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The Leeson effect

Phase noise and frequency stability in oscillators

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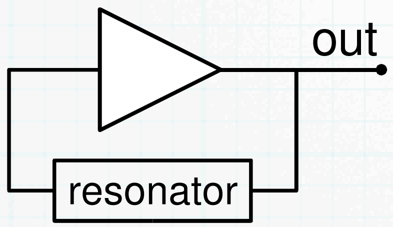
Contents

(*) covered by other tutorials

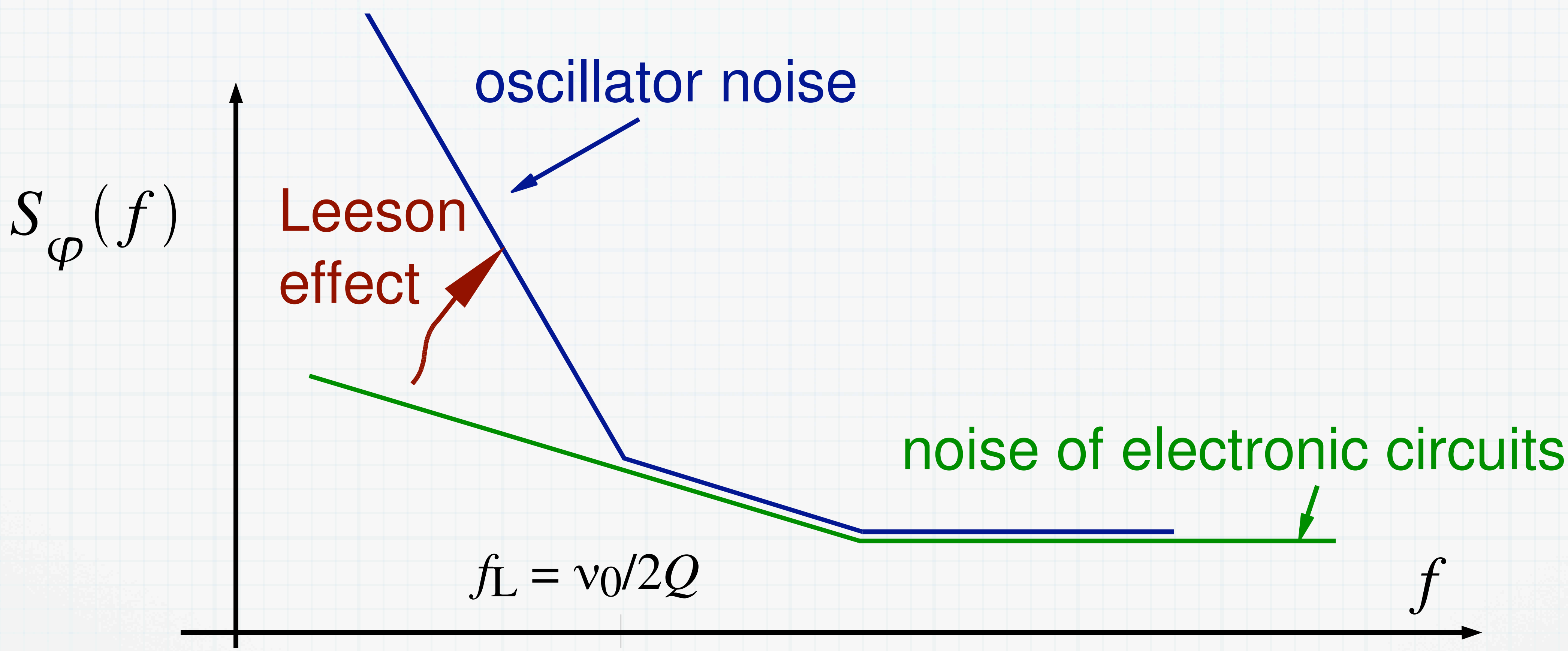
home page http://rubiola.org

The Leeson effect in a nutshell

David B. Leeson, A simple model for feed back oscillator noise, Proc. IEEE 54(2):329 (Feb 1966)



$$\underbrace{S_\varphi(f)}_{\text{oscillator noise}} = \left[1 + \left(\frac{\nu_0}{2Q}\right)^2 \frac{1}{f^2}\right] \underbrace{S_\psi(f)}_{\text{ampli noise}}$$



Notation

ν --> carrier frequency

f --> Fourier analysis

ω either $2\pi\nu$ or $2\pi f$

τ either

measurement time

relaxation time

-- optional --

Phase noise & friends

Notation

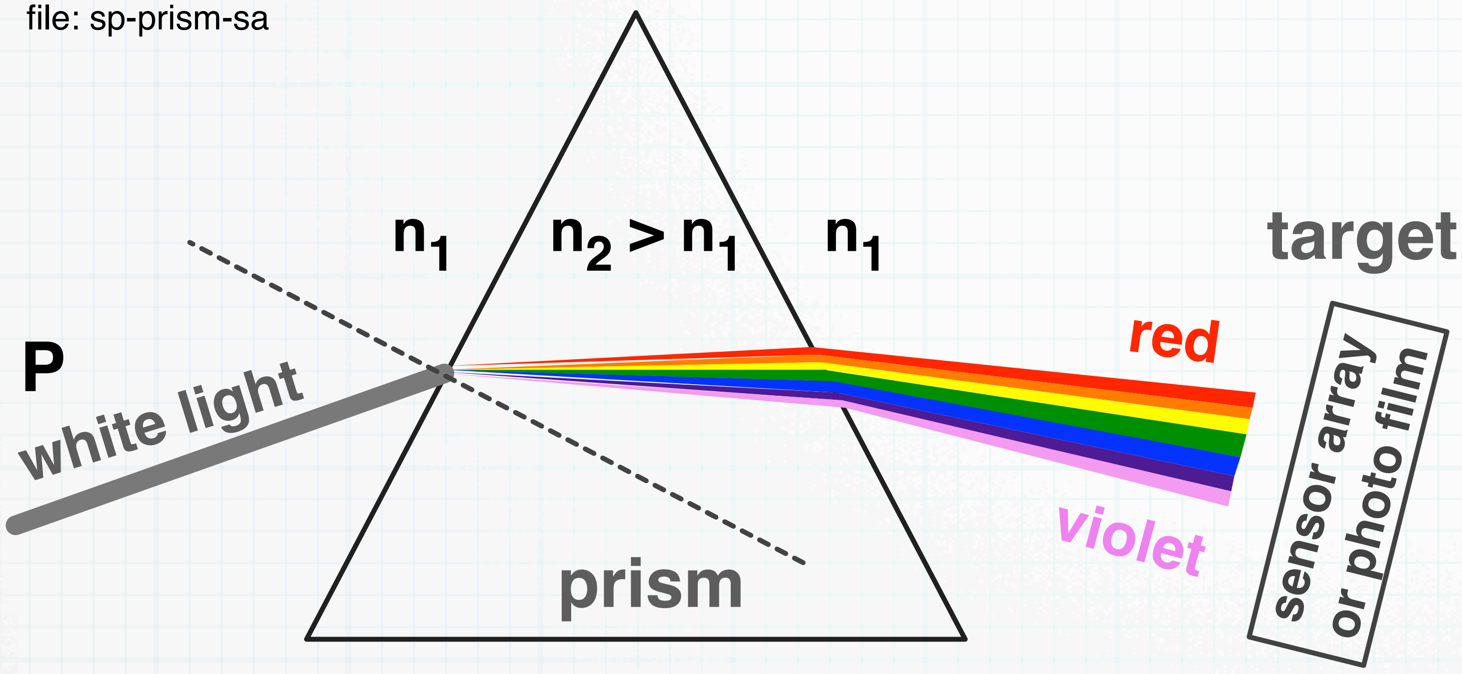
ν --> carrier frequency

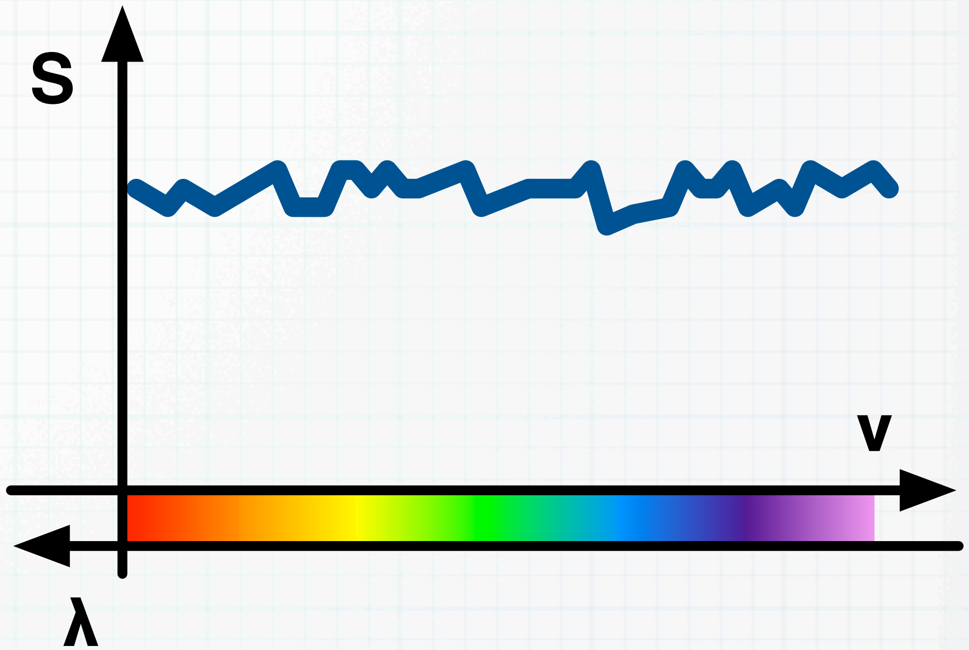
f --> Fourier analysis

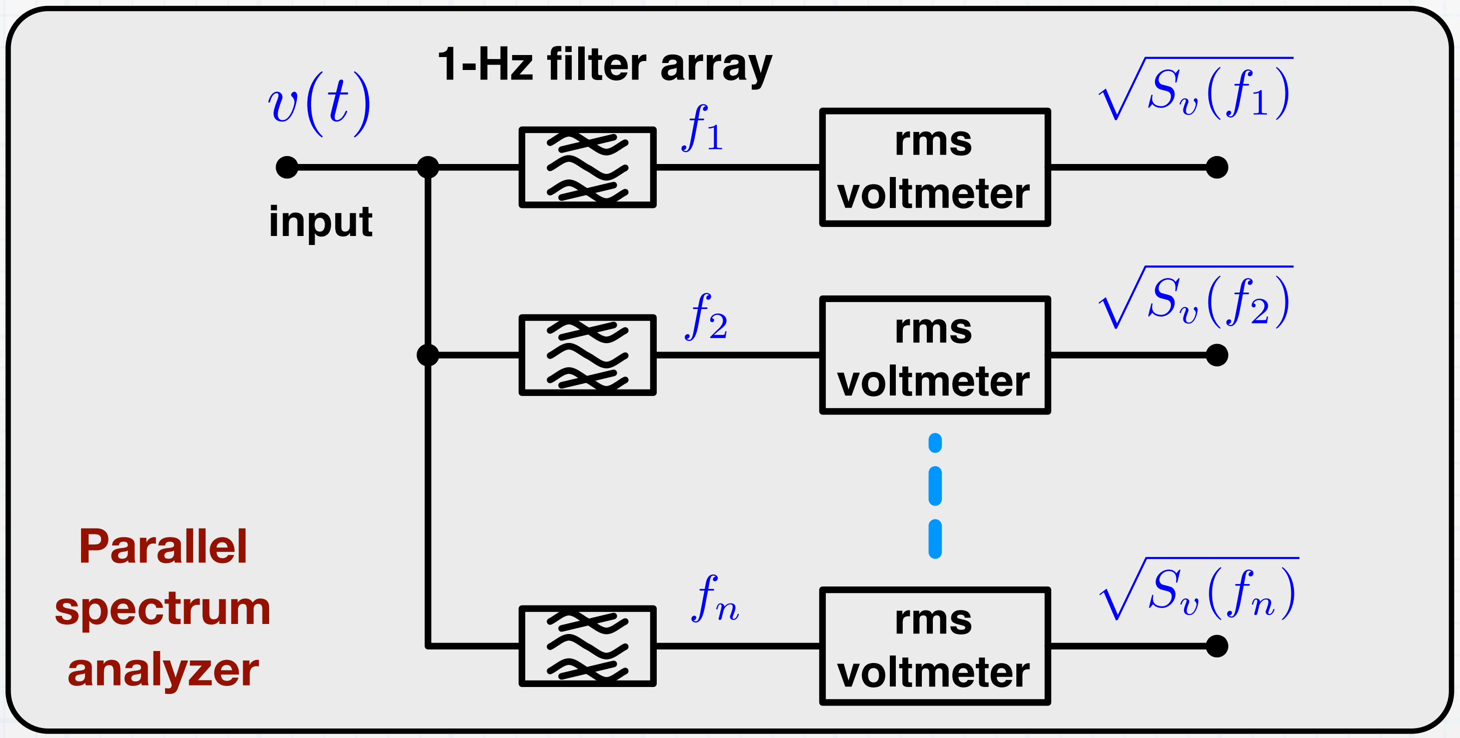
ω either $2\pi\nu$ or $2\pi f$

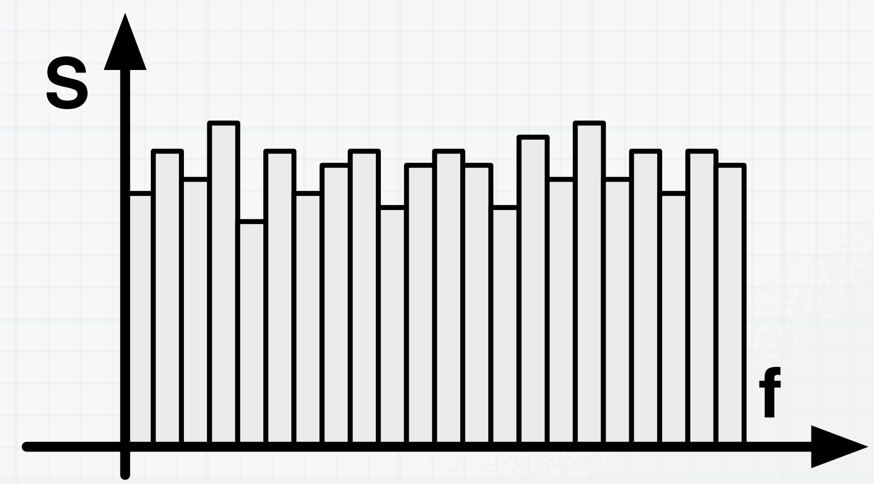
Physical concept of PSD

file: sp-prism-sa







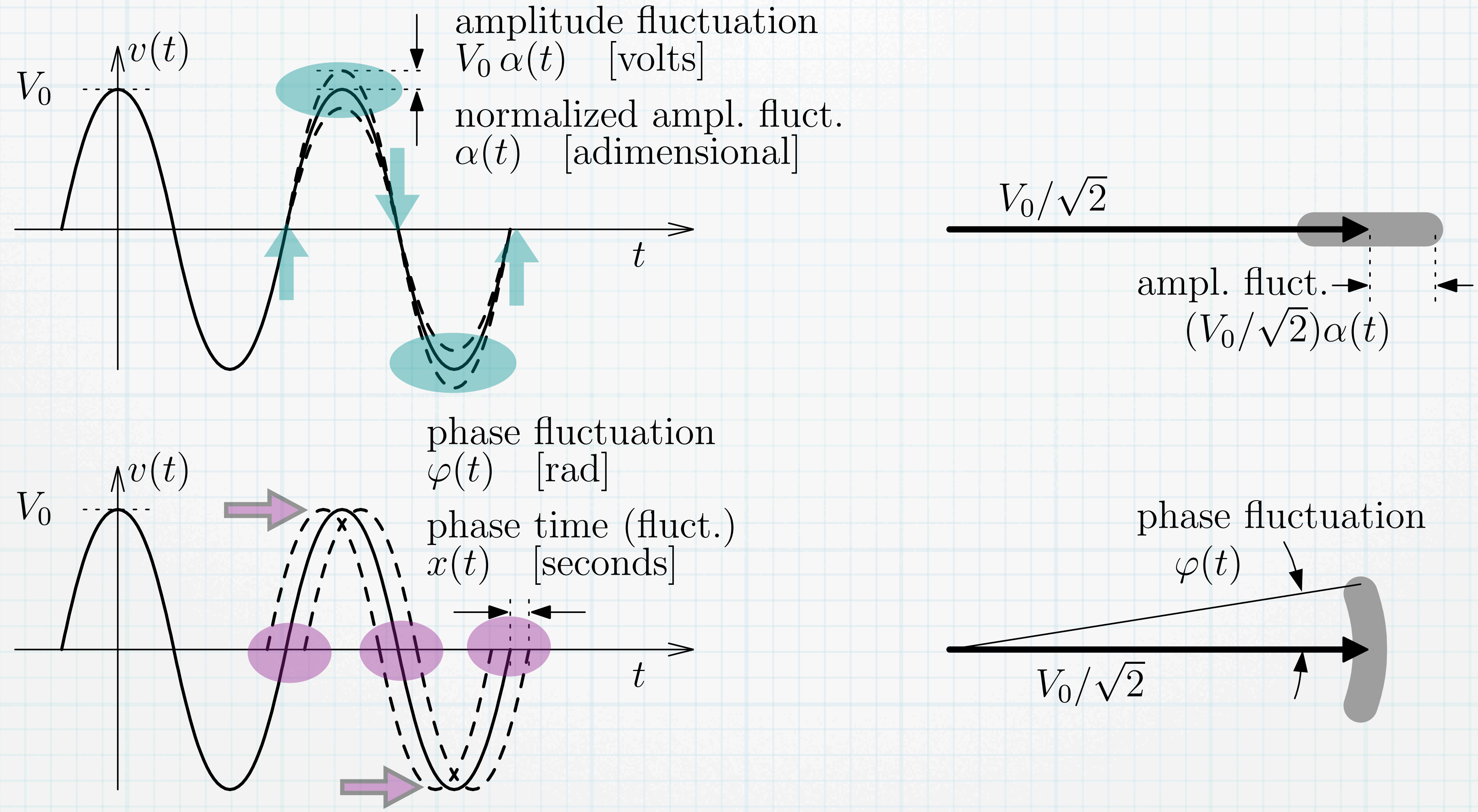


Clock signal affected by noise

Time Domain **Phasor Representation**

α(t)

φ(t)



polar coordinates $$v(t) = V_0\,[1+\alpha(t)]\cos[\omega_0 t + \varphi(t)]$$

Cartesian coordinates $$v(t) = V_0 \cos\omega_0 t + n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t$$

under low noise approximation

$$|n_c(t)| \ll V_0 \quad \text{and} \quad |n_s(t)| \ll V_0$$

It holds that

$$\alpha(t) = \frac{n_c(t)}{V_0} \quad \text{and} \quad \varphi(t) = \frac{n_s(t)}{V_0}$$

Physical quantities

$$v(t) = V_0\,[1+\alpha(t)]\cos\left[2\pi\nu_0 t + \varphi(t)\right]$$

Allow φ(t) to exceed ±π and count the number of turns, so that φ(t) describes the clock fluctuation in full

radian

phase noise $\boxed{\varphi(t)}$

$$(\Delta\nu)(t) = \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

Hertz

$\boxed{(\Delta\nu)(t)}$ frequency fluctuation

$$x = \frac{\varphi}{2\pi\nu_0}$$

$$y(t) = \frac{1}{2\pi\nu_0}\frac{d\varphi(t)}{dt}$$

$$y(t) = \frac{1}{\nu_0}(\Delta\nu)(t)$$

$$y(t) = \frac{dx(t)}{dt}$$

phase-time (fluctuation) $\boxed{x(t)}$

second

$\boxed{y(t)}$ fractional-frequency fluctuation

dimensionless

Phase noise & friends

$$v(t) = V_p\,[1+\alpha(t)]\cos\left[2\pi\nu_0 t + \varphi(t)\right]$$

random phase fluctuation

$$S_\varphi(f) = \text{PSD of } \varphi(t)$$

power spectral density

it is measured as

$$S_\varphi(f) = \frac{1}{T}\mathbb{E}\left\{\Phi(f)\Phi^*(f)\right\} \quad \text{(expectation)}$$

$$S_\varphi(f) \approx \frac{1}{T}\left\langle\Phi(f)\Phi^*(f)\right\rangle_m \quad \text{(average)}$$

$$\mathcal{L}(f) = \frac{1}{2}S_\varphi(f) \ \ \text{dBc}$$

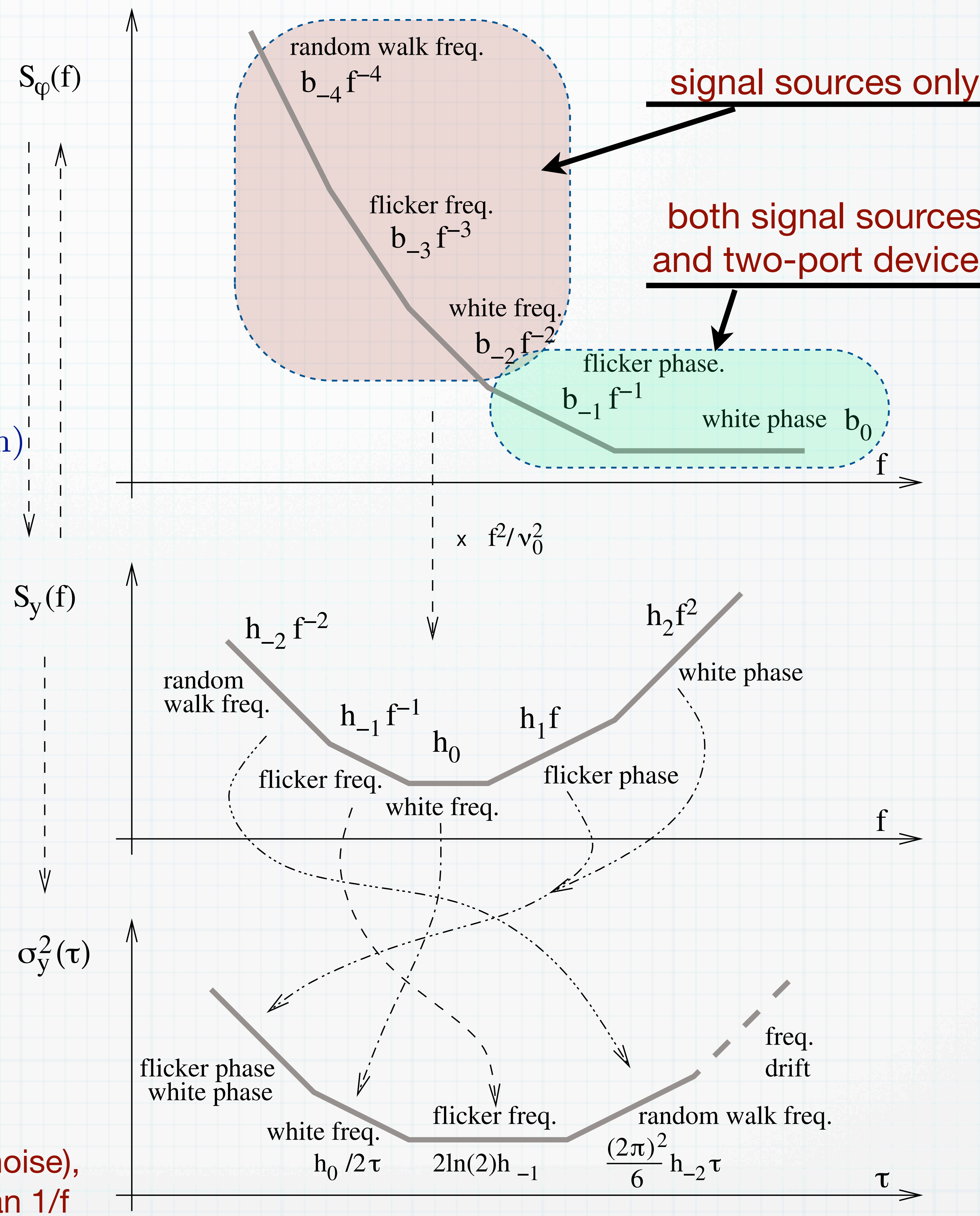
random fractional-frequency fluctuation

$$y(t) = \frac{\dot\varphi(t)}{2\pi\nu_0} \quad \Rightarrow \quad S_y = \frac{f^2}{\nu_0^2}S_\varphi(f)$$

Allan variance
(two-sample wavelet-like variance)

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\frac{1}{2}\left[\overline{y}_{k+1} - \overline{y}_k\right]^2\right\} .$$

approaches a half-octave bandpass filter (for white noise),
hence it converges even with processes steeper than 1/f



Skip

Allan variance

definition

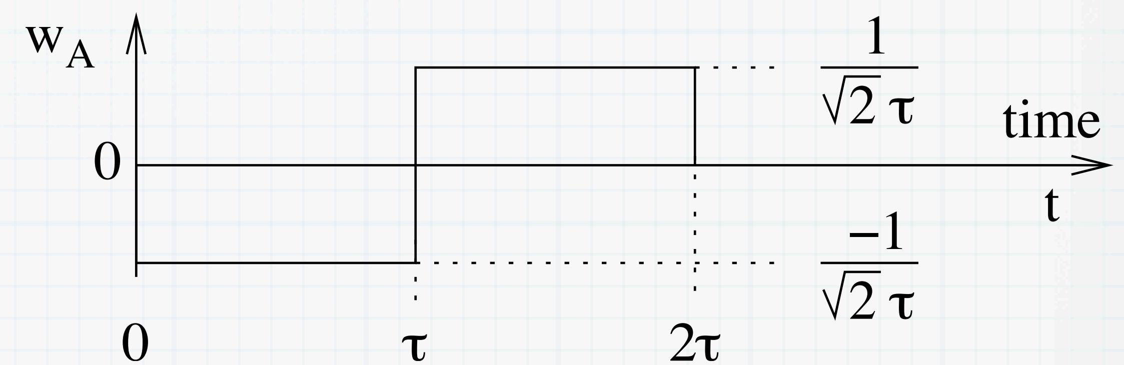
$$\sigma_y^2(\tau) = \mathbb{E}\left\{\frac{1}{2}\left[\overline{y}_{k+1} - \overline{y}_k\right]^2\right\}$$

