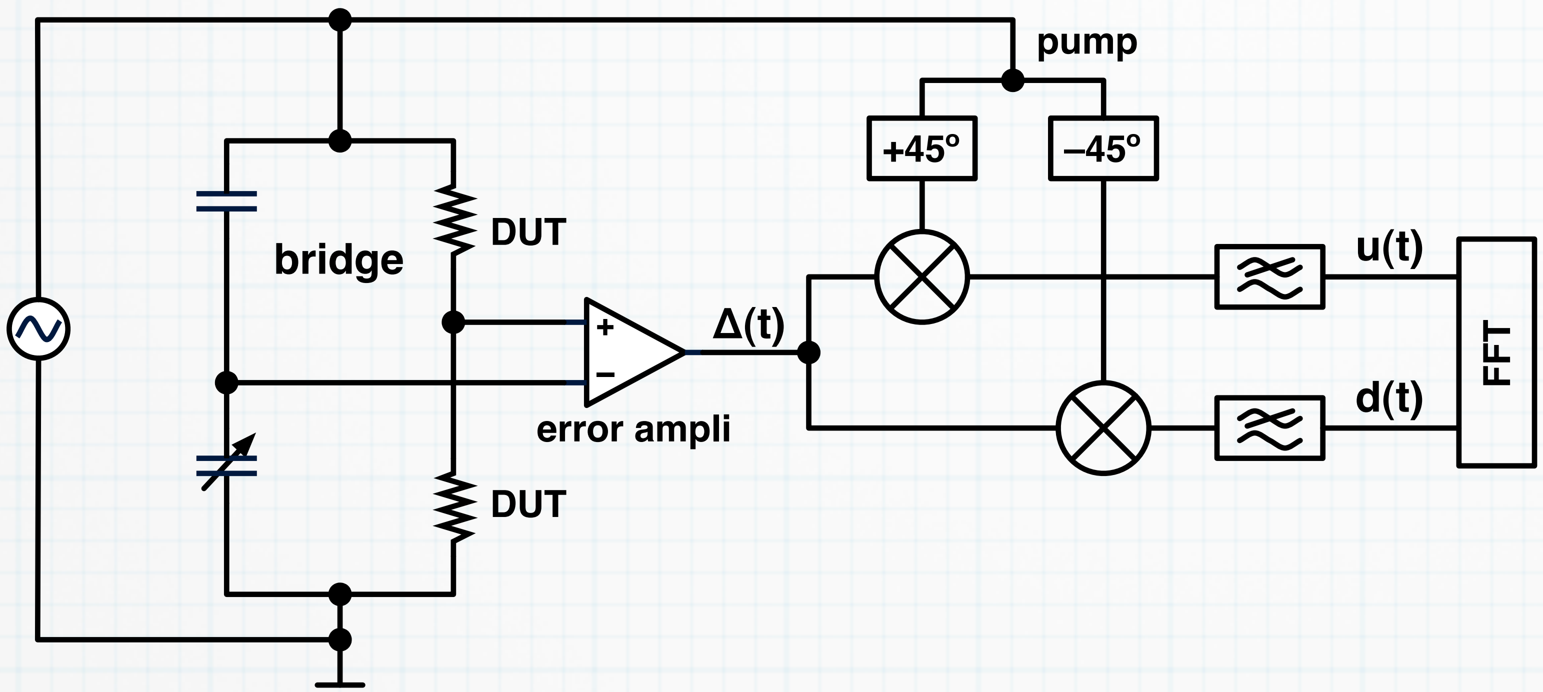


FIG. 9. Experimental frequency spectrum of the current noise from DUT resistances of 100 kΩ and 500 MΩ (continuous line) compared with the limits (dashed line) given by the instrument and set by residual correlated noise components.