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\frac{1}{2}\left[\frac{1}{\tau}\int_{(k+1)\tau}^{(k+2)\tau} y(t)\,dt - \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} y(t)\,dt\right]^2\right\}$$

wavelet-like variance

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\left[\int_{-\infty}^{+\infty} y(t)\,w_A(t)\,dt\right]^2\right\}$$

$$w_A = \begin{cases} -\frac{1}{\sqrt{2}\tau} & 0 < t < \tau \\ \frac{1}{\sqrt{2}\tau} & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases}$$



energy

$$E\{w_A\} = \int_{-\infty}^{+\infty} w_A^2(t)\,dt = \frac{1}{\tau}$$

the Allan variance differs from a wavelet variance in the normalization on power, instead of on energy

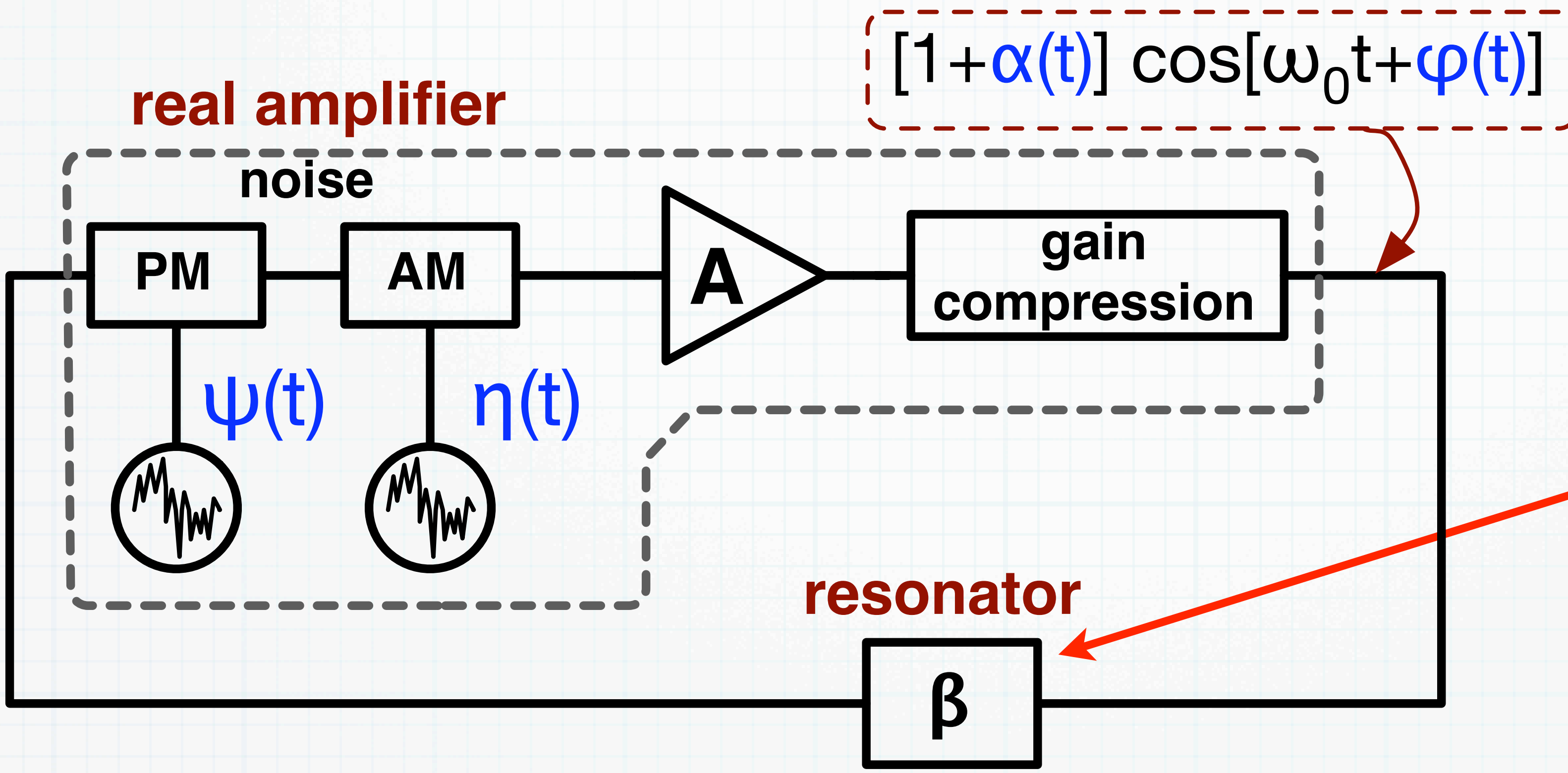
Relationships between spectra and variances

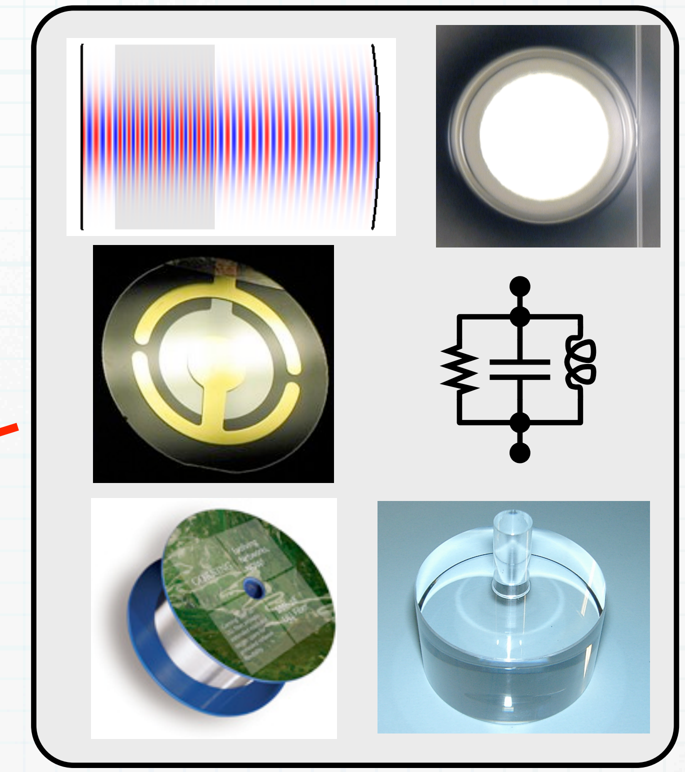
noise type	$S_\varphi(f)$	$S_y(f)$	$S_\varphi \leftrightarrow S_y$	$\sigma_y^2(\tau)$	$\mathrm{mod}\,\sigma_y^2(\tau)$
white PM	b_0	$h_2 f^2$	$h_2 = \dfrac{b_0}{\nu_0^2}$	$\dfrac{3 f_H h_2}{(2\pi)^2}\tau^{-2}$ $2\pi\tau f_H \gg 1$	$\dfrac{3 f_H \tau_0 h_2}{(2\pi)^2}\,\tau^{-3}$
flicker PM	$b_{-1} f^{-1}$	$h_1 f$	$h_1 = \dfrac{b_{-1}}{\nu_0^2}$	$[1.038 + 3\ln(2\pi f_H \tau)]\,\dfrac{h_1}{(2\pi)^2}\,\tau^{-2}$	$0.084\, h_1 \tau^{-2}$ $n \gg 1$
white FM	$b_{-2} f^{-2}$	h_0	$h_0 = \dfrac{b_{-2}}{\nu_0^2}$	$\dfrac{1}{2} h_0\,\tau^{-1}$	$\dfrac{1}{4} h_0\,\tau^{-1}$
flicker FM	$b_{-3} f^{-3}$	$h_{-1} f^{-1}$	$h_{-1} = \dfrac{b_{-3}}{\nu_0^2}$	$2\ln(2)\; h_{-1}$	$\dfrac{27}{20}\ln(2)\; h_{-1}$
random walk FM	$b_{-4} f^{-4}$	$h_{-2} f^{-2}$	$h_{-2} = \dfrac{b_{-4}}{\nu_0^2}$	$\dfrac{(2\pi)^2}{6} h_{-2}\tau$	$0.824\,\dfrac{(2\pi)^2}{6} h_{-2}\,\tau$
linear frequency drift $\dot{y}$				$\dfrac{1}{2}\,(\dot{y})^2\,\tau^2$	$\dfrac{1}{2}\,(\dot{y})^2\,\tau^2$

f_H is the high cutoff frequency, needed for the noise power to be finite.

Heuristic explanation of the Leeson effect

General oscillator model

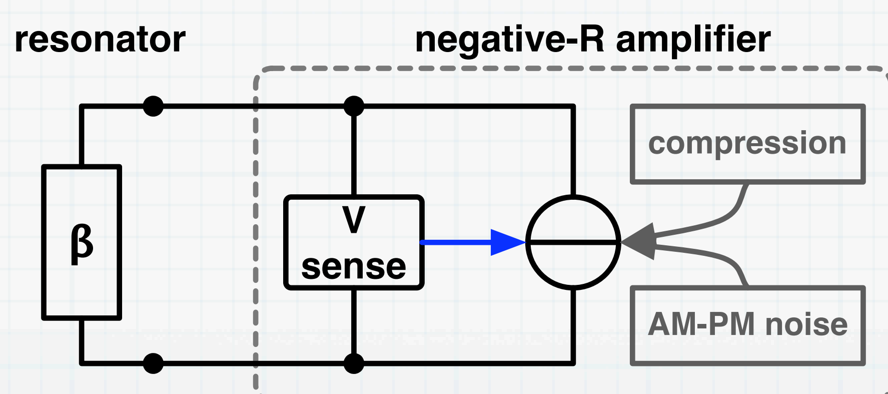




Barkhausen condition $A\beta = 1$ at ω_0

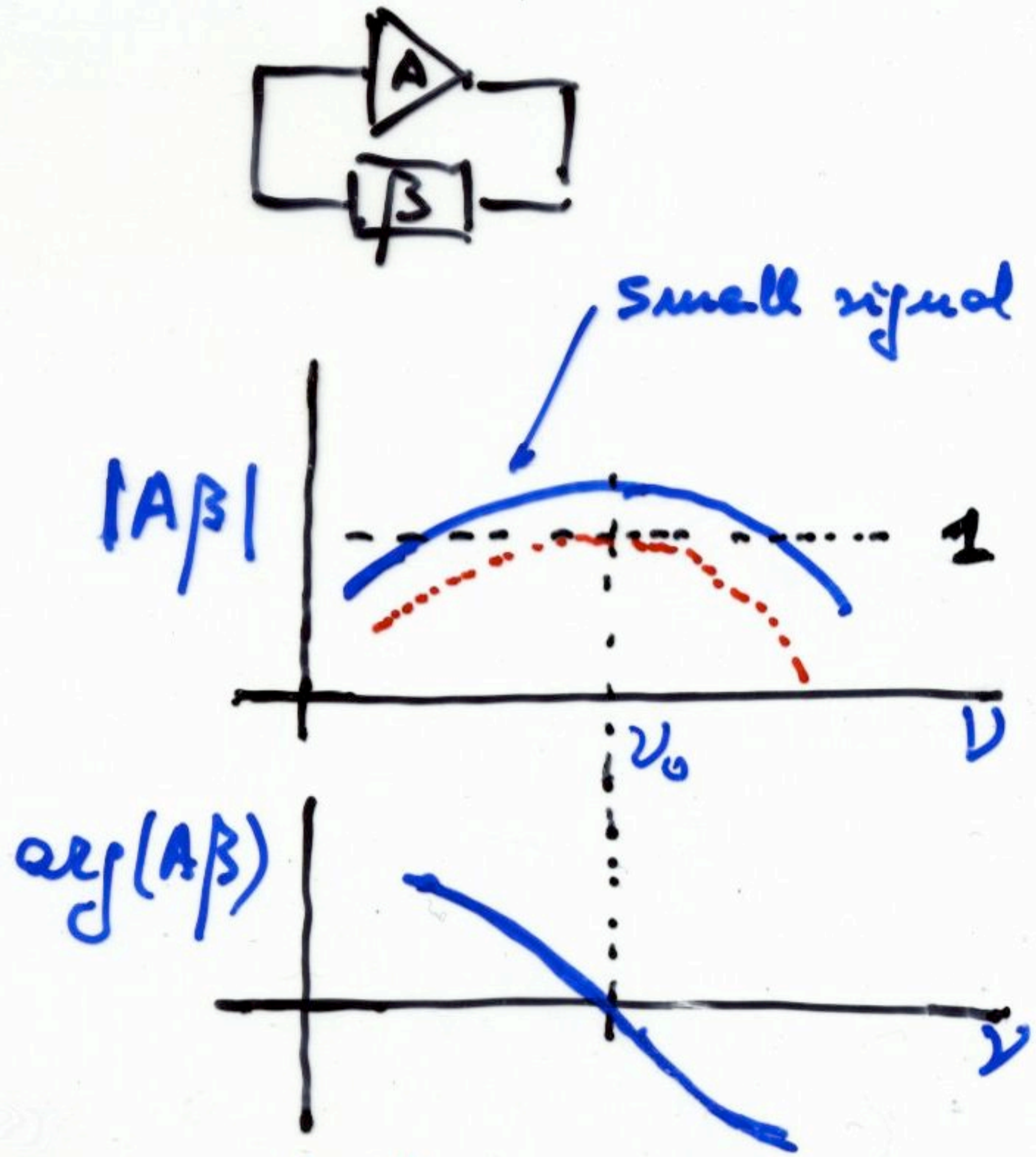
(phase matching)

The model also describes the negative-R oscillator



AR

BARKHAUSEN CONDITION



let $A = \text{const}$

β: 2nd order diff. eq resonator

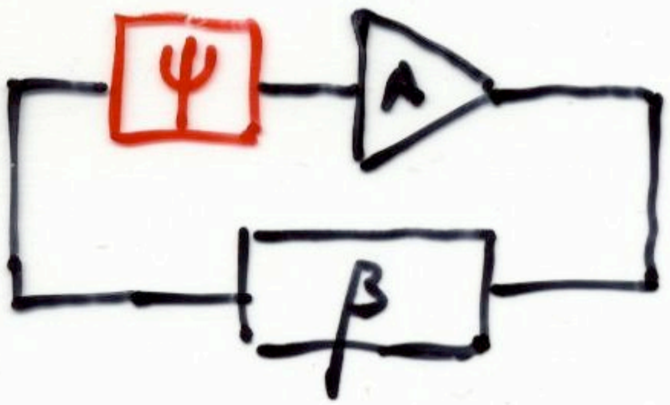
$$\arg(\beta) = -\arctan Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\simeq -2Q\,\frac{\omega - \omega_0}{\omega_0}$$

$$-2Q\,\frac{\Delta\omega}{\omega_0}$$

- $\arg(\beta)$ sets the oscillation frequency
- saturation fixes $|A\beta| = 1$

TUNING AN OSCILLATOR



add a phase ψ

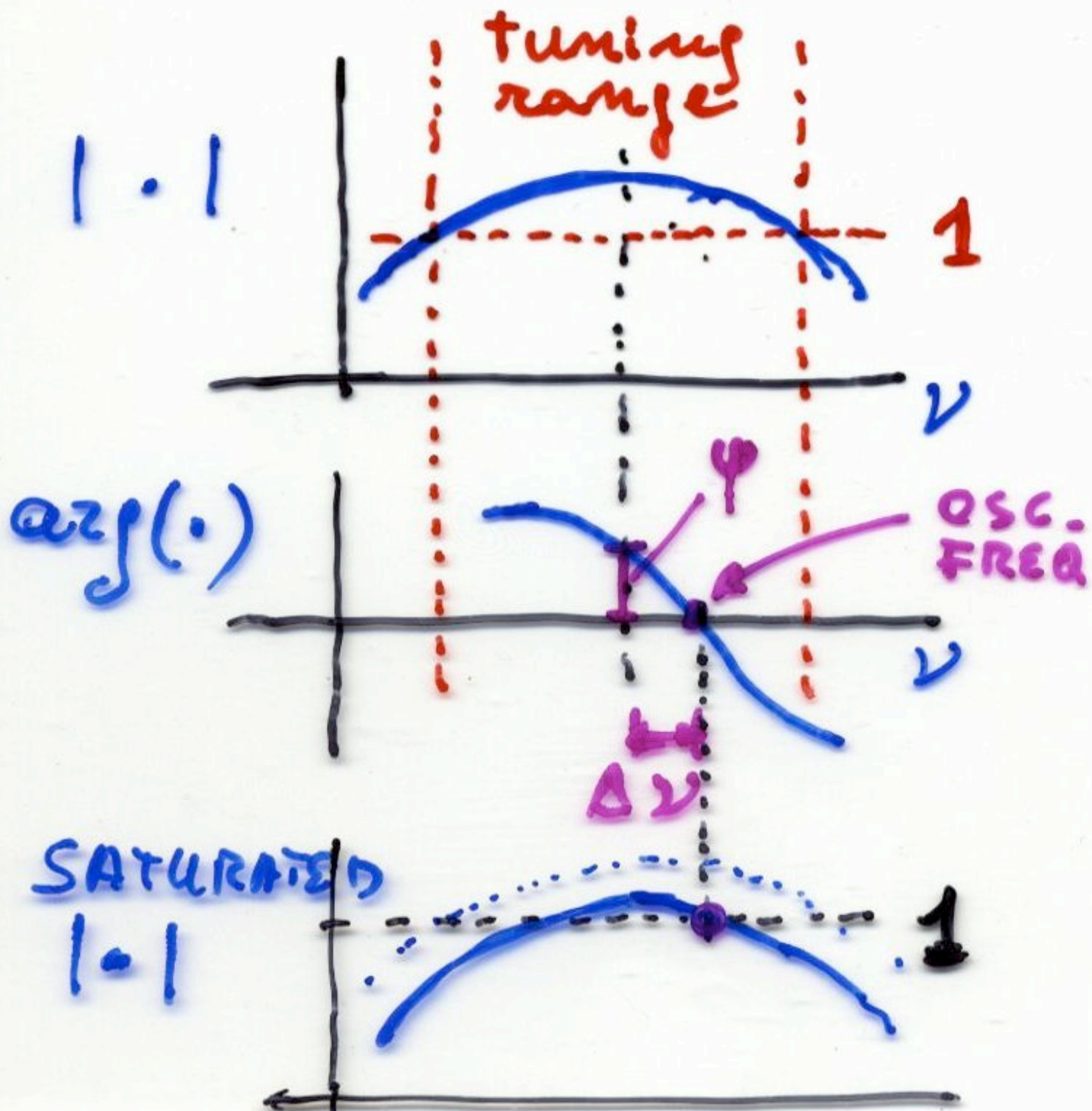
$$+\arg(\beta) + \psi = 0$$

$$+\arg(\beta) = -\psi$$

approx:

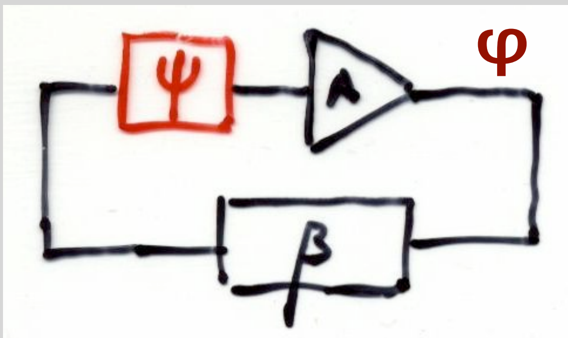
$$2Q\,\frac{\Delta\omega}{\omega_0} = \psi$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta\nu}{\nu_0} = \frac{\psi}{2Q}$$



Heuristic derivation of the Leeson formula

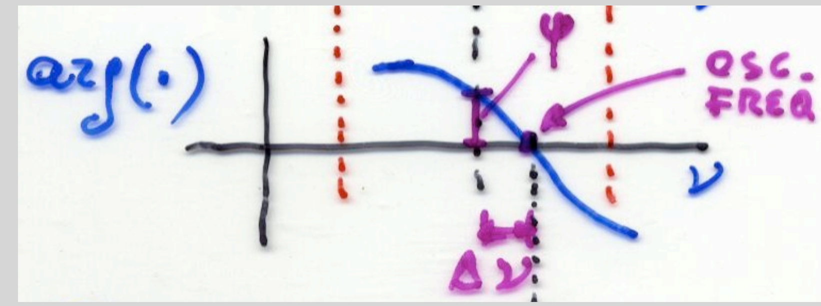
fast fluctuation: no feedback



$$\varphi(t) = \psi(t)$$

$$S_\varphi(f) = S_\psi(f)$$

slow fluctuations: ψ ⇒ Δν conversion



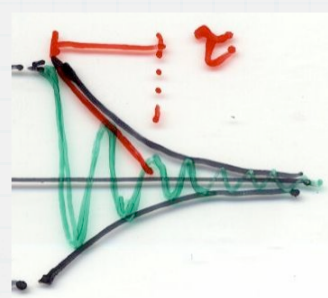
$$\Delta\nu = \frac{\nu_0}{2Q}\,\psi$$ **static**

$$S_{\Delta\nu}(f) = \left(\frac{\nu_0}{2Q}\right)^2 S_\psi(f)$$

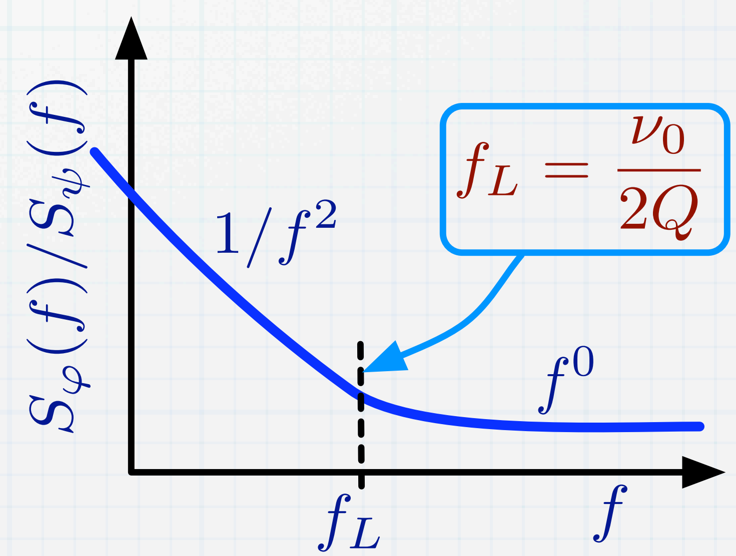
$$S_\varphi(f) = \frac{1}{f^2}\left(\frac{\nu_0}{2Q}\right)^2 S_\psi(f)$$ **integral**

fast or slow?

$$\tau = \frac{Q}{\pi\nu_0}$$

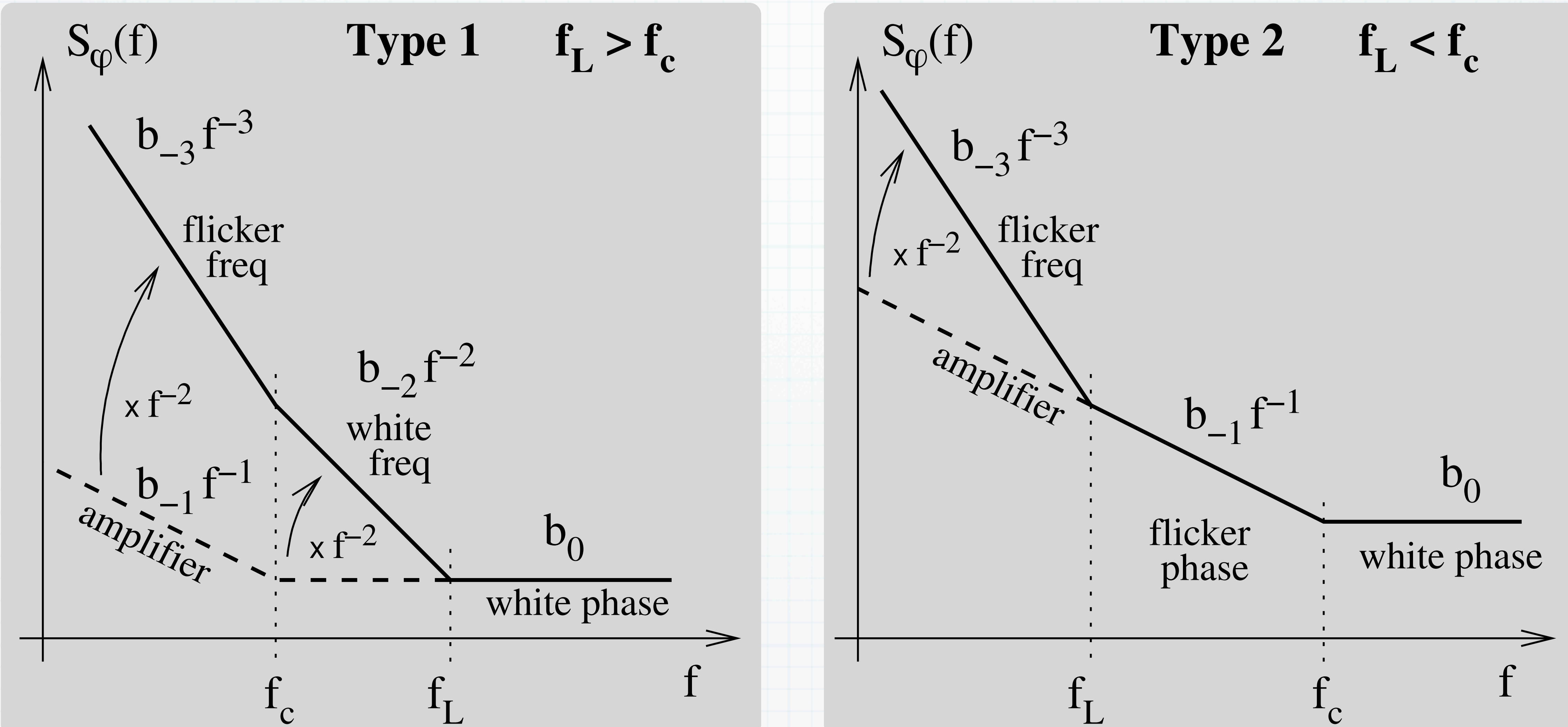


$$S_\varphi(f) = \left[1 + \frac{1}{f^2}\left(\frac{\nu_0}{2Q}\right)^2\right] S_\psi(f)$$