M. Sampietro & al, Rev. Sci. Instrum 70(5) p.2520-2525, may 1999

Electro-migration in thin films



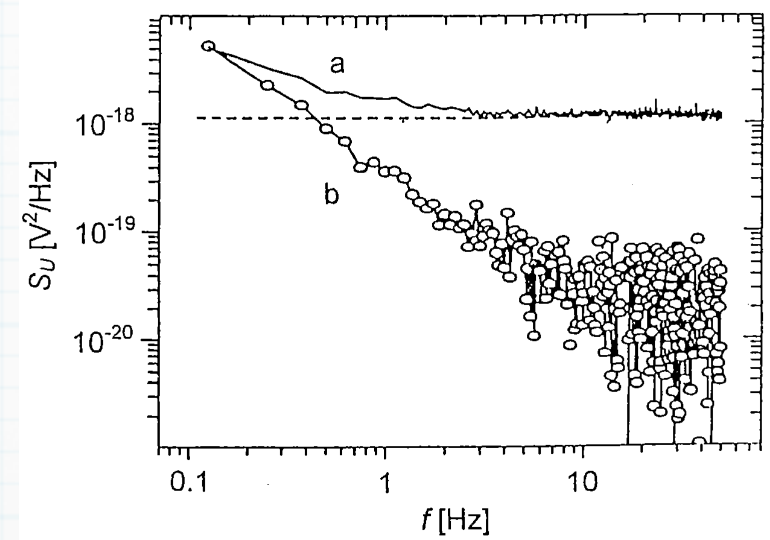
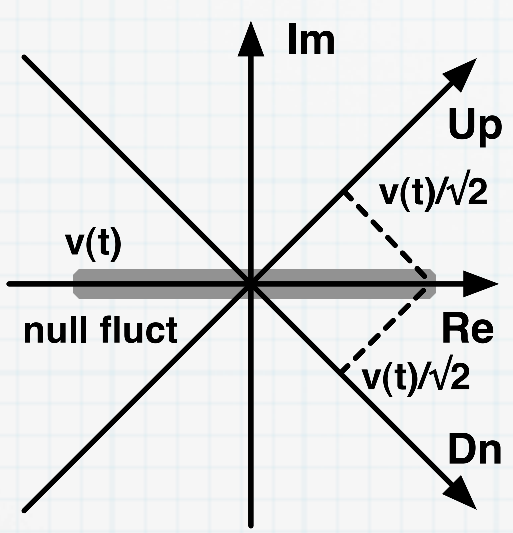


Fig. 1 $1/f$ noise of an $AlSi_{0.01}Cu_{0.002}$ thin film measured at room temperature (a) without and (b) with the phase-sensitive ac correlation technique. The Johnson noise level is indicated by the dashed line.



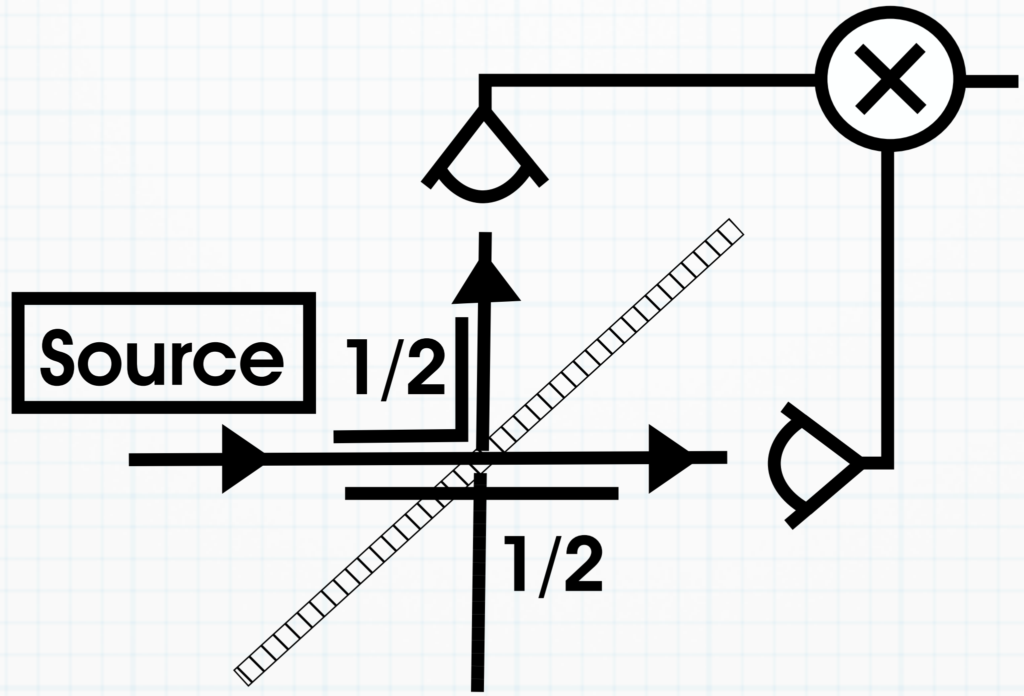
- Random noise: X' and X" (real and imag part) of a signal are statistically independent
- The detection on two orthogonal axes eliminates the amplifier noise.
  This work with a single amplifier!
- The DUT noise is detected