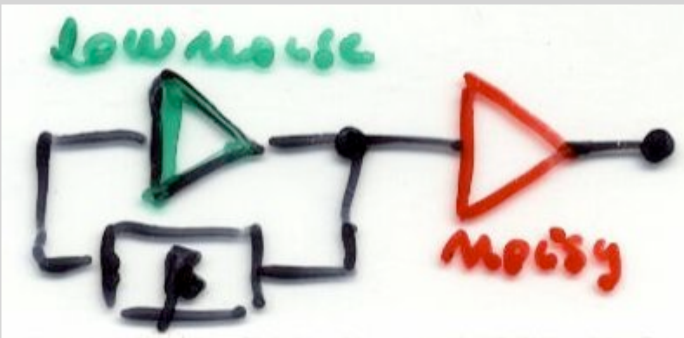
Though obtained with simplifications, this result turns out to be is exact

Oscillator noise
-- real sustaining amplifier --



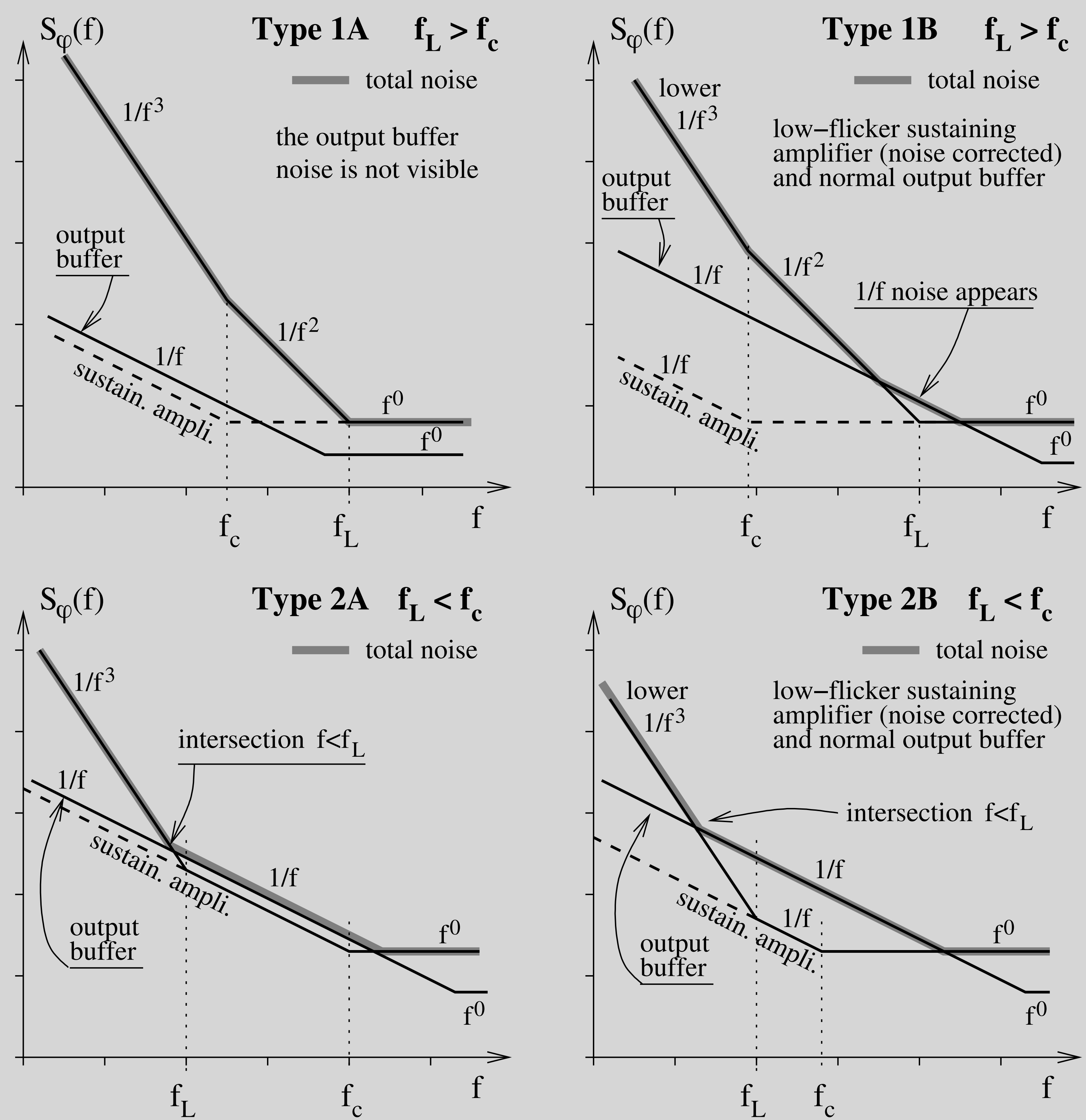
The sustaining-amplifier noise is $S_\varphi(f) = b_0 + b_{-1}/f$ (white and flicker)

The effect of the output buffer



Cascading two amplifiers, flicker noise adds as

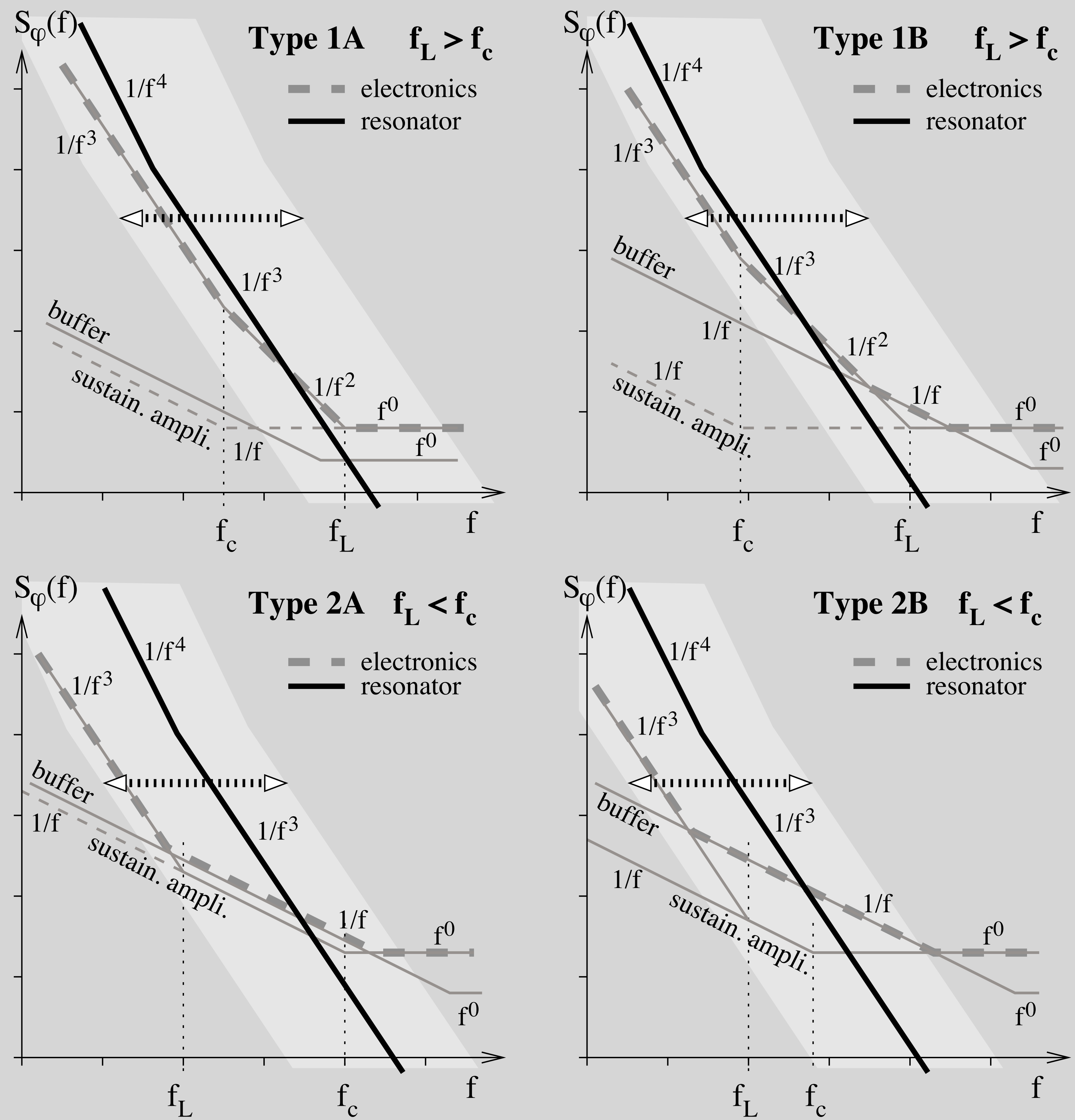
$$S_\varphi(f) = [S_\varphi(f)]_1 + [S_\varphi(f)]_2$$



Figures are from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press

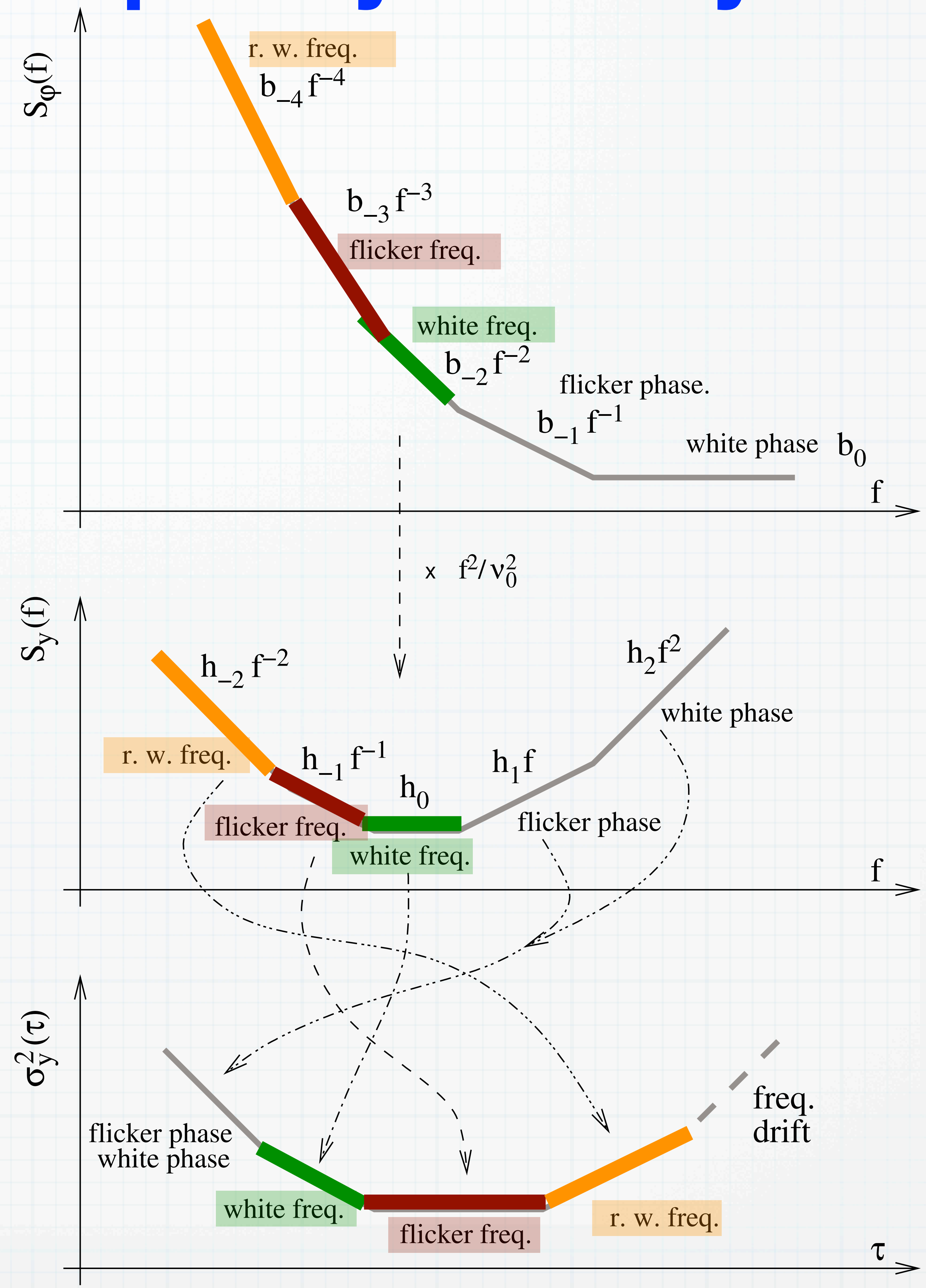
The resonator natural frequency fluctuates

- **The oscillator tracks the resonator natural frequency, hence its fluctuations**
- **The fluctuations of the resonator natural frequency contain 1/f and 1/f² (frequency flicker and random walk), thus 1/f³ and 1/f⁴ of the oscillator phase**
- **The resonator bandwidth does not apply to the natural-frequency fluctuation. (Tip: an oscillator can be frequency modulated ar a rate >> f_L)**



Figures are from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press

Phase noise –> frequency stability



phase noise

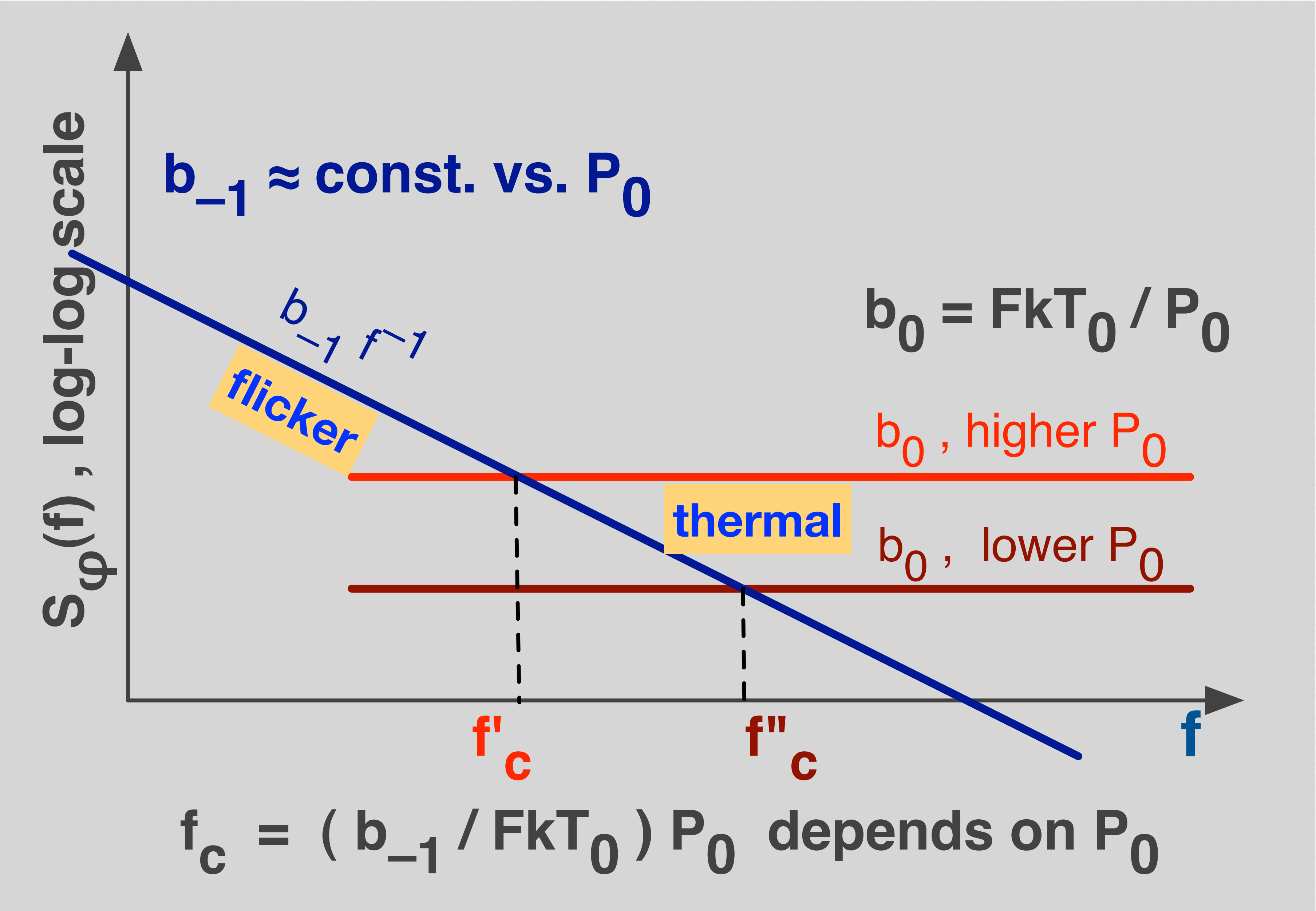
frequency noise

Allan variance

white	$\sigma^2(\tau) = h_0/2\tau$
flicker	$\sigma^2(\tau) = 2\ln(2)\, h_{-1}$
r.walk	$\sigma^2(\tau) = ((2\pi)^2/6)\, h_0\tau$

AM-PM noise in amplifiers

Amplifier white and flicker noise



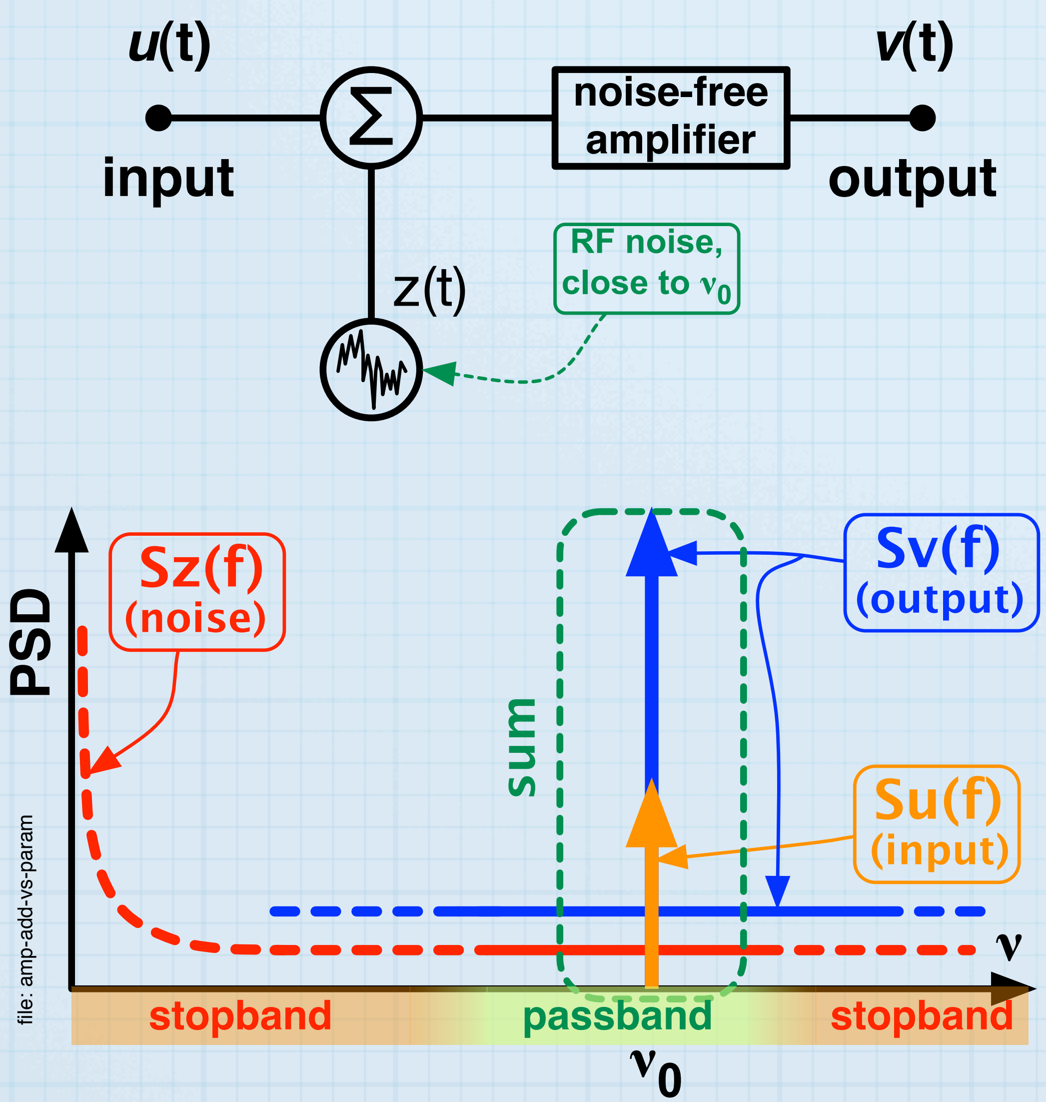
Thermal noise $kT_0 = 4\times10^{-21}$ W/Hz (–174 dBm/Hz)
Noise figure F

photodetector $b_{-1} \approx -120$ dBrad²/Hz Rubiola & al. IEEE Trans. MTT (& JLT) 54 (2) p.816–820 (2006)

typical amplifier phase noise			
RATE	GaAs HBT microwave	SiGe HBT microwave	Si bipolar HF/UHF
fair	−100		−120
good	−110	−120	−130
best	−120	−130	−150
	unit dBrad^2/Hz		

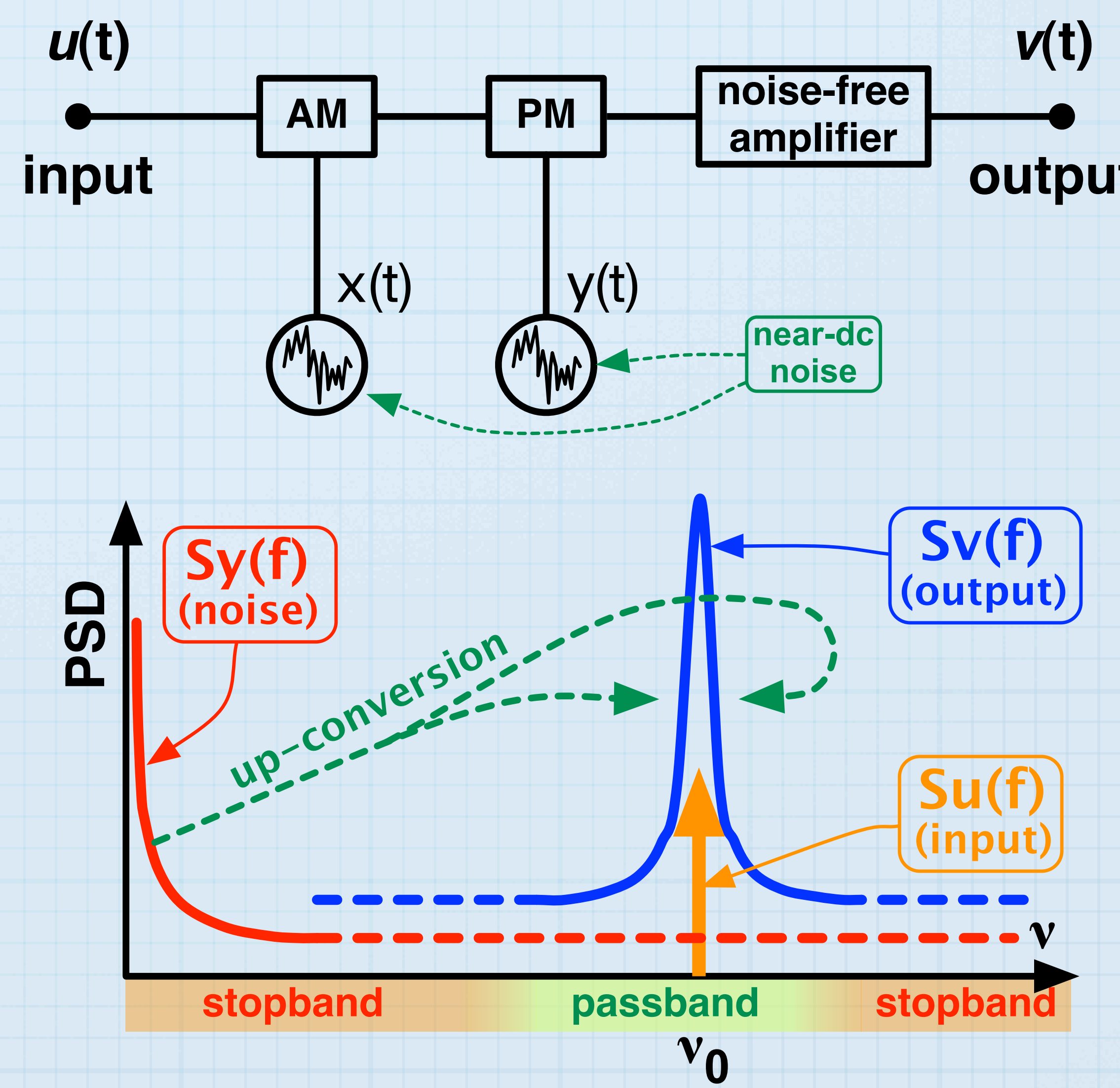
The difference between additive and parametric noise

additive noise



the noise sidebands are
independent of the carrier

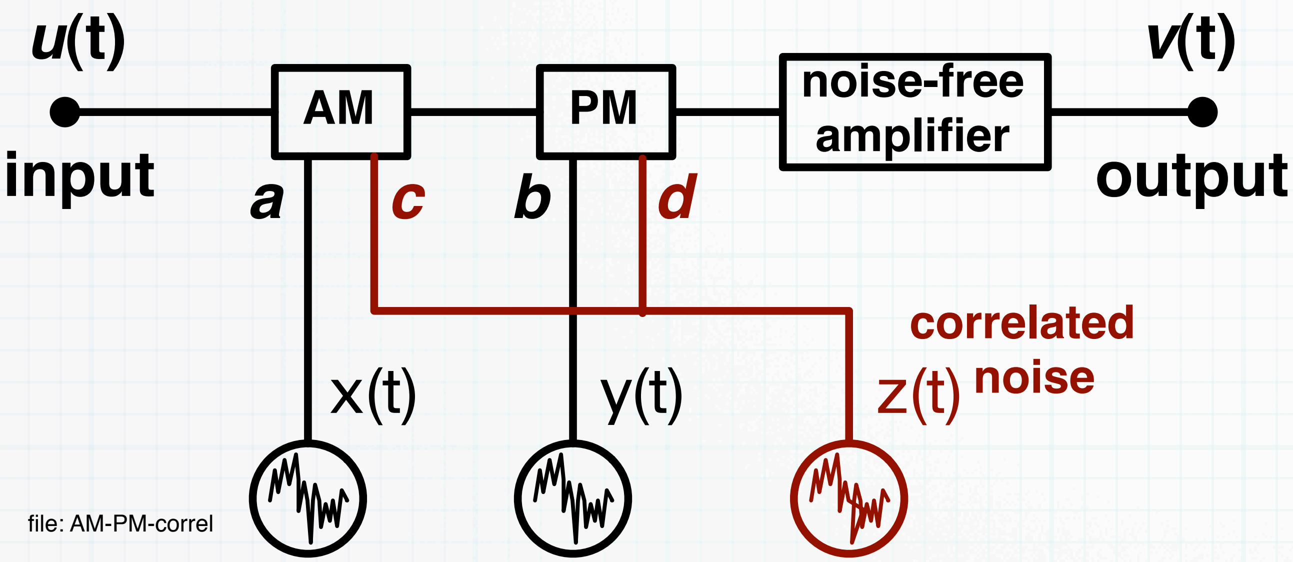
parametric noise

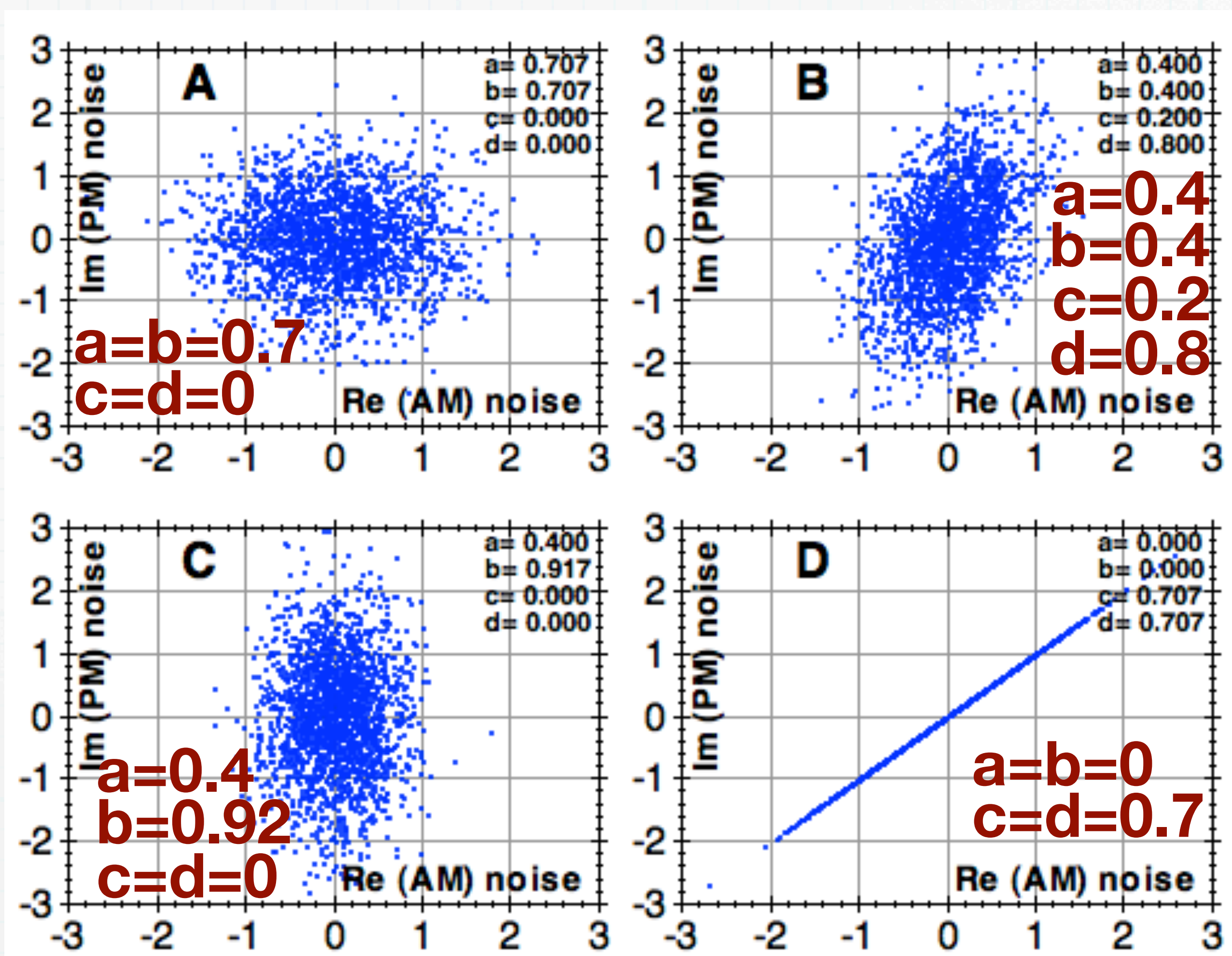


the noise sidebands are
proportional to the carrier

Correlation between AM and PM noise

R. Boudot, E. Rubiola, arXiv:1001.2047v1, Jan 2010. Also IEEE T MTT (submitted)





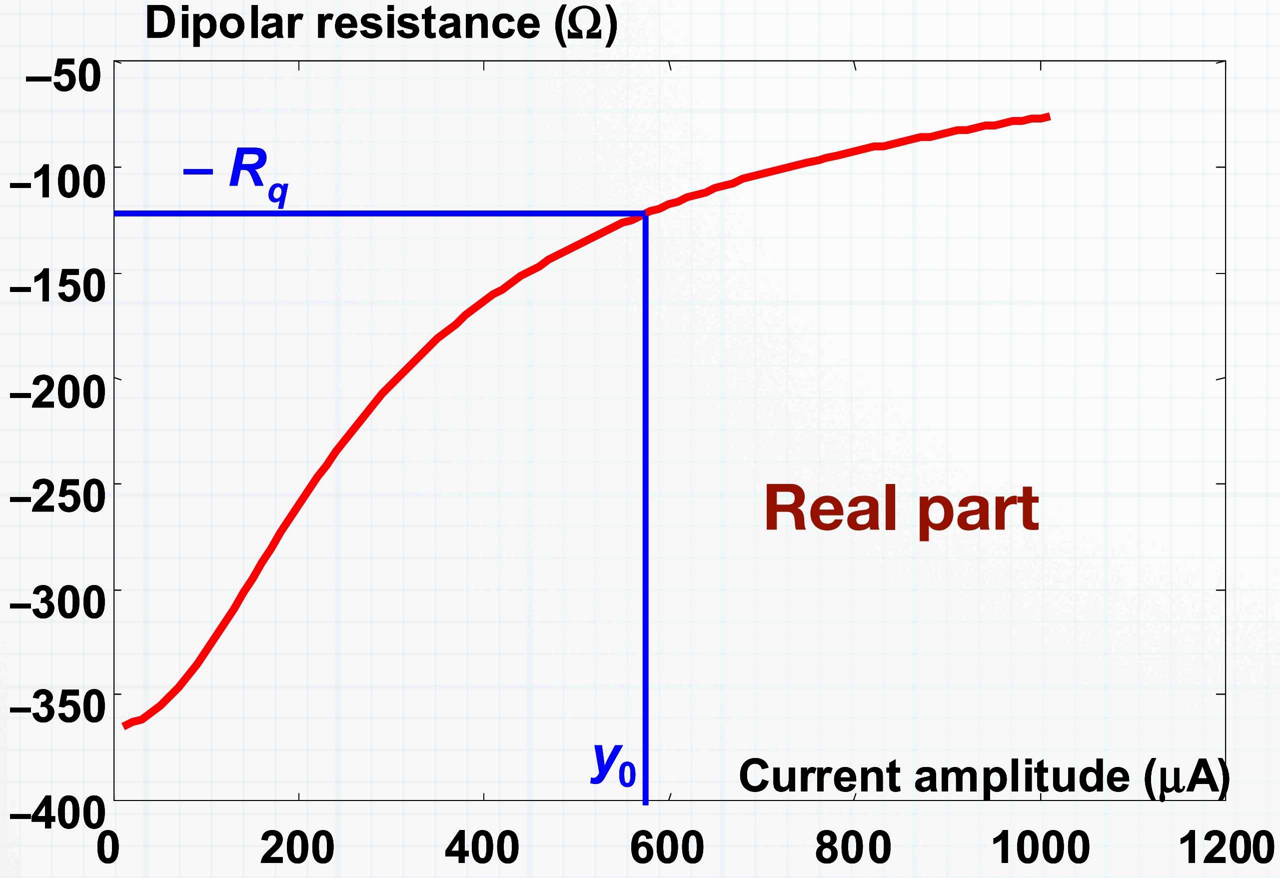
$a^2 + b^2 + c^2 + d^2 = 1$

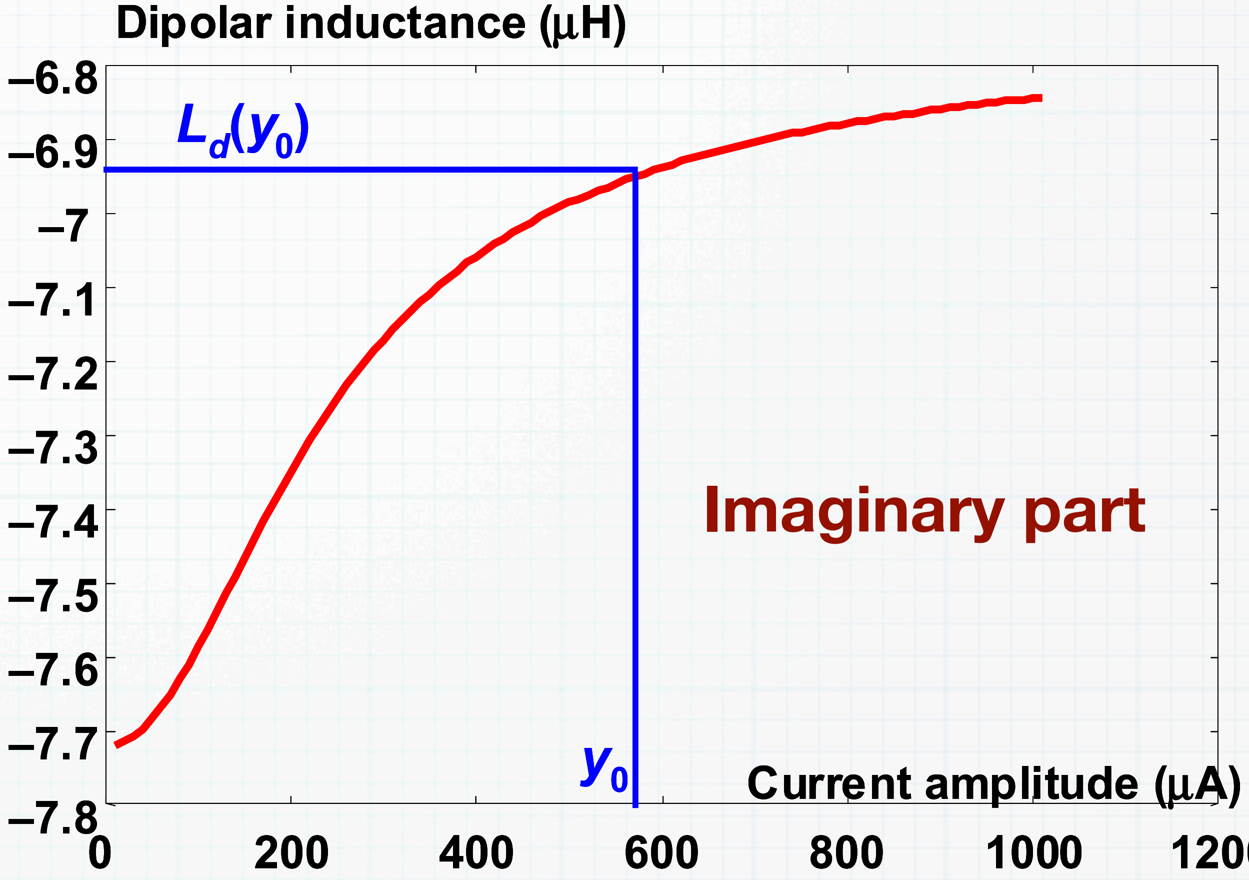
The need for this model comes from the physics of popular amplifiers

- Bipolar transistor. The fluctuation of the carriers in the base region acts on the base thickness, thus on the gain, and on the capacitance of the reverse-biased base-collector junction.
- Field-effect transistor. The fluctuation of the carriers in the channel acts on the drain-source current, and also on the gate-channel capacitance because the distance between the \`electrodes' is affected by the channel thickness.
- Laser amplifier. The fluctuation of the pump power acts on the density of the excited atoms, and in turn on gain, on maximum power, and on refraction index.

AM and PM fluctuations are correlated because originate from the same near-dc random process

Amplitude-phase coupling in amplifiers





Oscillation amplitude is hidden in the current

- In the gain-compression region, RF amplitude affects the phase
- The consequence is that AM noise turns into PM noise
- Well established fact in quartz oscillators (Colpitts and other schemes)
- Similar phenomenon occurs in other types of (sustaining) amplifier

R. Brendel & E. Rubiola, Proc. 2007 IFCS p.1099-1105, Geneva CH, 28 may - 1 Jun 2007

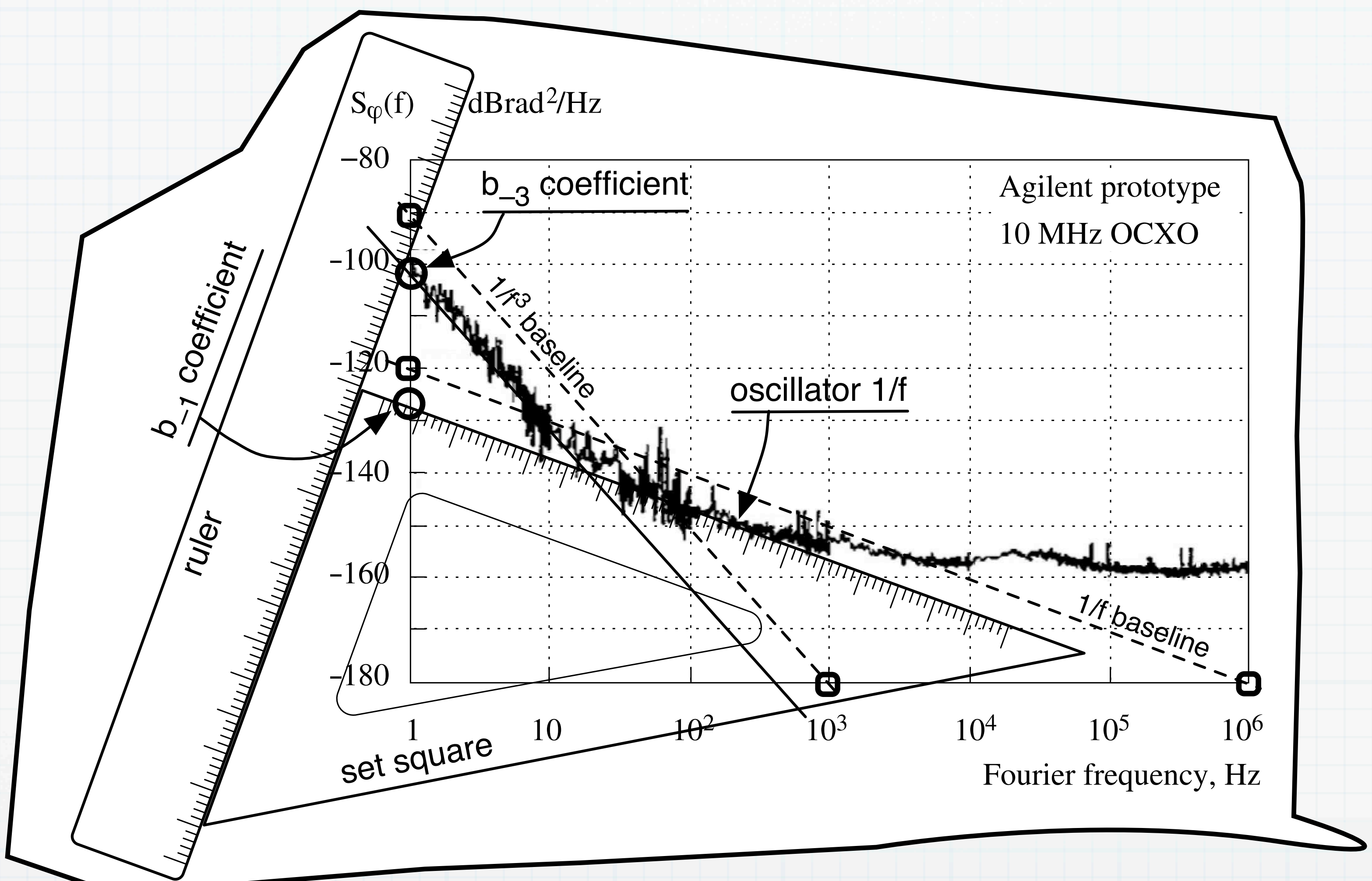
Oscillator Hacking

Analysis of commercial oscillators

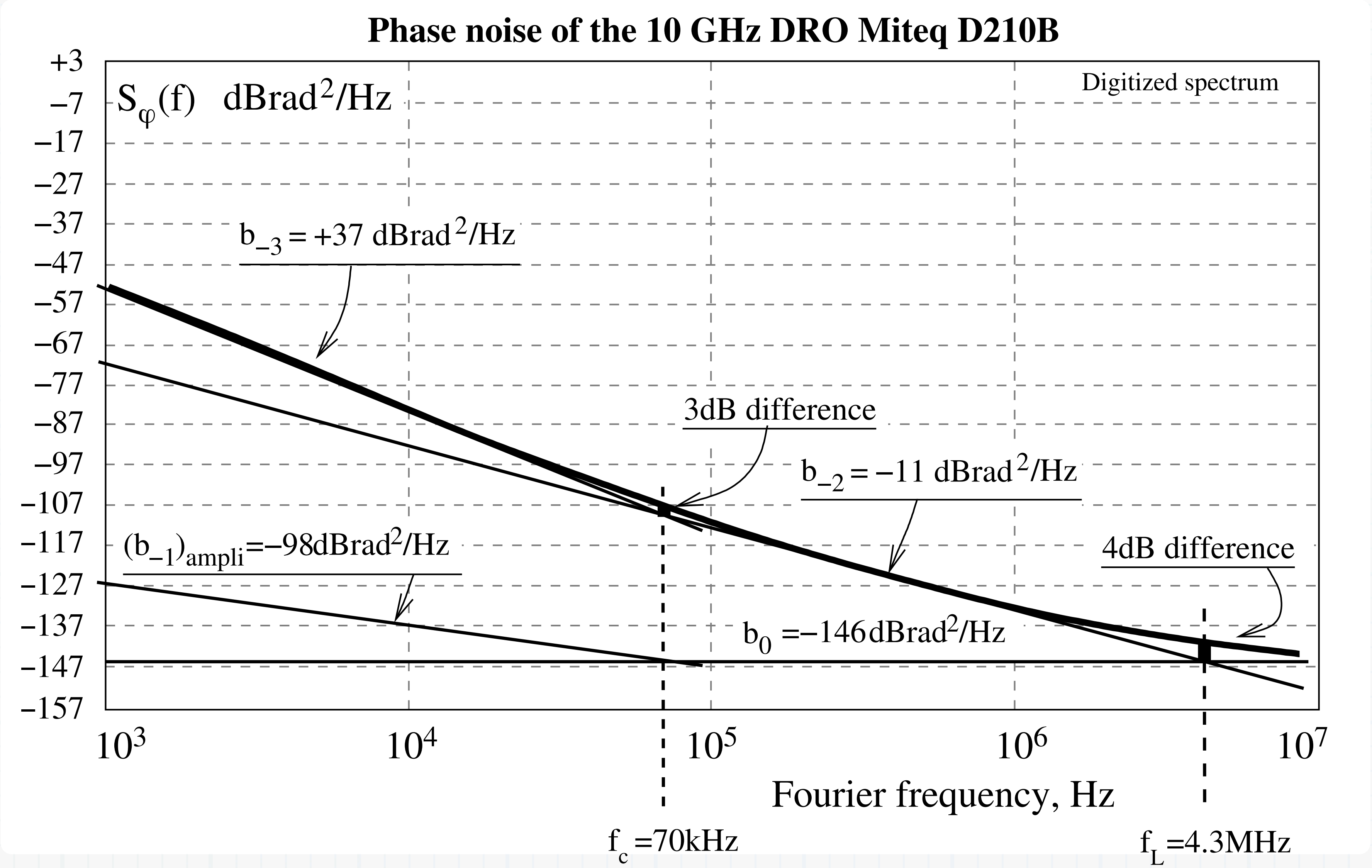
The purpose of this section is to help to understand the oscillator inside from the phase noise spectra, plus some technical information. I have chosen some commercial oscillators as an example.

The conclusions about each oscillator represent only my understanding based on experience and on the data sheets published on the manufacturer web site.

You should be aware that this process of interpretation is not free from errors. My conclusions were not submitted to manufacturers before writing, for their comments could not be included.