$$S_{ud}(f) = \frac{1}{2}\left[S_\alpha(f) - S_\varphi(f)\right]$$

A. Seeger, H. Stoll, 1/f noise and defects in thin metal films, proc. ICNF p.162-167, Hong Kong 23-26 aug 1999

RF/microwave version: E. Rubiola, V. Giordano, H. Stoll, IEEE Transact. IM 52(1) pp.182-188, feb 2003

Hanbury Brown - Twiss effect

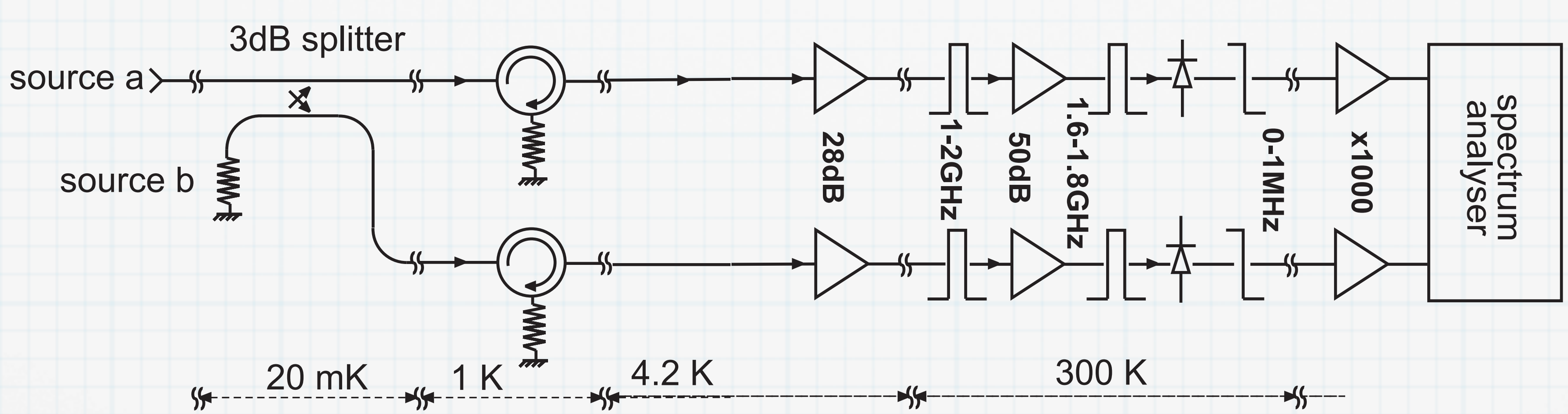


in single-photon regime, anti-correlation shows up

R. Hanbury Brown, R. Q. Twiss, Correlation between photons in two coherent beams of light, Nature 177 (1956) 27-29

Also observed at microwave frequencies

C. Glattli & al. (2004), PRL 93(5) 056801, Jul 2004



$kT = 2.7\times10^{-25}$ J at 20 mK, $h\nu = 1.12\times10^{-24}$ J at 1.7 GHz, $kT/h\nu = -6.1$ dB

Conclusions

- Rejection of the instrument noise
- AM noise, RIN, etc. –> validation of the instrument without a reference low-noise source
- Display quantities
  - $\langle \mathrm{Re}\{S_{yx}\} \rangle_m$ is faster and more accurate
  - $\langle \mathrm{Im}\{S_{yx}\} \rangle_m$ gives the background noise
  - $\max\{\langle S_{yx} \rangle_m, 0_+\}$ provide easier readout
- Applications in many fields of metrology

The cross spectrum method is magic

Correlated noise sometimes makes magic difficult

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