Miteq D210B, 10 GHz DRO



From the table

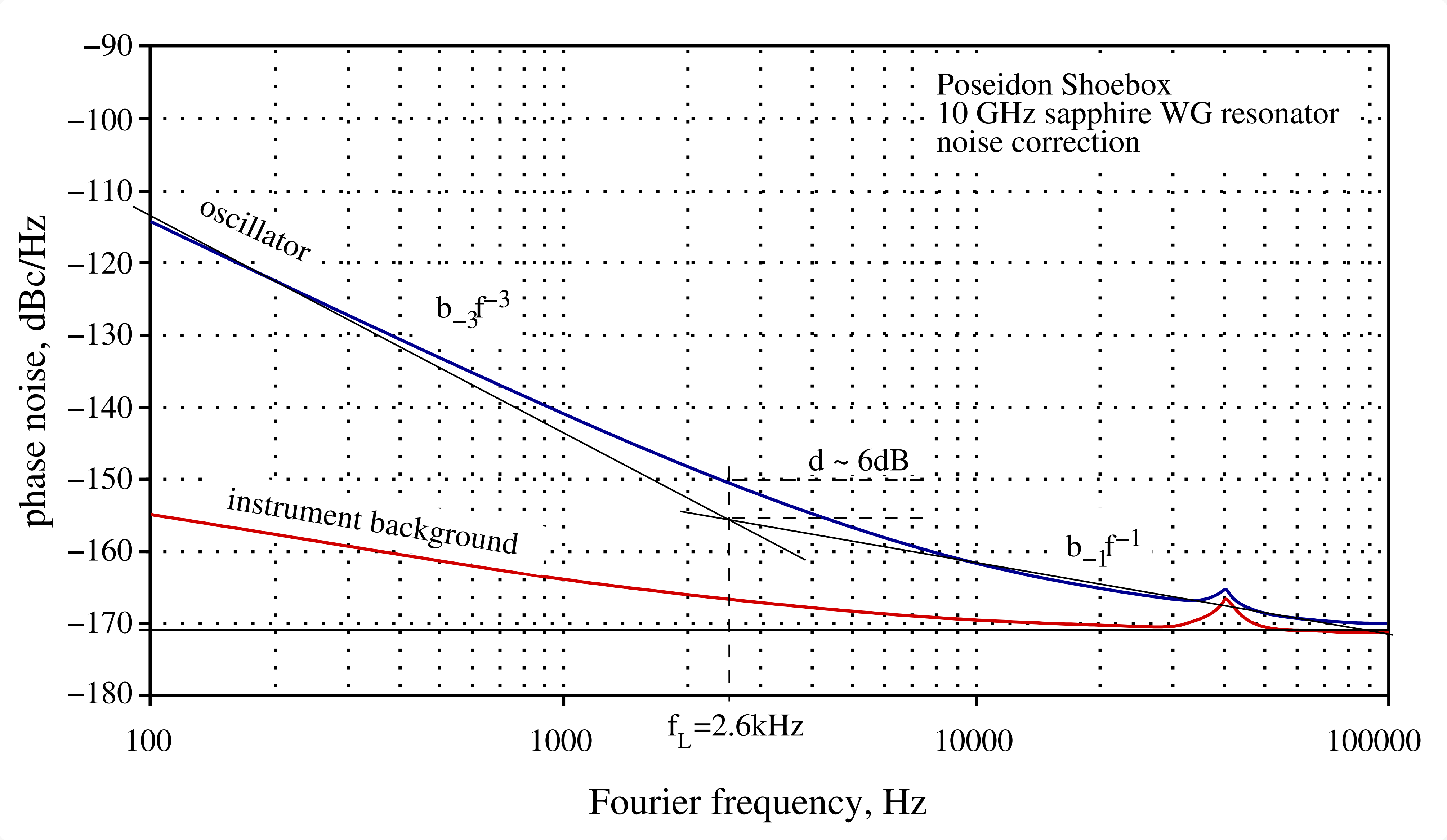
$\sigma^2_y = h_0/2\tau + 2\ln(2)h_{-1}$

$h_0 = b_{-2}/\nu^2_0$

$h_{-1} = b_{-3}/\nu^2_0$

- $kT_0 = 4\times10^{-21}$ W/Hz (–174 dBm/Hz)
- floor –146 dBrad²/Hz, guess F = 1.25 (1 dB) => P_0 = 2 μW (–27 dBm)
- f_L = 4.3 MHz, $f_L = \nu_0/2Q$ => Q = 1160
- f_c = 70 kHz, $b_{-1}/f = b_0$ => $b_{-1} = 1.8\times10^{-10}$ (–98 dBrad²/Hz) [sust.ampli]
- $h_0 = 7.9\times10^{-22}$ and $h_{-1} = 5\times10^{-17}$ => $\sigma_y = 2\times10^{-11}/\sqrt{\tau} + 8.3\times10^{-9}$

Poseidon Scientific Instruments – Shoebox 10 GHz sapphire whispering-gallery (1)

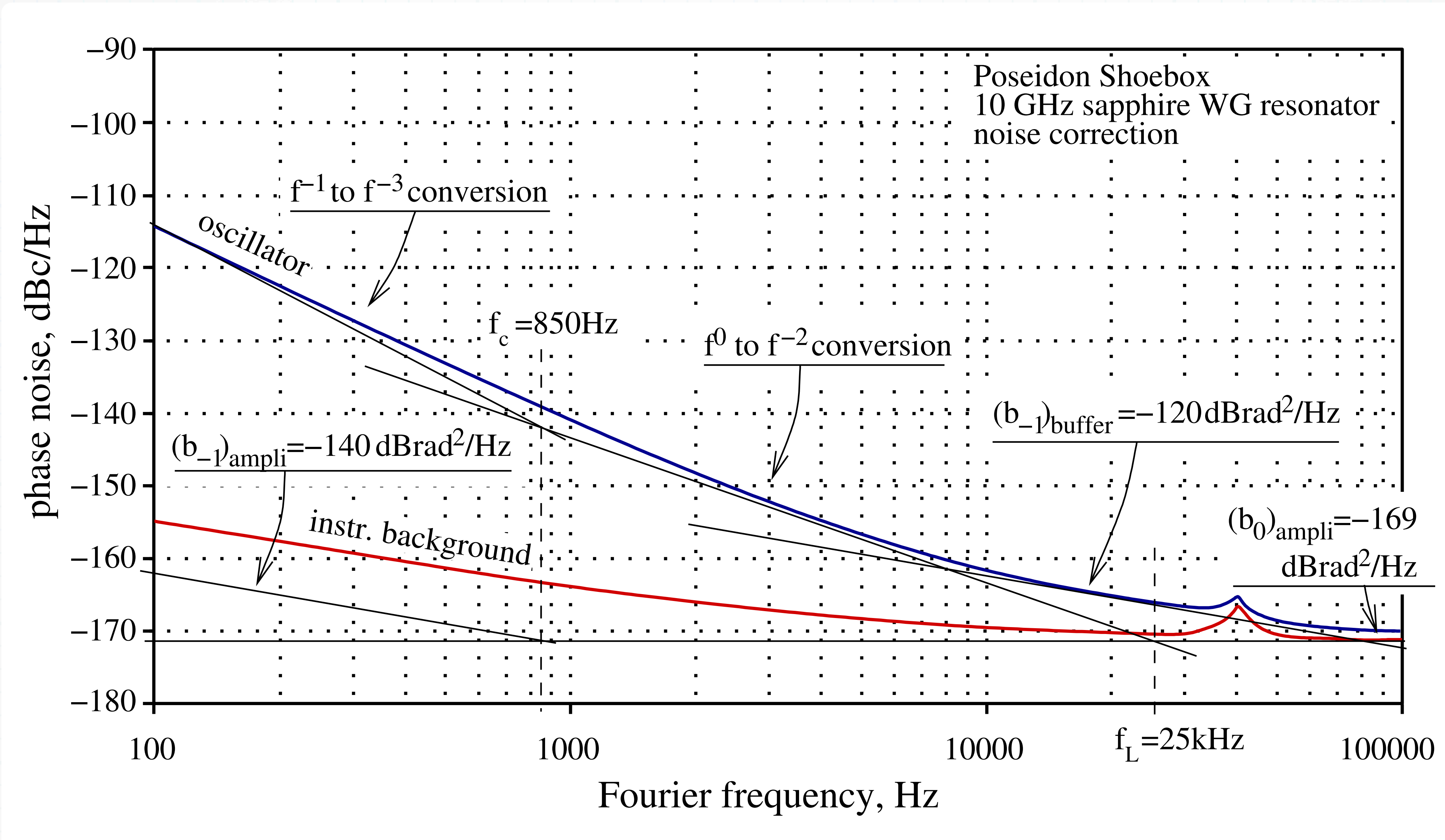


$f_L = \nu_0/2Q = 2.6$ kHz => $Q = 1.8\times10^6$

This incompatible with the resonator technology.
Typical Q of a sapphire whispering gallery resonator:
2×10^5 @ 295K (room temp), 3×10^7 @ 77K (liquid N), 4×10^9 @ 4K (liquid He).
In addition, d ~ 6 dB does not fit the power-law.

The interpretation shown is wrong, and the Leeson frequency is somewhere else

Poseidon Scientific Instruments – Shoebox 10 GHz sapphire whispering-gallery (2)



The 1/f noise of the output buffer is higher than that of the sustaining amplifier (a complex amplifier with interferometric noise reduction / or a Pound control)

In this case both 1/f and 1/f² are present

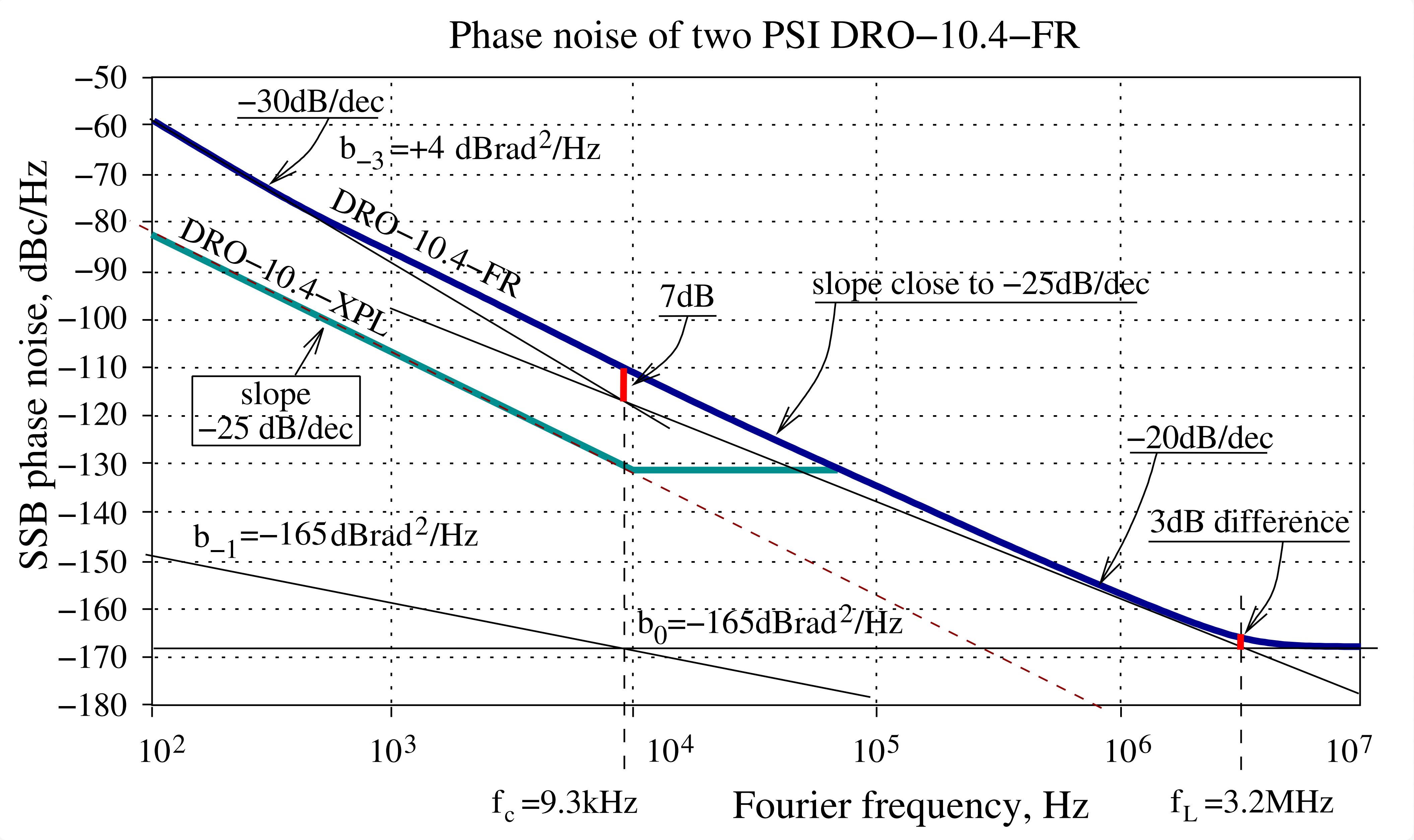
white noise –169 dBrad²/Hz, guess F = 5 dB (interferometer) => P_0 = 0 dBm

buffer flicker –120 dBrad²/Hz @ 1 Hz => good microwave amplifier

$f_L = \nu_0/2Q$ = 25 kHz => Q = 2×10⁵ (quite reasonable)

f_c = 850 Hz => flicker of the interferometric amplifier –139 dBrad²/Hz @ 1 Hz

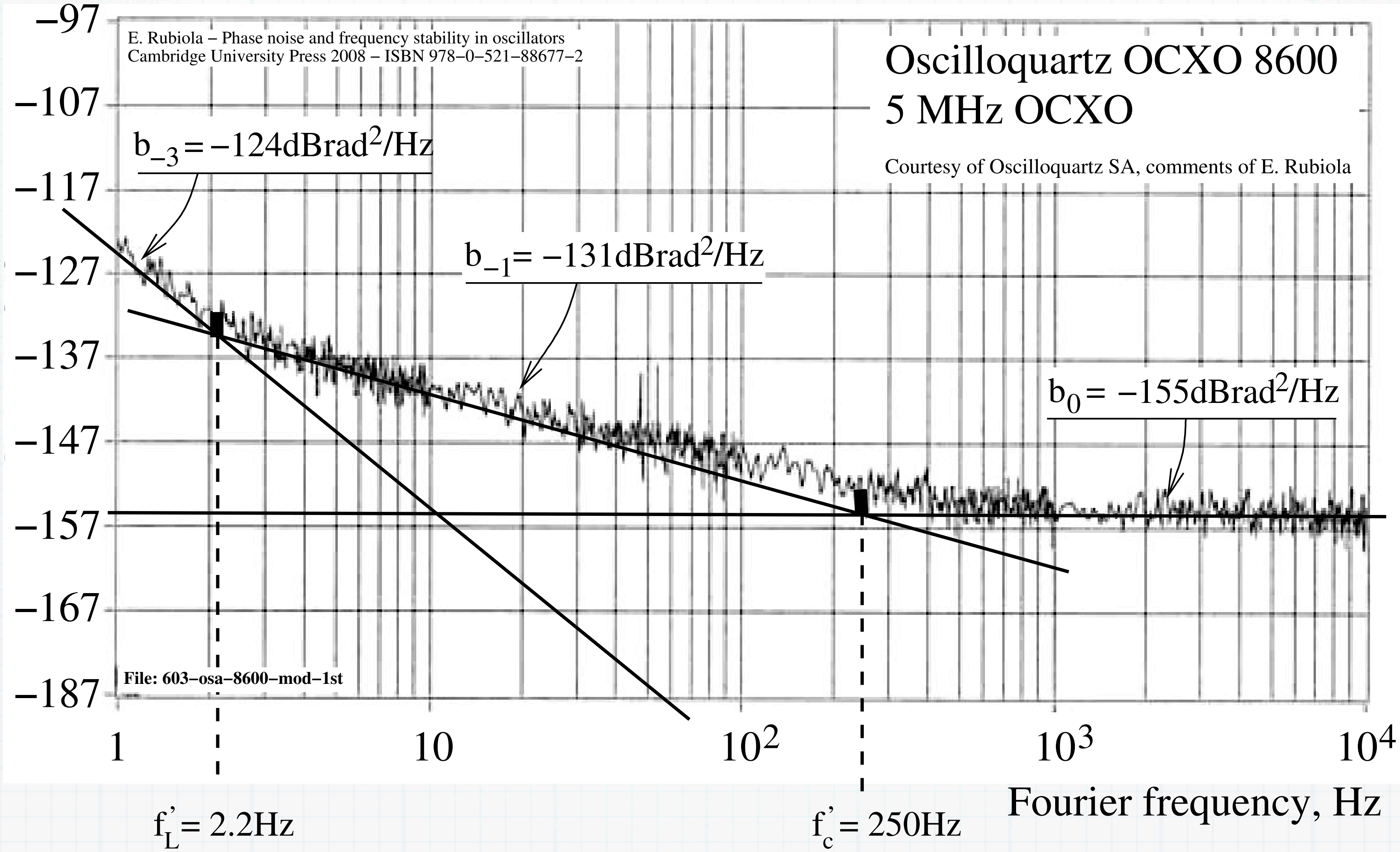
Poseidon Scientific Instruments 10 GHz dielectric resonator oscillator (DRO)



- floor –165 dBrad²/Hz, guess F = 1.25 (1 dB) => P_0 = 160 μW (–8 dBm)
- f_L = 3.2 MHz, $f_L = \nu_0/2Q$ => Q = 625
- f_c = 9.3 kHz, $b_{-1}/f = b_0$ => $b_{-1} = 2.9\times10^{-13}$ (–125 dBrad²/Hz) [sust.ampli, too low]

Slopes are not in agreement with the theory

Example – Oscilloquartz 8600 (wrong)



ANALYSIS

1 – floor $S_{\varphi 0}$ = –155 dBrad²/Hz, guess F = 1 dB → P_0 = –18 dBm

2 – ampli flicker S_φ = –132 dBrad²/Hz @ 1 Hz → good RF amplifier

3 – merit factor $Q = \nu_0/2f_L = 5{\cdot}10^6/5 = 10^6$ (seems too low)

4 – take away some flicker for the output buffer:

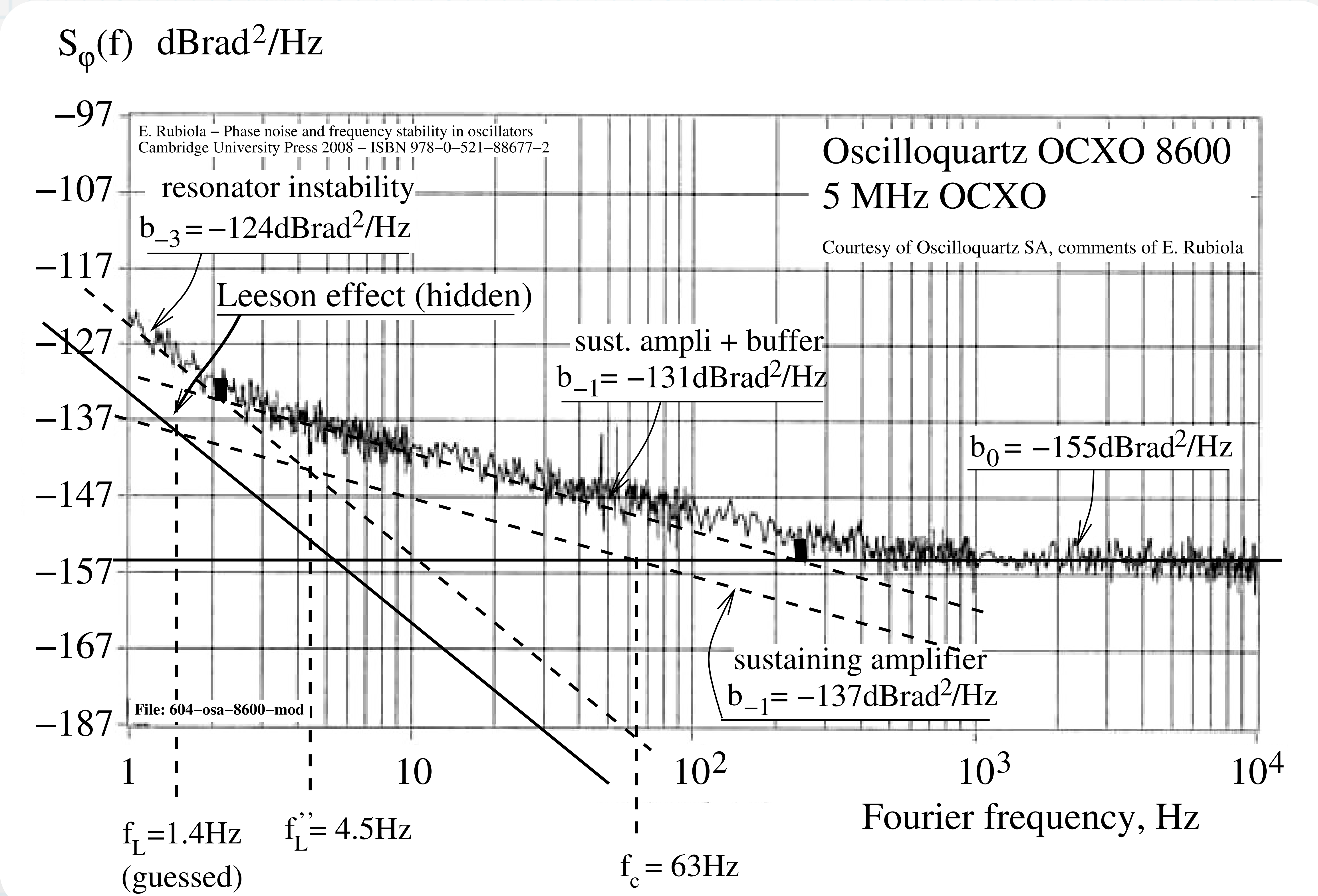
* flicker in the oscillator core is lower than –132 dBrad²/Hz @ 1 Hz
* f_L is higher than 2.5 Hz
* the resonator Q is lower than 10^6

This is inconsistent with the resonator technology (expect $Q > 10^6$).

The true Leeson frequency is lower than the frequency labeled as f_L

The $1/f^3$ noise is attributed to the fluctuation of the quartz resonant frequency

Example – Oscilloquartz 8600 (right)

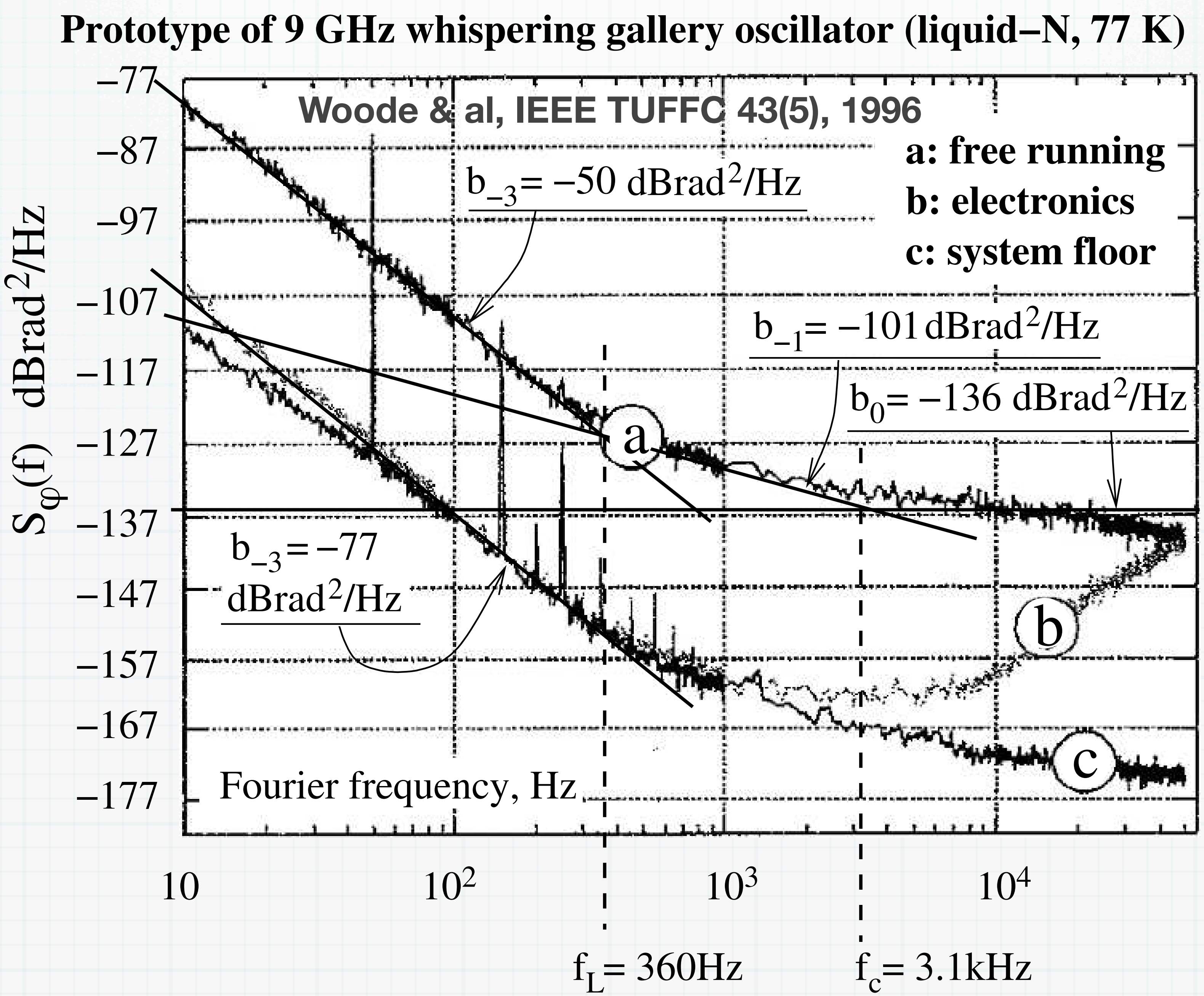


F=1dB b_0 => P_0=–18 dBm

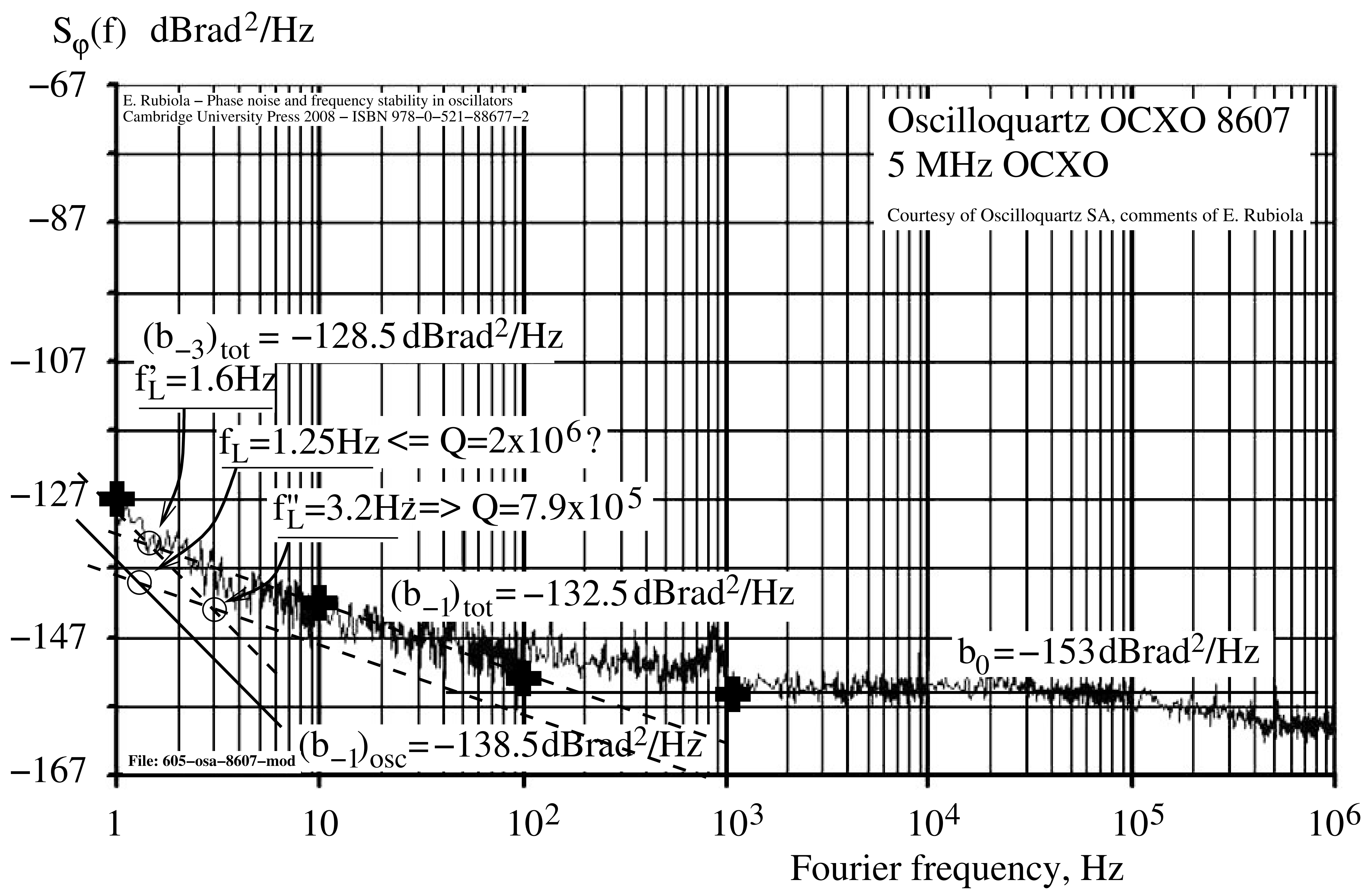
$(b_{-3})_{osc}$ => σ_y=1.5x10^{-13}, Q=5.6x10^5 (too low)

Q $\stackrel{?}{=}$ 1.8x10^6 => σ_y=4.6x10^{-14} Leeson (too low)

Whispering gallery oscillator, liquid-N_2 temp



Example – Oscilloquartz 8607

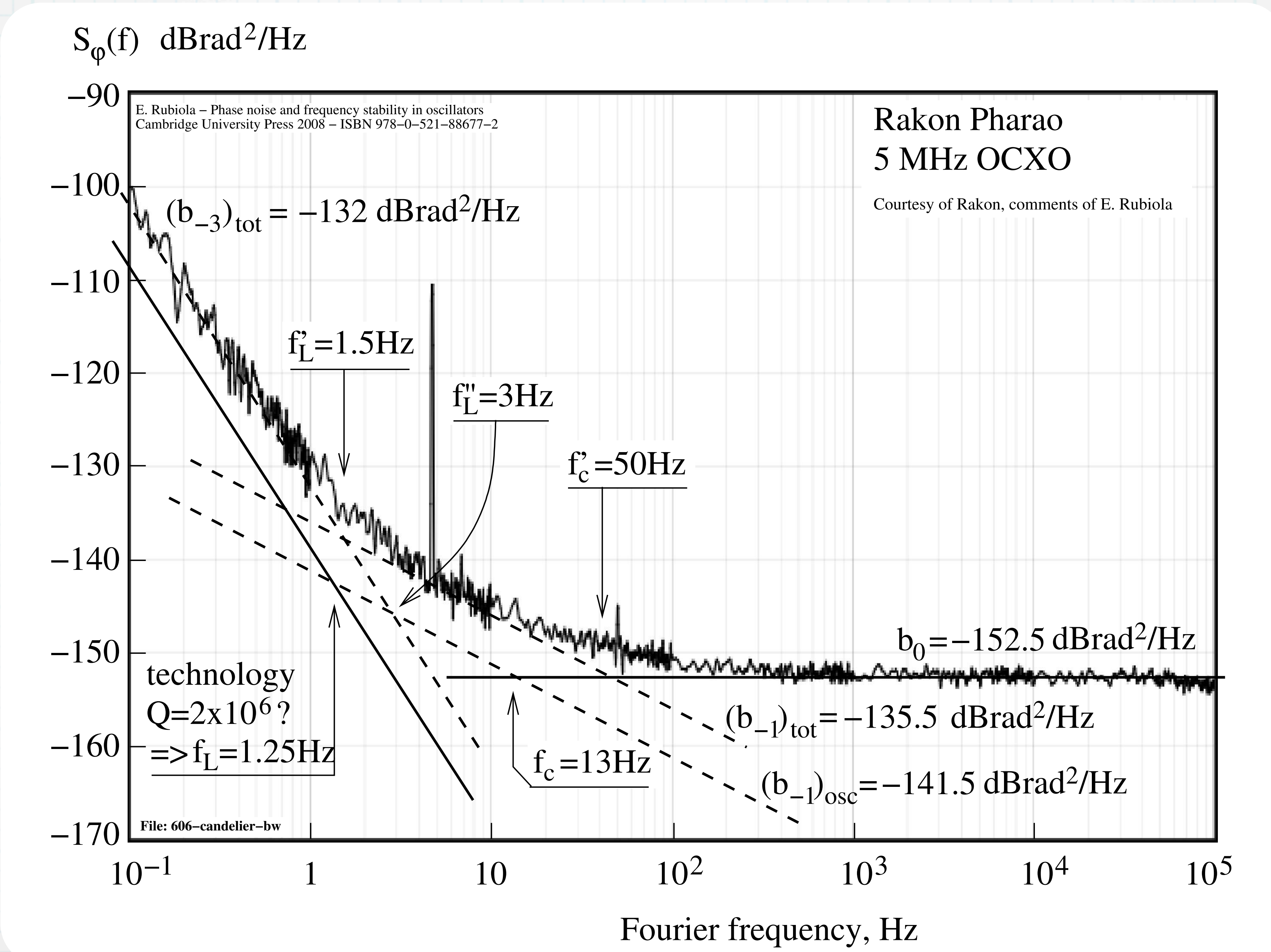


F=1dB b_0 => P_0=–20 dBm

$(b_{-3})_{osc}$ => σ_y=8.8x10^{-14}, Q=7.8x10^5 (too low)

Q $\stackrel{?}{=}$ 2x10^6 => σ_y=3.5x10^{-14} Leeson (too low)

Example – CMAC Pharao

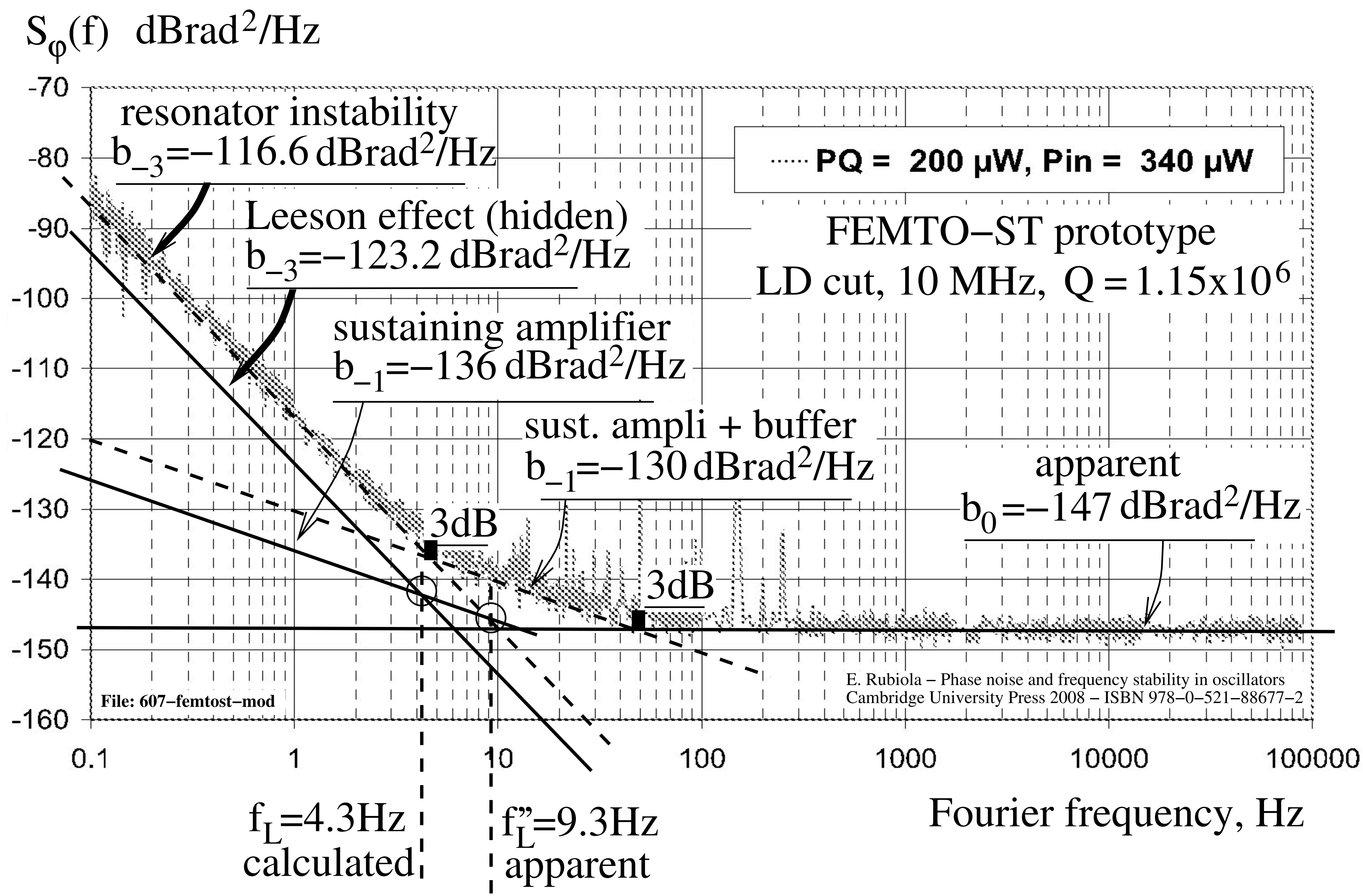


F=1dB b_0 => P_0=–20.5 dBm **$(b_{-3})_{osc}$ => $\sigma_y=5.9\times10^{-14}$, Q=8.4x10^5 (too low)**

$Q \overset{?}{=} 2\times10^6$ => $\sigma_y=2.5\times10^{-14}$ Leeson (too low)

Skip

Example – FEMTO-ST prototype

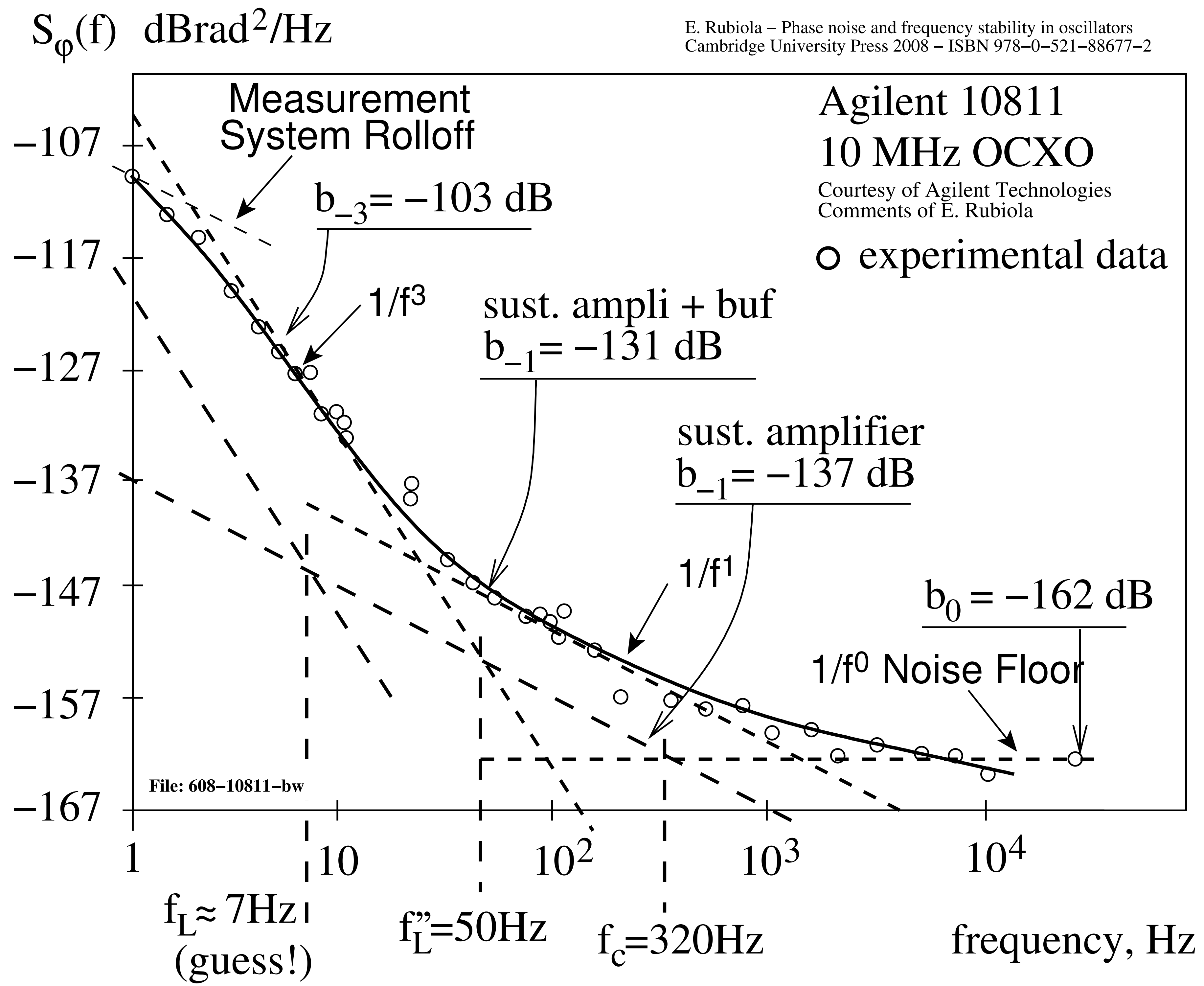


F=1dB b_0 => P_0=–26 dBm $(b_{-3})_{osc}$ => $\sigma_y=1.7\times10^{-13}$, $Q=5.4\times10^{5}$ (too low)

(there is a problem) $Q=1.15\times10^{6}$ => $\sigma_y=8.1\times10^{-14}$ Leeson (too low)

Skip

Example – Agilent 10811

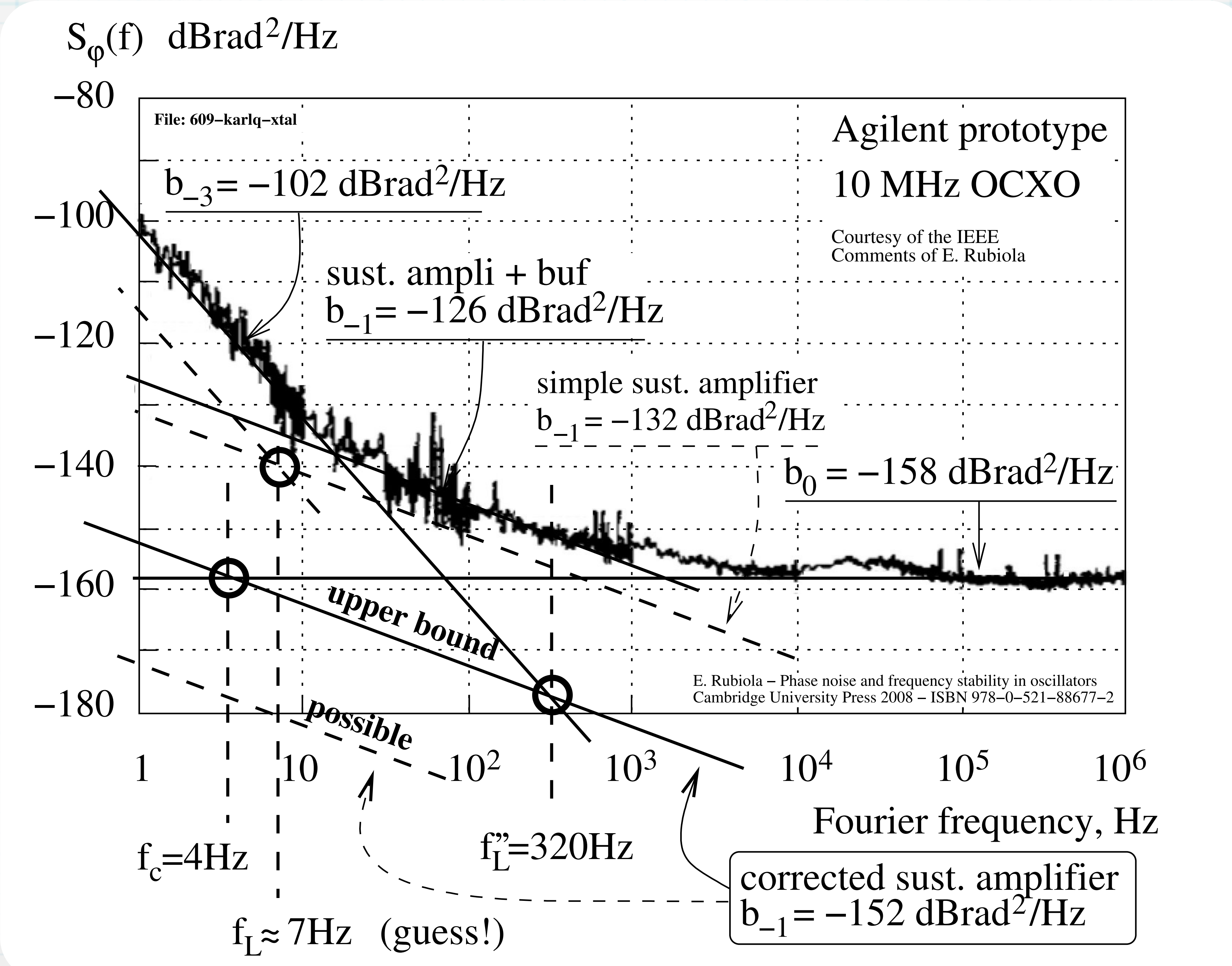


F=1dB b_0 => P_0=–11 dBm

$(b_{-3})_{osc}$ => σ_y=8.3x10^{-13}, Q=1x10^5 (too low)

Q $\stackrel{?}{=}$ 7x10^5 => σ_y=1.2x10^{-13} Leeson (too low)

Example – Agilent prototype



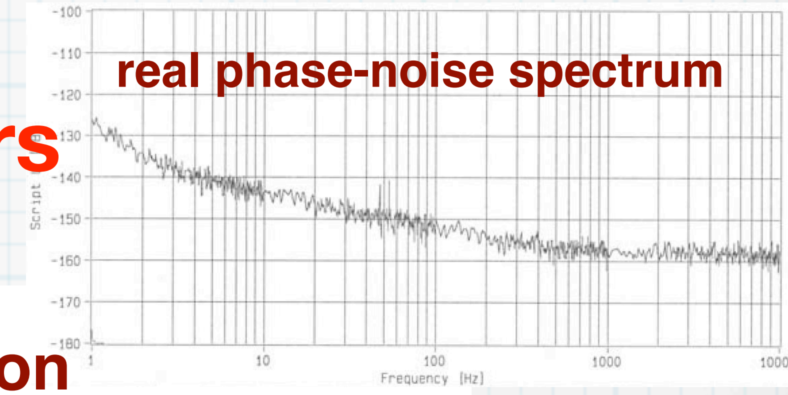
F=1dB b_0 => P_0=–12 dBm

$(b_{-3})_{osc}$ => $\sigma_y=9.3\times10^{-13}$ Q=1.6×10^5

Q $\stackrel{?}{=}$ 7×10^5 => $\sigma_y=2.1\times10^{-13}$ (Leeson)

Skip

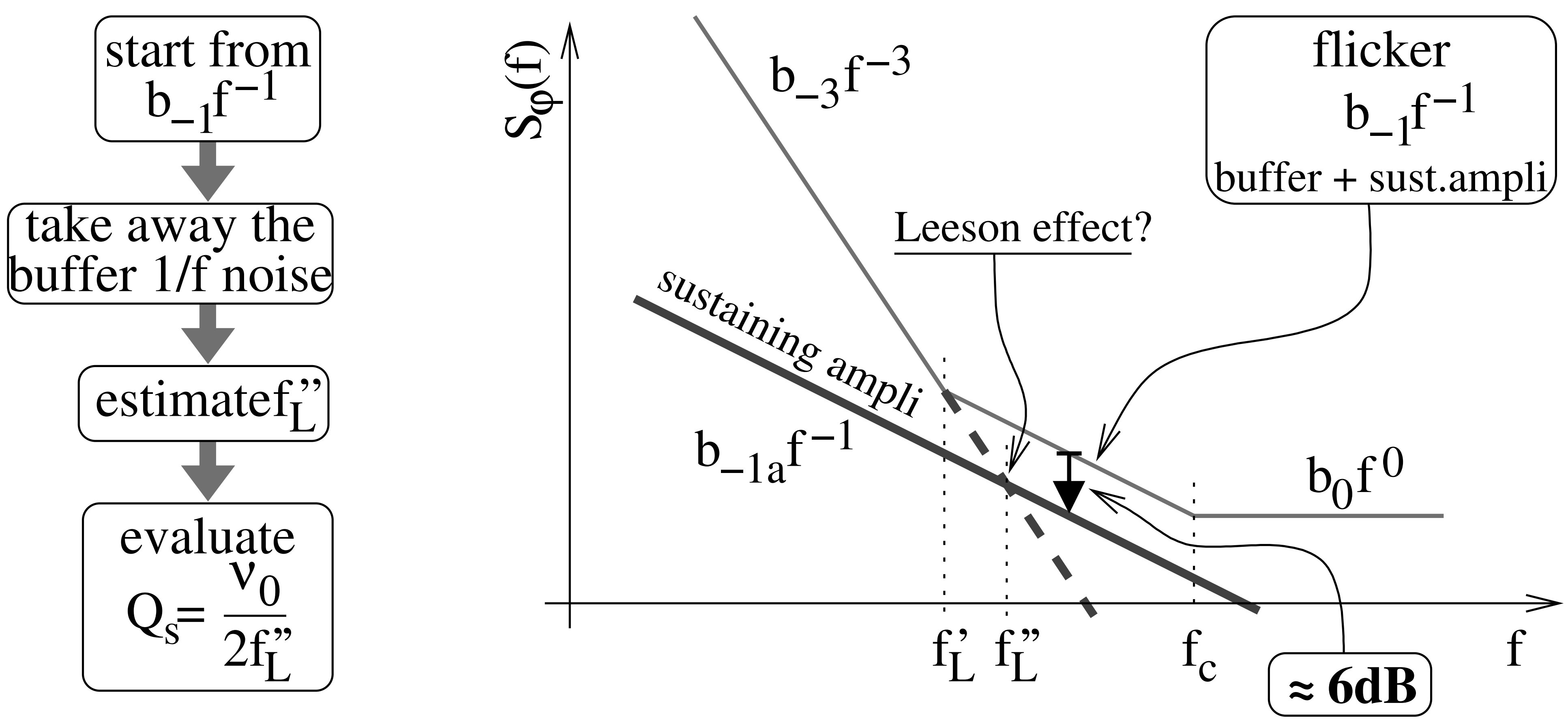
Interpretation of $S_\varphi(f)$ [1]



Only quartz-crystal oscillators

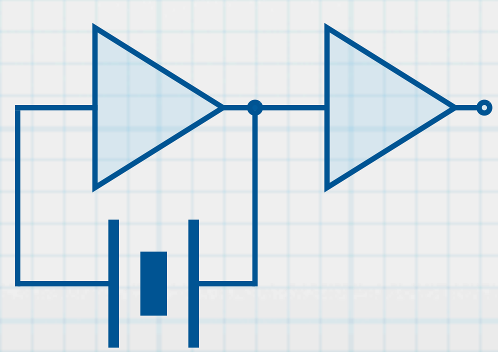
E. Rubiola – Phase noise and frequency stability in oscillators
Cambridge University Press 2008 – ISBN 978–0–521–88677–2

after parametric estimation



File: 602a–xtal–interpretation

Sanity check:
– power P_0 at amplifier input
– Allan deviation σ_y (floor)



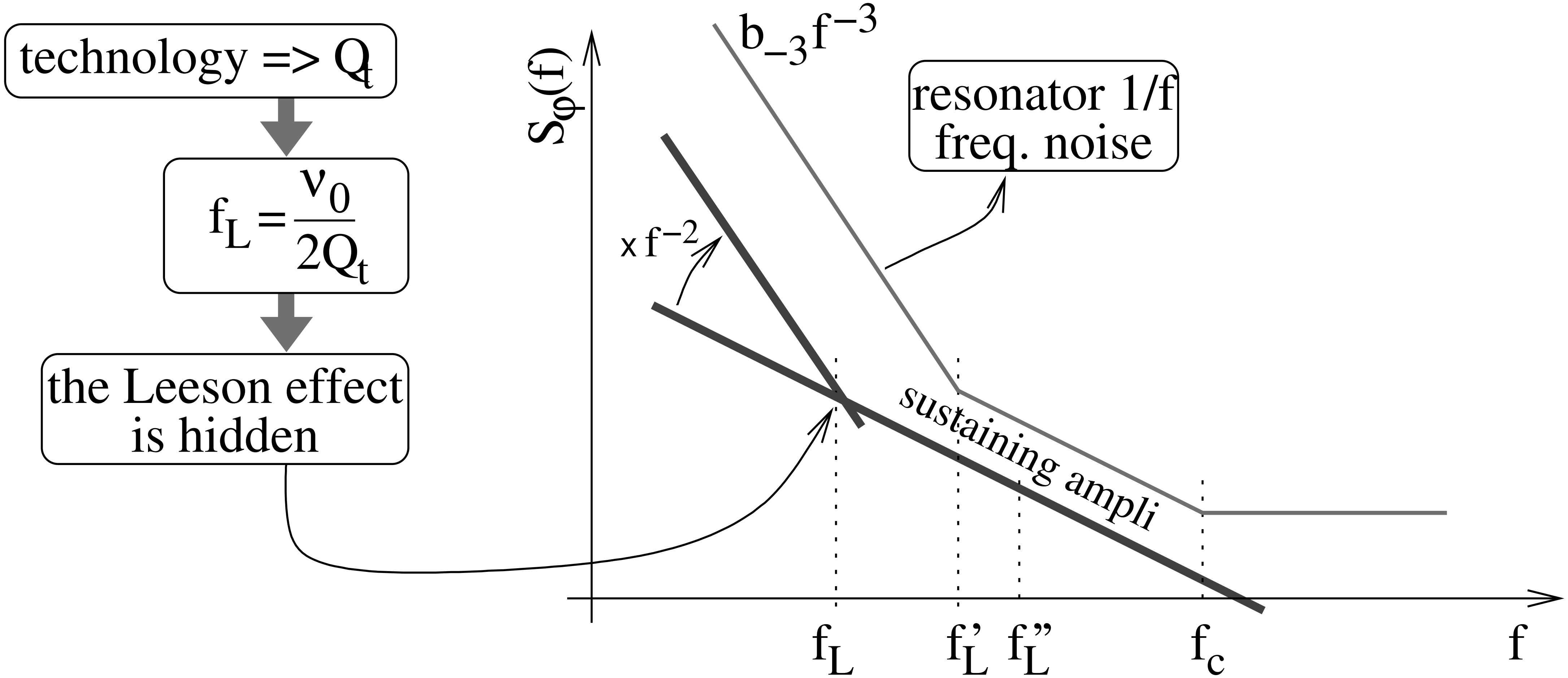
2–3 buffer stages => the sustaining amplifier contributes ≲ 25% of the total 1/f noise

Skip

Interpretation of $S_\varphi(f)$ [2]

Only quartz-crystal oscillators

E. Rubiola – Phase noise and frequency stability in oscillators
Cambridge University Press 2008 – ISBN 978–0–521–88677–2

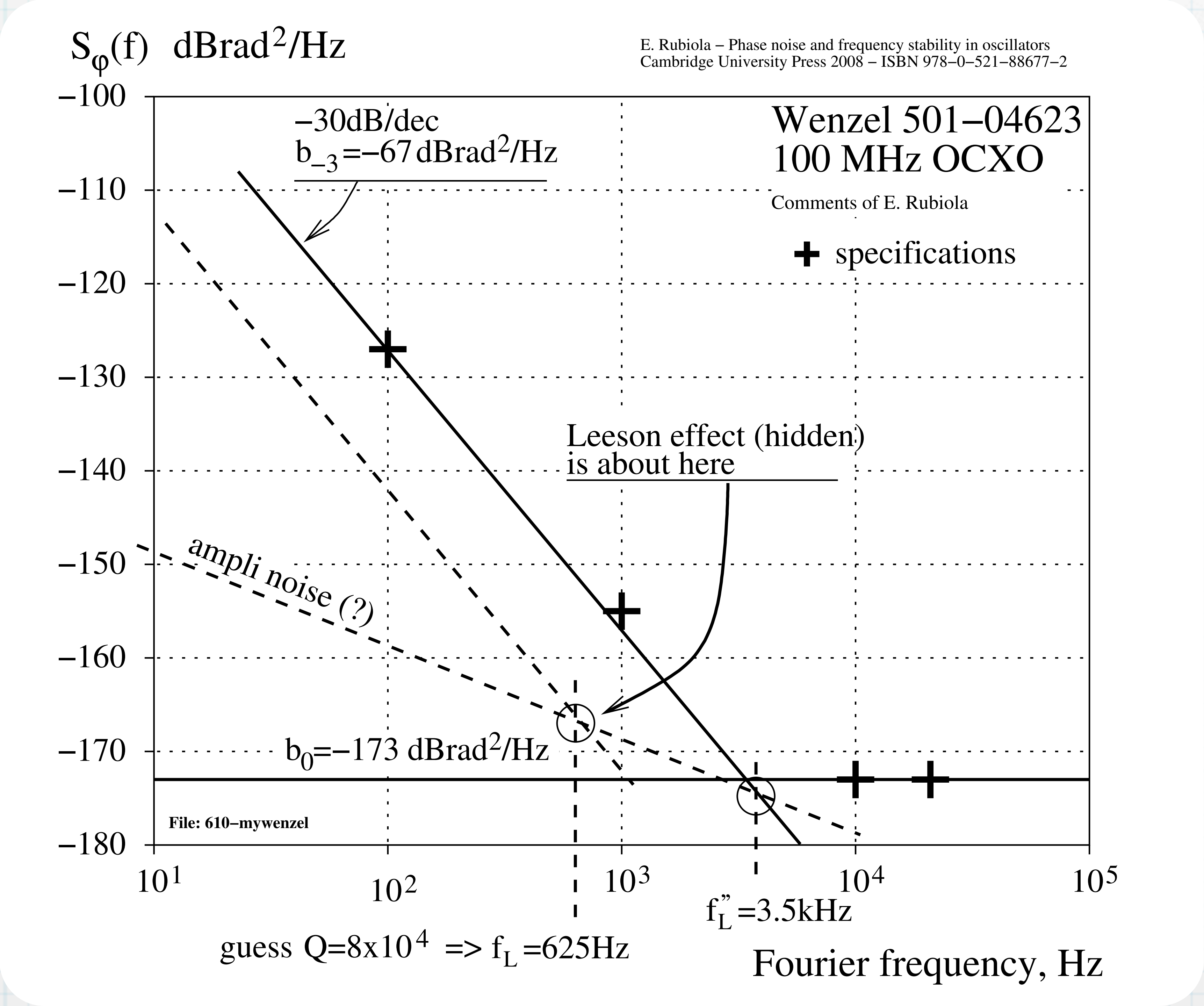


File: 602b–xtal–interpretation

Technology suggests a quality factor Q_t. In all xtal oscillators we find $Q_t \gg Q_s$

Example – Wenzel 501-04623

Figures are from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press



Data are from the manufacturer web site. Interpretation and mistakes are of the authors.

Estimating $(b_{-1})_{ampli}$ is difficult because there is no visible 1/f region

F=1dB b_0 => P_0=0 dBm

$(b_{-3})_{osc}$ => $\sigma_y=5.3\times10^{-12}$ Q=1.4x10^4

$Q \stackrel{?}{=} 8\times10^4$ => $\sigma_y=9.3\times10^{-13}$ (Leeson)

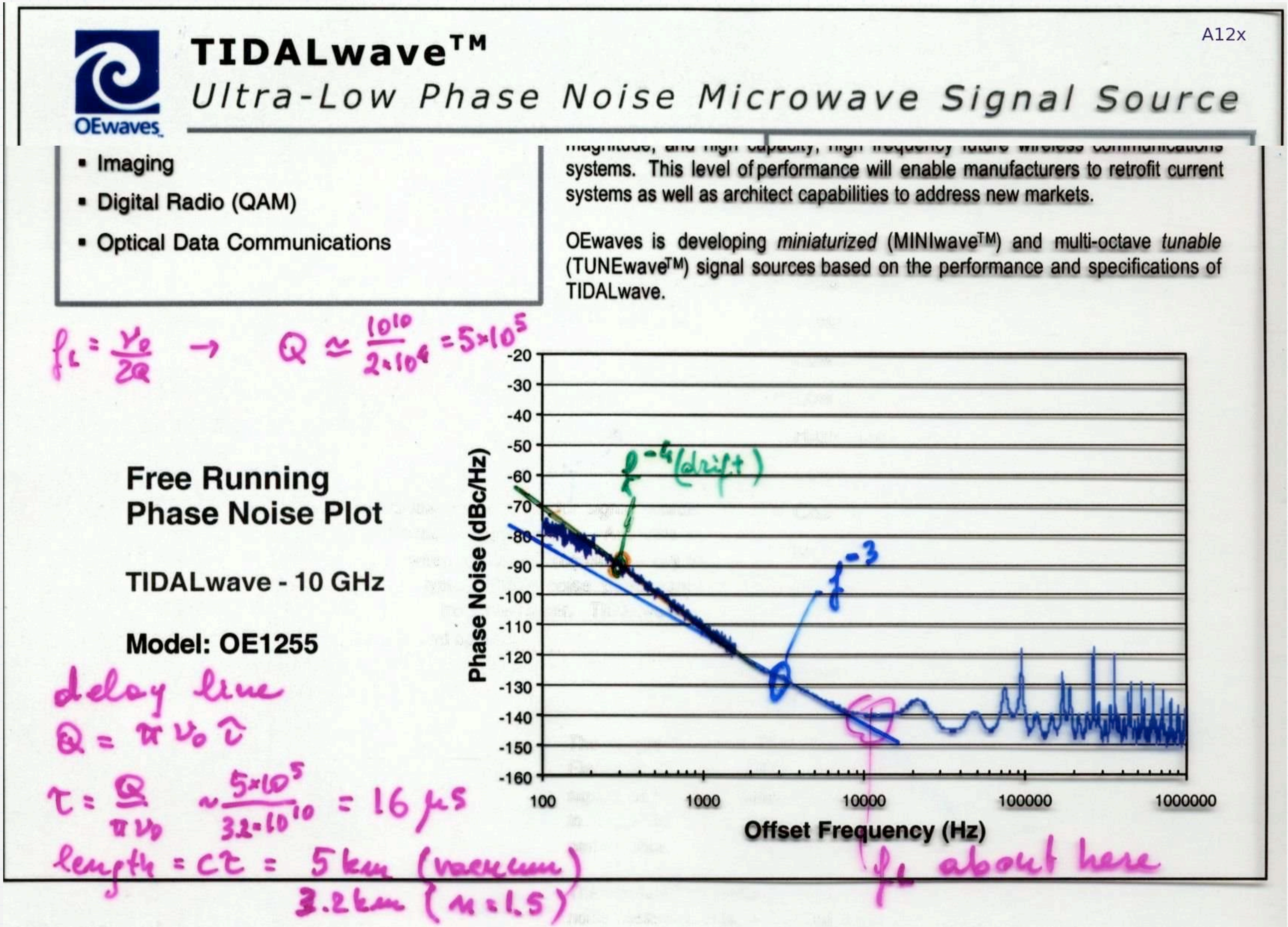
Quartz-oscillator summary

Oscillator	ν_0	$(b_{-3})_{\text{tot}}$	$(b_{-1})_{\text{tot}}$	$(b_{-1})_{\text{amp}}$	f'_L	f''_L	Q_s	Q_t	f_L	$(b_{-3})_{\text{L}}$	R	Note
Oscilloquartz 8600	5	-124.0	-131.0	-137.0	2.24	4.5	5.6×10^5	1.8×10^6	1.4	-134.1	10.1	(1)
Oscilloquartz 8607	5	-128.5	-132.5	-138.5	1.6	3.2	7.9×10^5	2×10^6	1.25	-136.5	8.1	(1)
Rakon Pharao	5	-132.0	-135.5	-141.1	1.5	3	8.4×10^5	2×10^6	1.25	-139.6	7.6	(2)
FEMTO-ST LD prot.	10	-116.6	-130.0	-136.0	4.7	9.3	5.4×10^5	1.15×10^6	4.3	-123.2	6.6	(3)
Agilent 10811	10	-103.0	-131.0	-137.0	25	50	1×10^5	7×10^5	7.1	-119.9	16.9	(4)
Agilent prototype	10	-102.0	-126.0	-132.0	16	32	1.6×10^5	7×10^5	7.1	-114.9	12.9	(5)
Wenzel 501-04623	100	-67.0	-132 ?	-138 ?	1800	3500	1.4×10^4	8×10^4	625	-79.1	15.1	(6)
unit	MHz	dB rad^2/Hz	dB rad^2/Hz	dB rad^2/Hz	Hz	Hz	(none)	(none)	Hz	dB rad^2/Hz	dB	

Notes
(1) Data are from specifications, full options about low noise and high stability.
(2) Measured by Rakon on a sample. Rakon confirmed that $2\times10^6 < Q < 2.2\times10^6$ in actual conditions.
(3) LD cut, built and measured in our laboratory, yet by a different team. Q_t is known.
(4) Measured by Hewlett Packard (now Agilent) on a sample.
(5) Implements a bridge scheme for the degeneration of the amplifier noise. Same resonator of the Agilent 10811.
(6) Data are from specifications.

$$R = \left.\frac{(\sigma_y)_{\text{oscill}}}{(\sigma_y)_{\text{Leeson}}}\right|_{\text{floor}} = \sqrt{\frac{(b_{-3})_{\text{tot}}}{(b_{-3})_L}} = \frac{Q_t}{Q_s} = \frac{f''_L}{f_L}$$

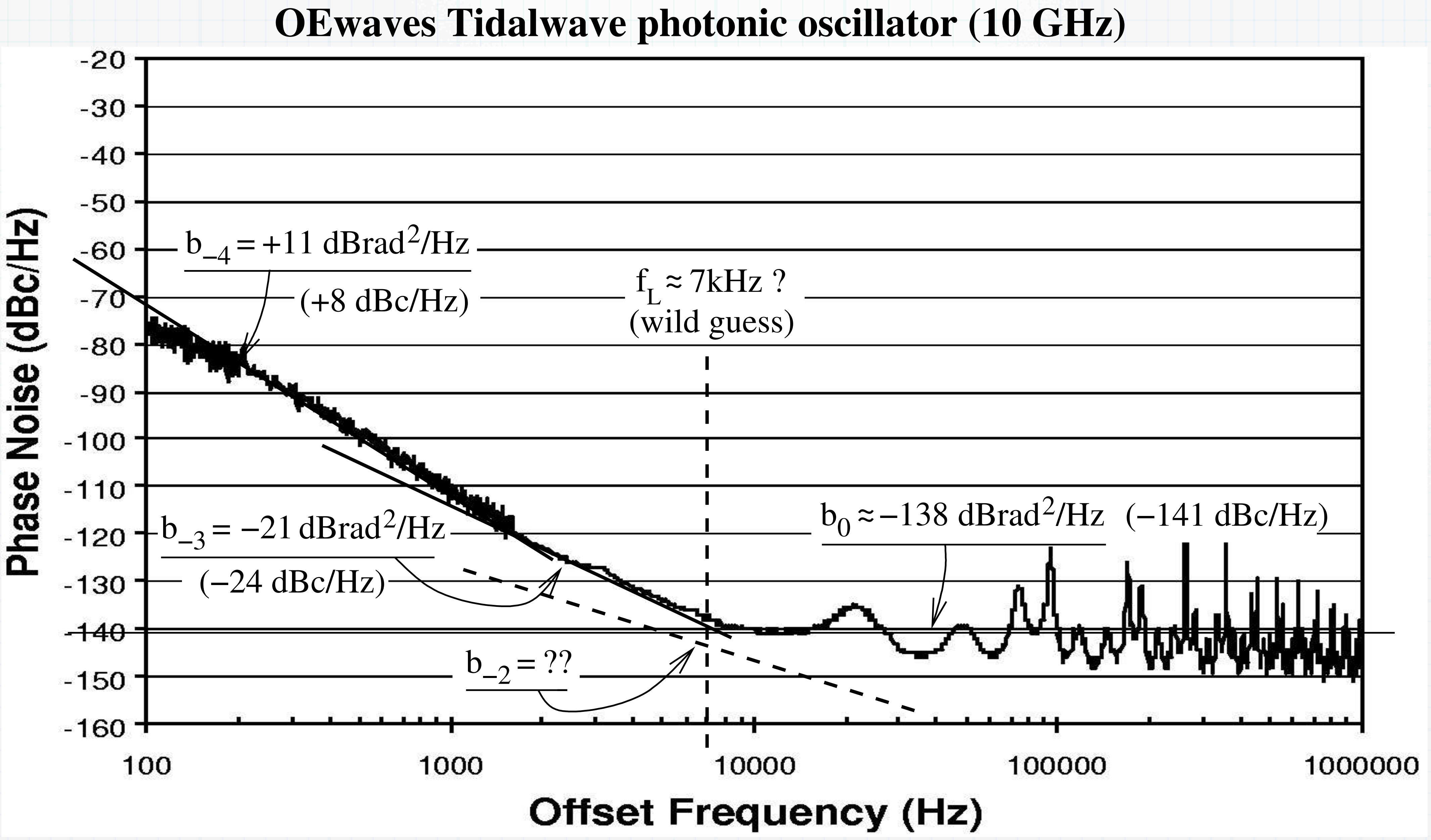
Opto-electronic oscillator



Courtesy of OEwaves (handwritten notes are mine). Cut from the oscillator specifications available at the URL http://www.oewaves.com/products/pdf/TDALwave_Datasheet_012104.pdf

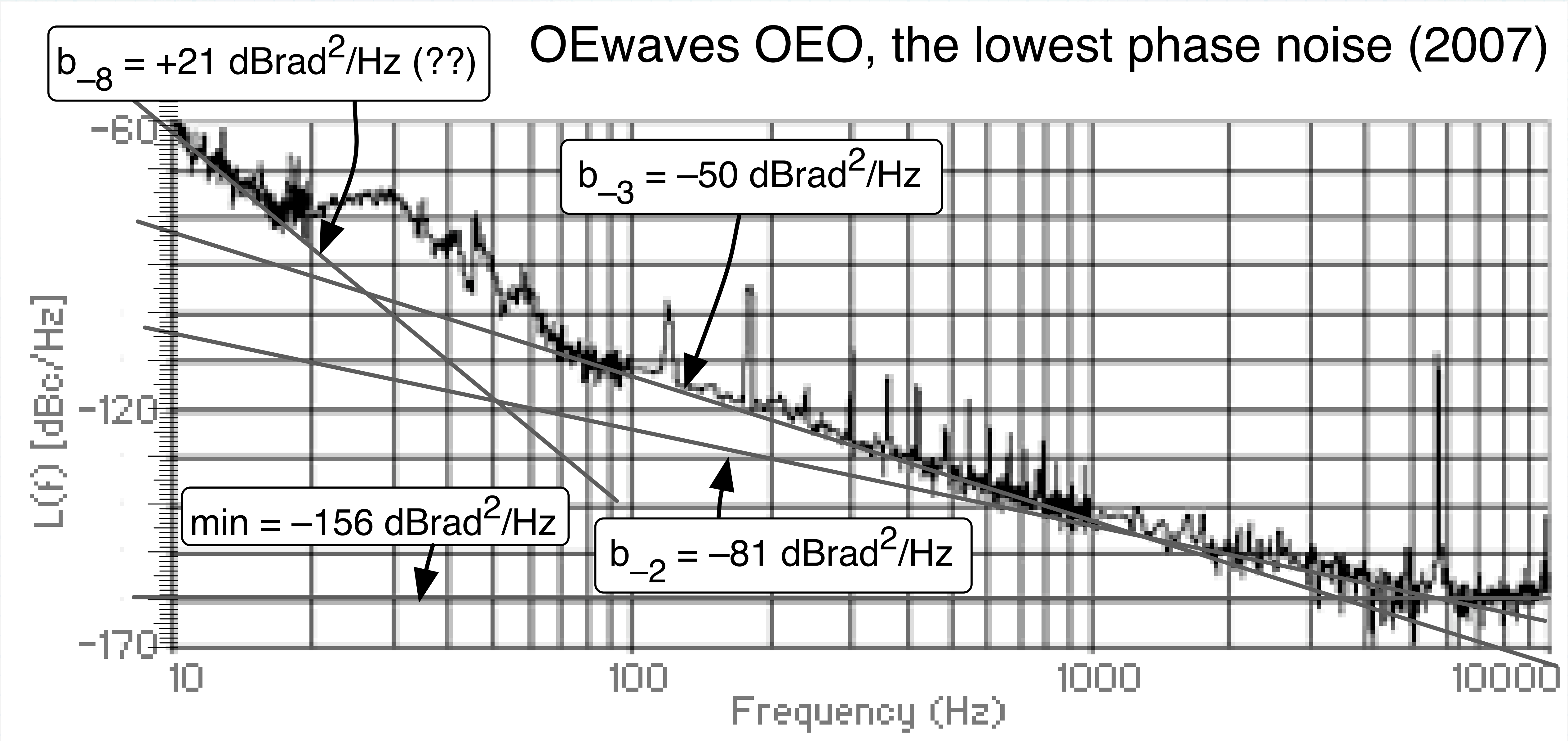
Opto-electronic oscillator

The spectrum is © OEwaves. The figure is from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press



Courtesy of OEwaves (handwritten notes are mine). Cut from the oscillator specifications available at the URL http://www.oewaves.com/products/pdf/TDALwave_Datasheet_012104.pdf

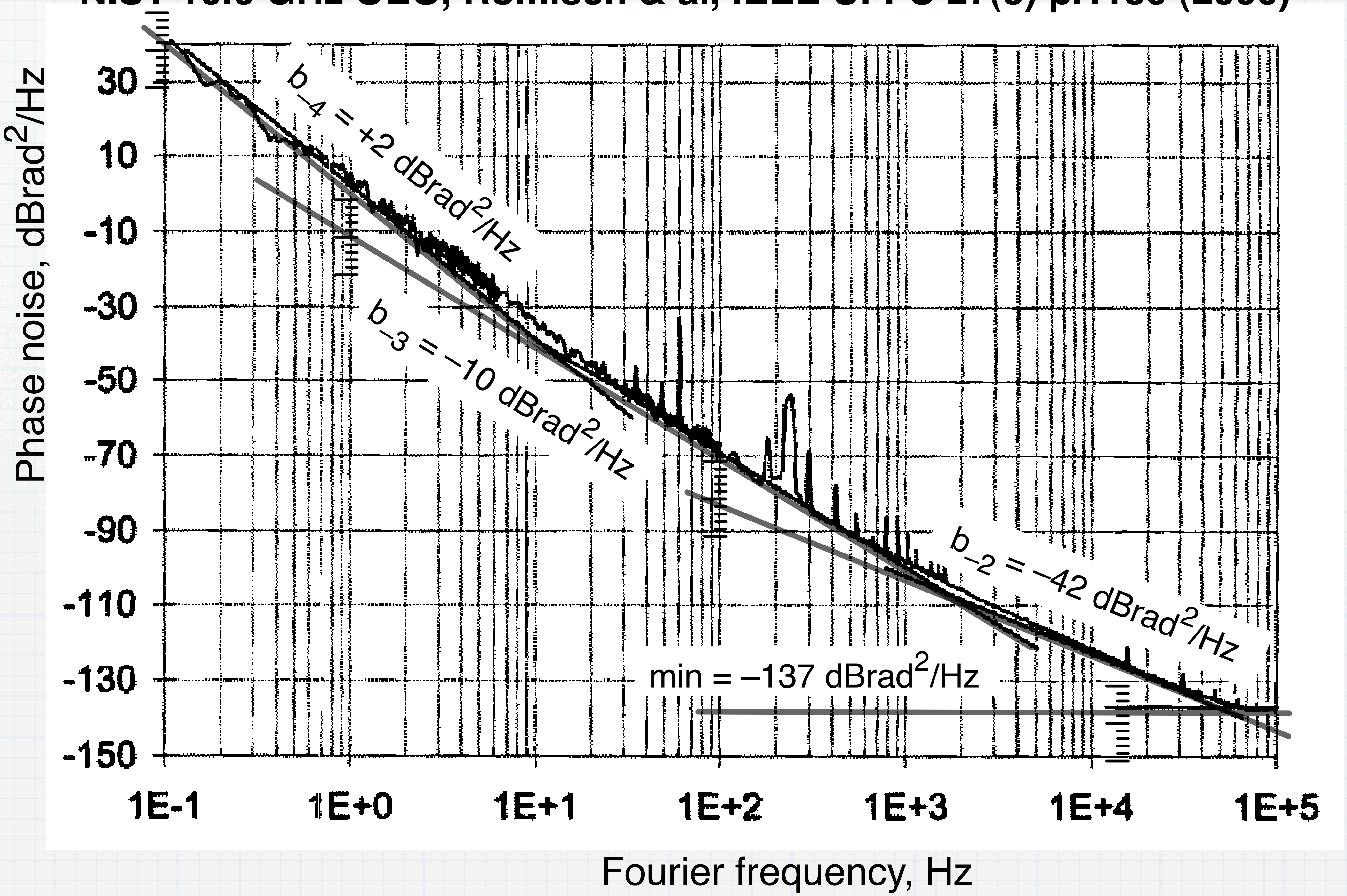
Opto-electronic oscillator



Courtesy of OEwaves, notes are mine

Opto-electronic oscillator

NIST 10.6 GHz OEO, Römisch & al, IEEE UFFC 27(5) p.1159 (2000)



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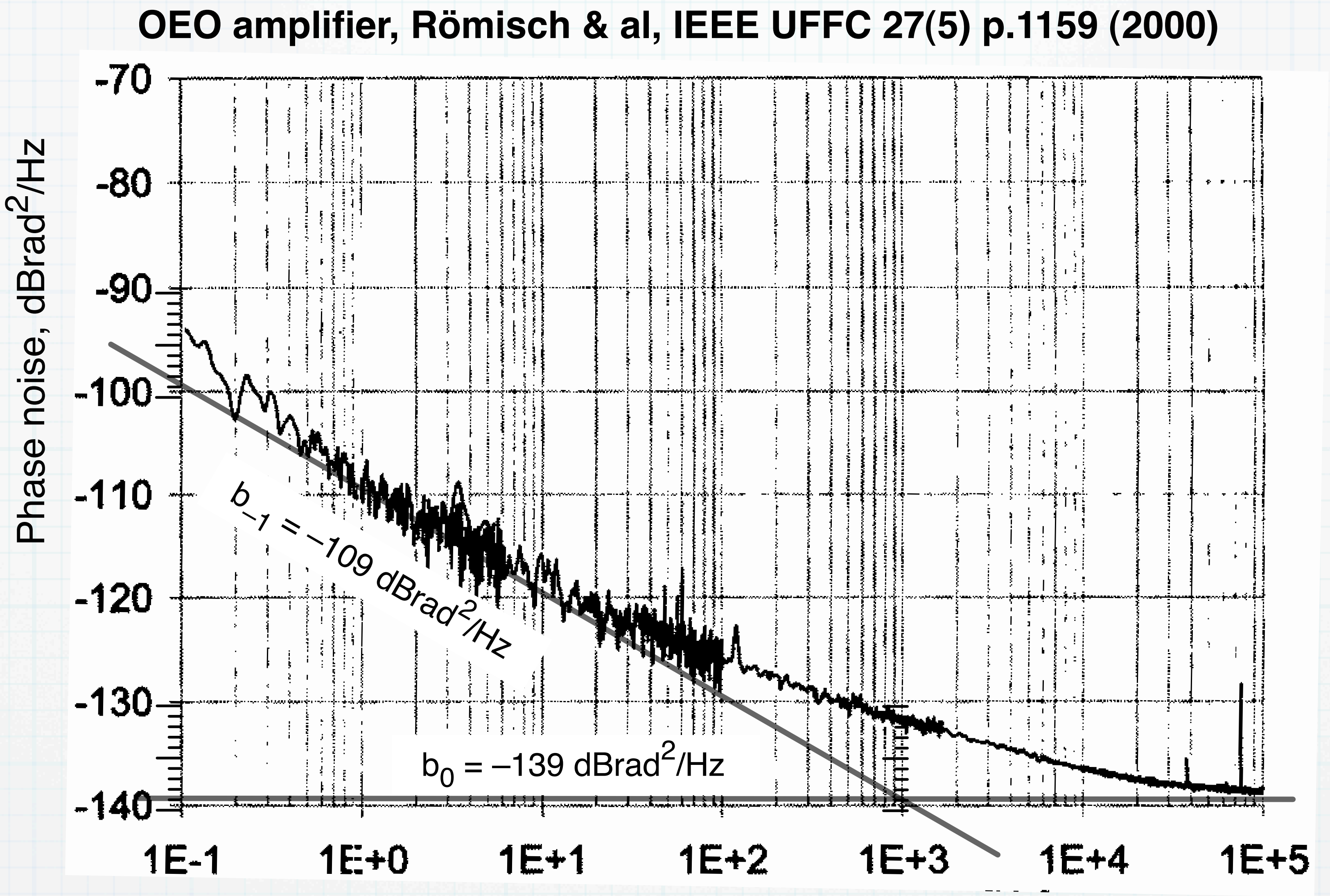
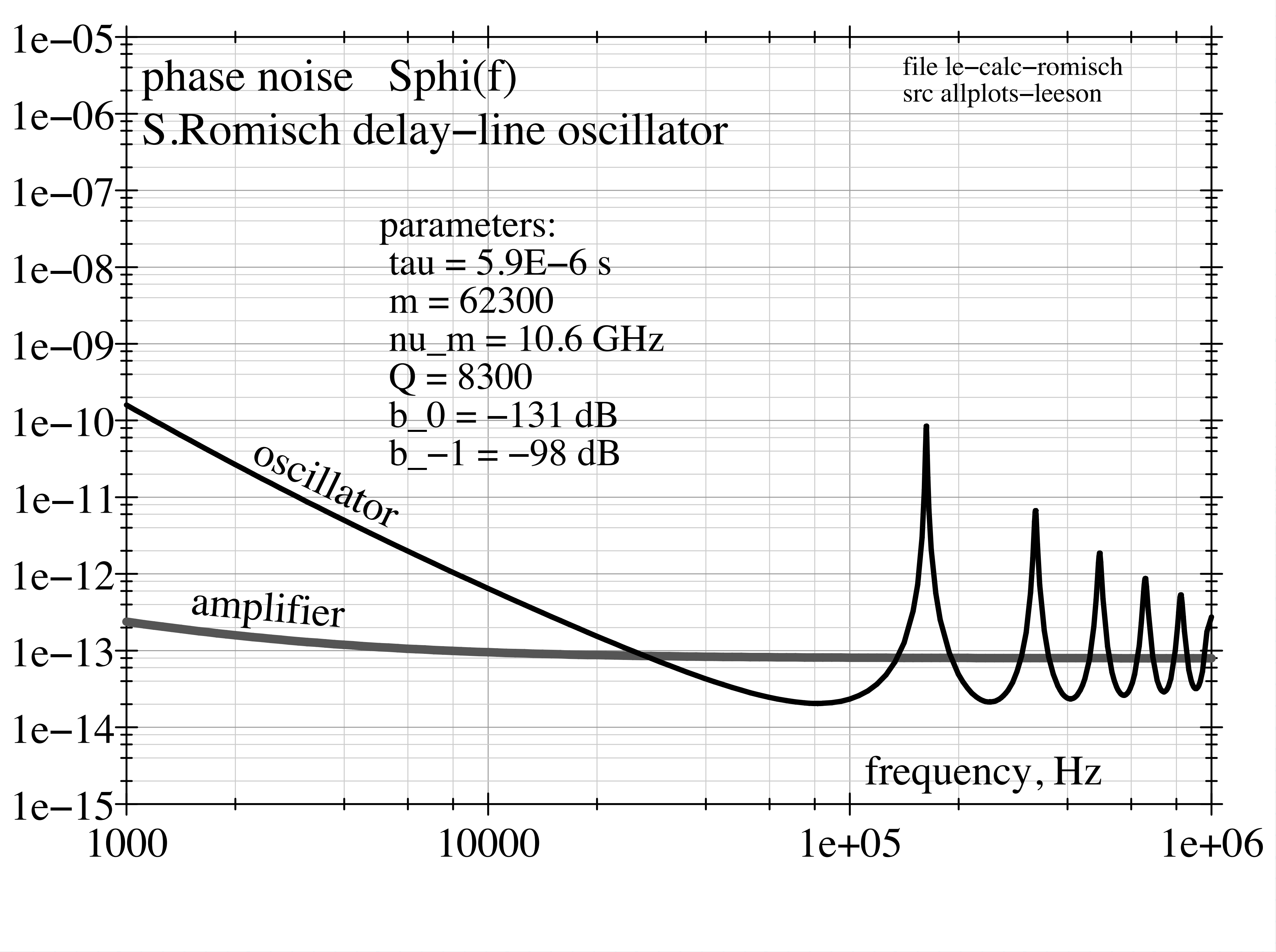
Opto-electronic oscillator (amplifier)

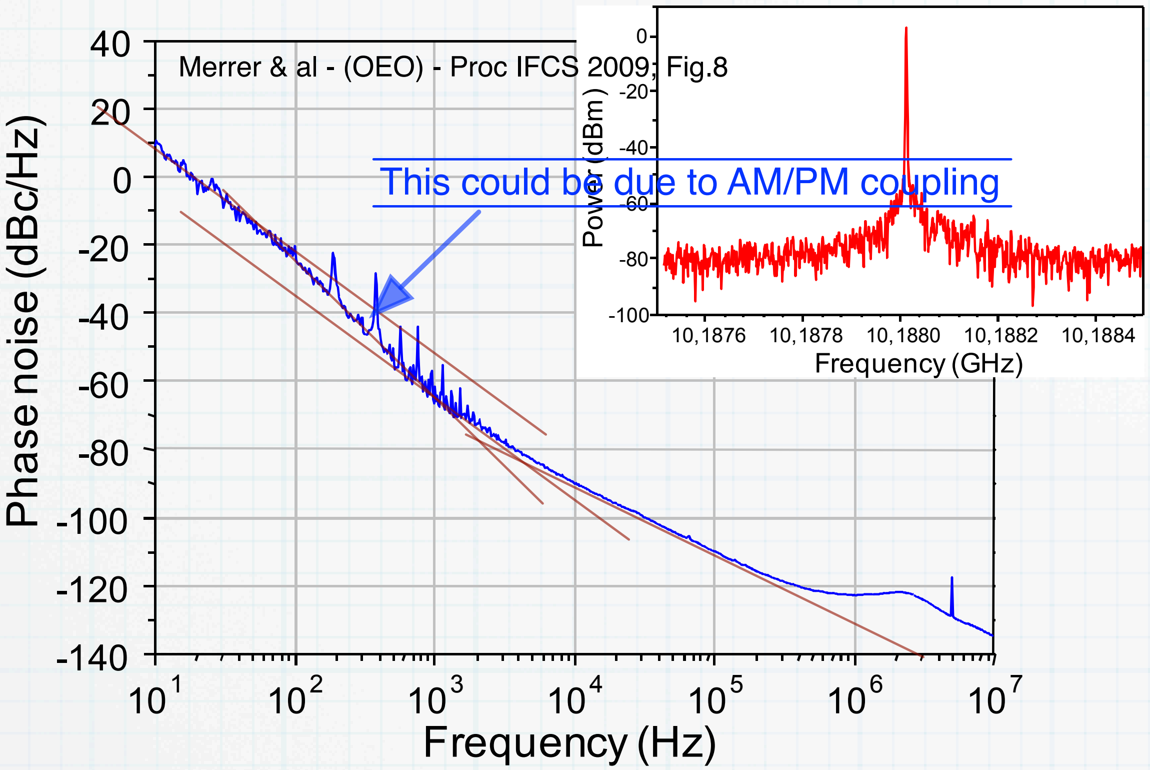


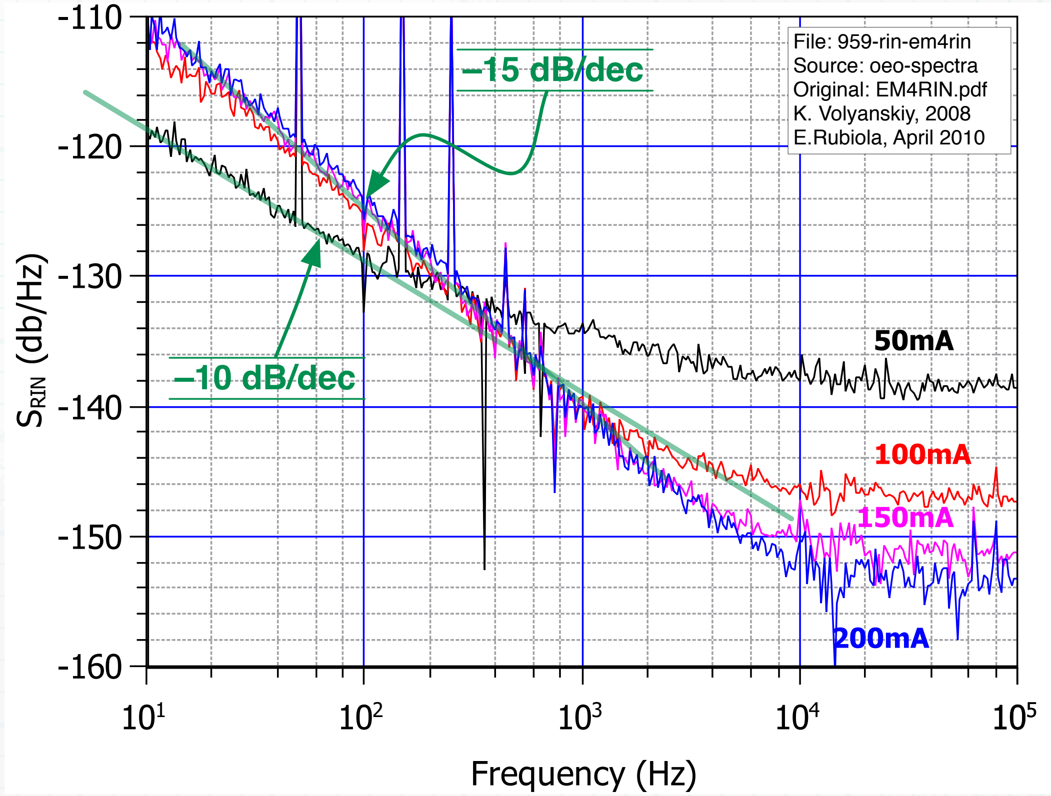
Opto-electronic oscillator simulation

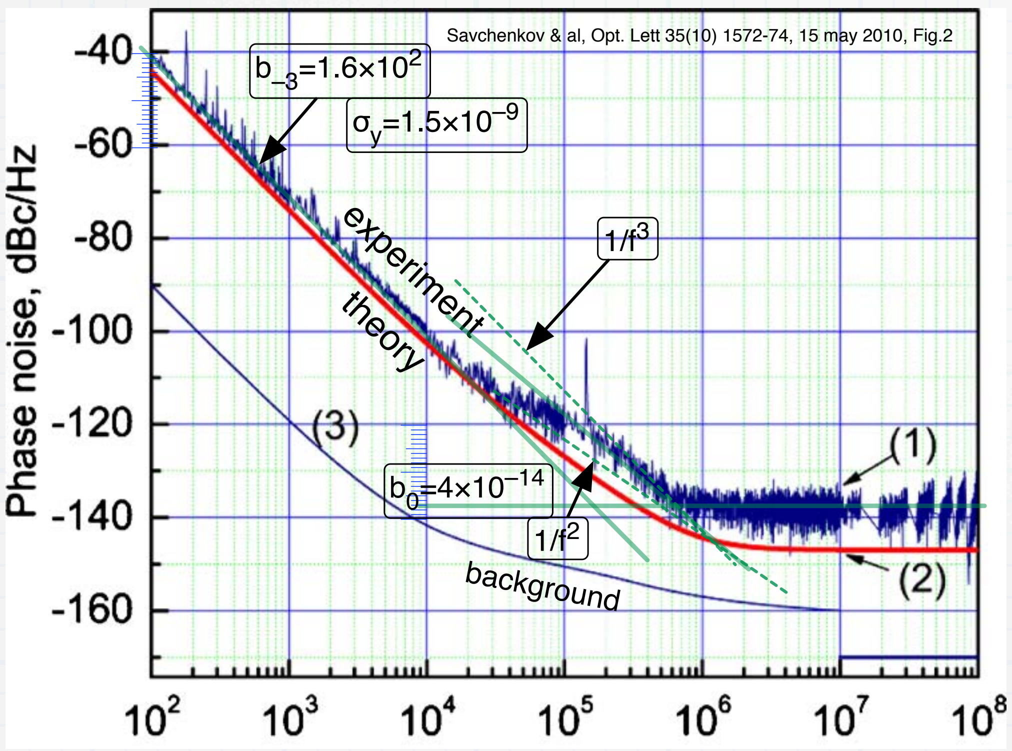
Figures are from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press

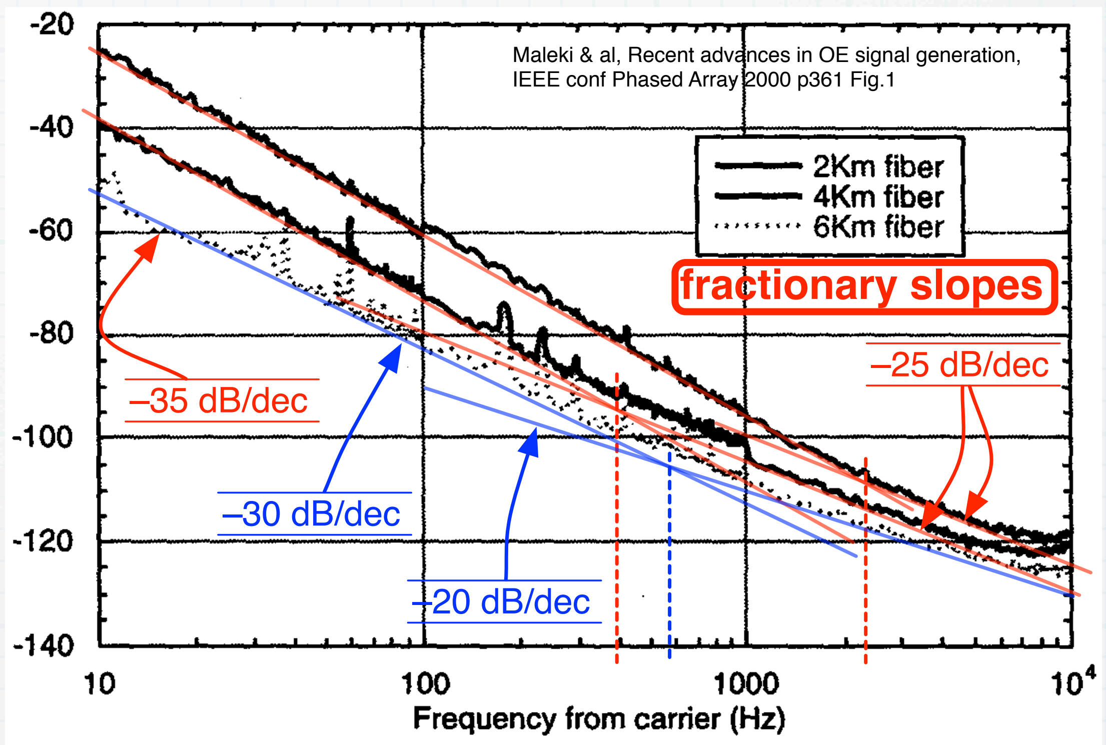


Things may not be that simple









Resonator theory

Resonator – time domain

$$\ddot{x} + \frac{\omega_n}{Q}\dot{x} + \omega_n^2 x = \frac{\omega_n}{Q}\dot{v}(t)$$

ω_n natural frequency

Q quality factor

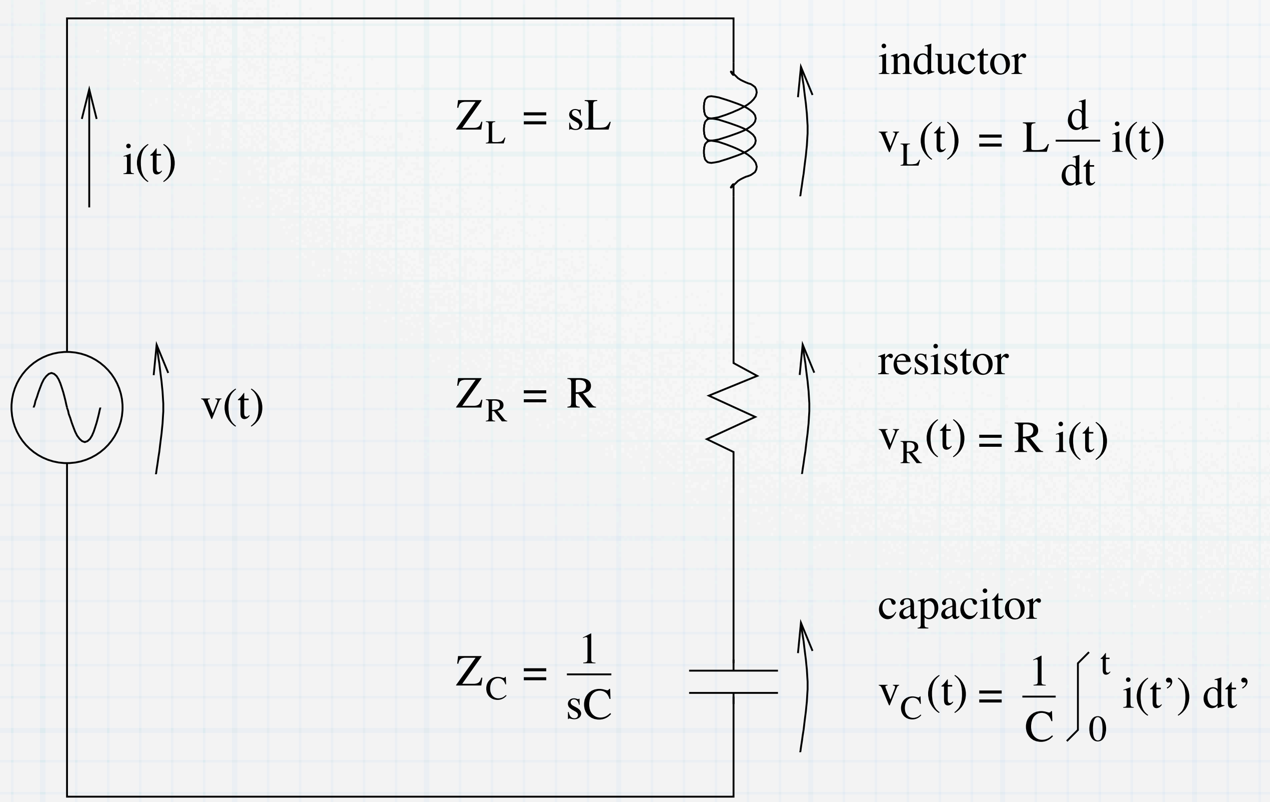
τ relaxation time

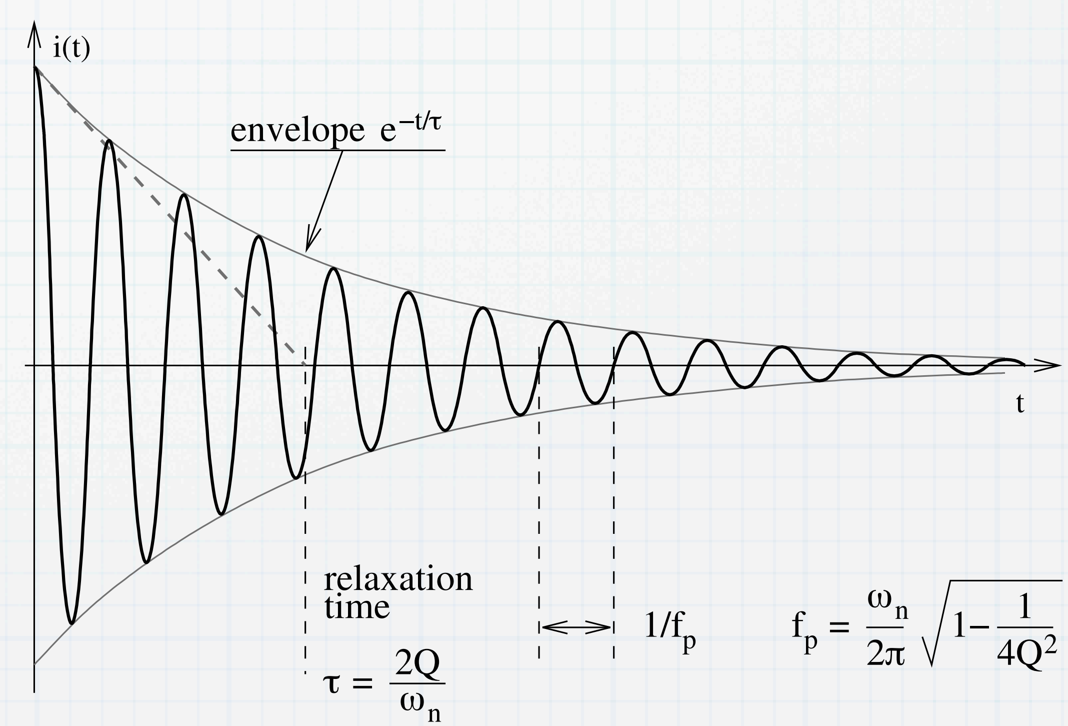
$$\tau = \frac{2Q}{\omega_n}$$

ω_p free-decay pseudofrequency

$$\omega_p = \omega_n\sqrt{1 - 1/4Q^2}$$

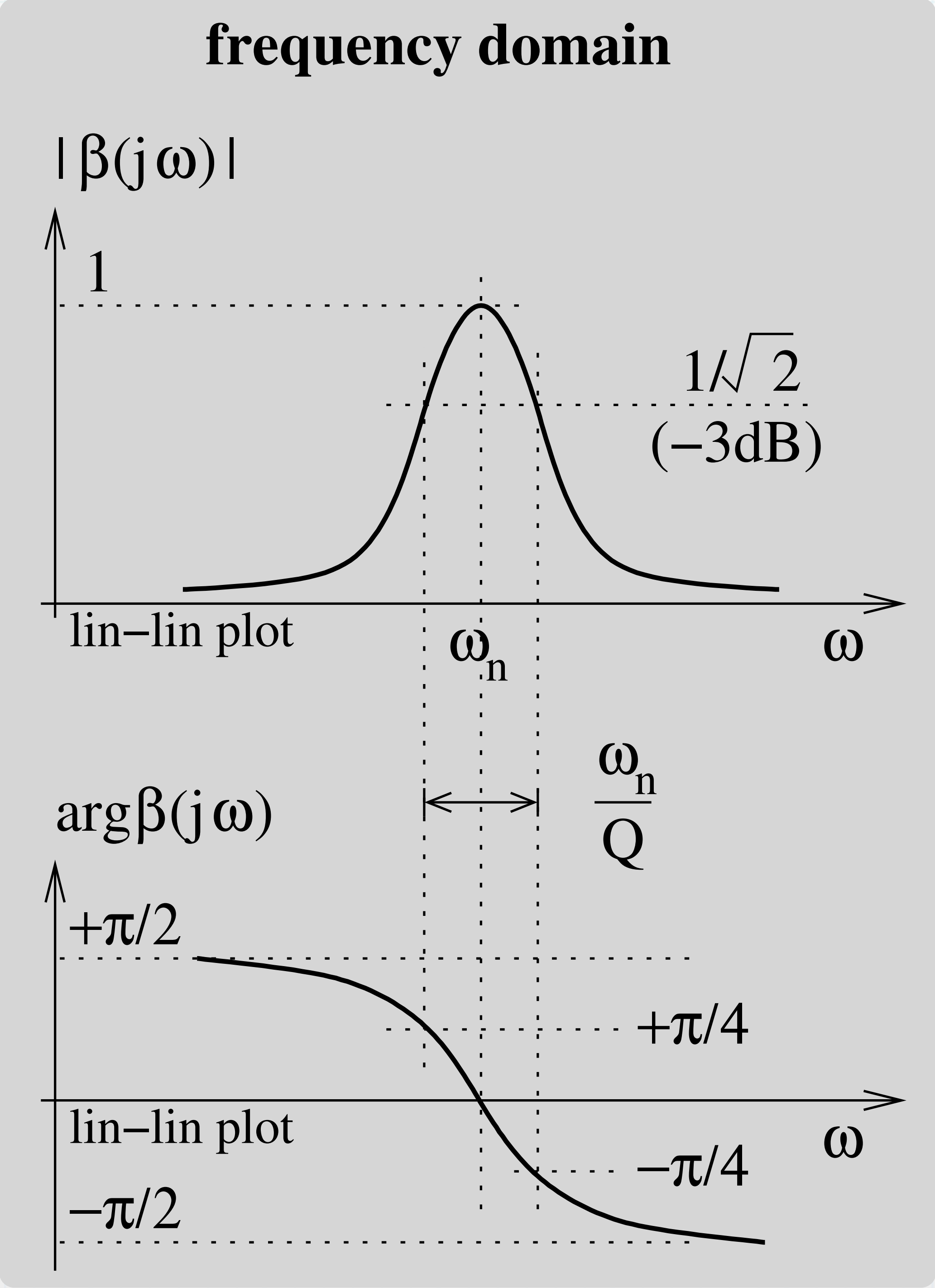
shorthand: f = ω/2π

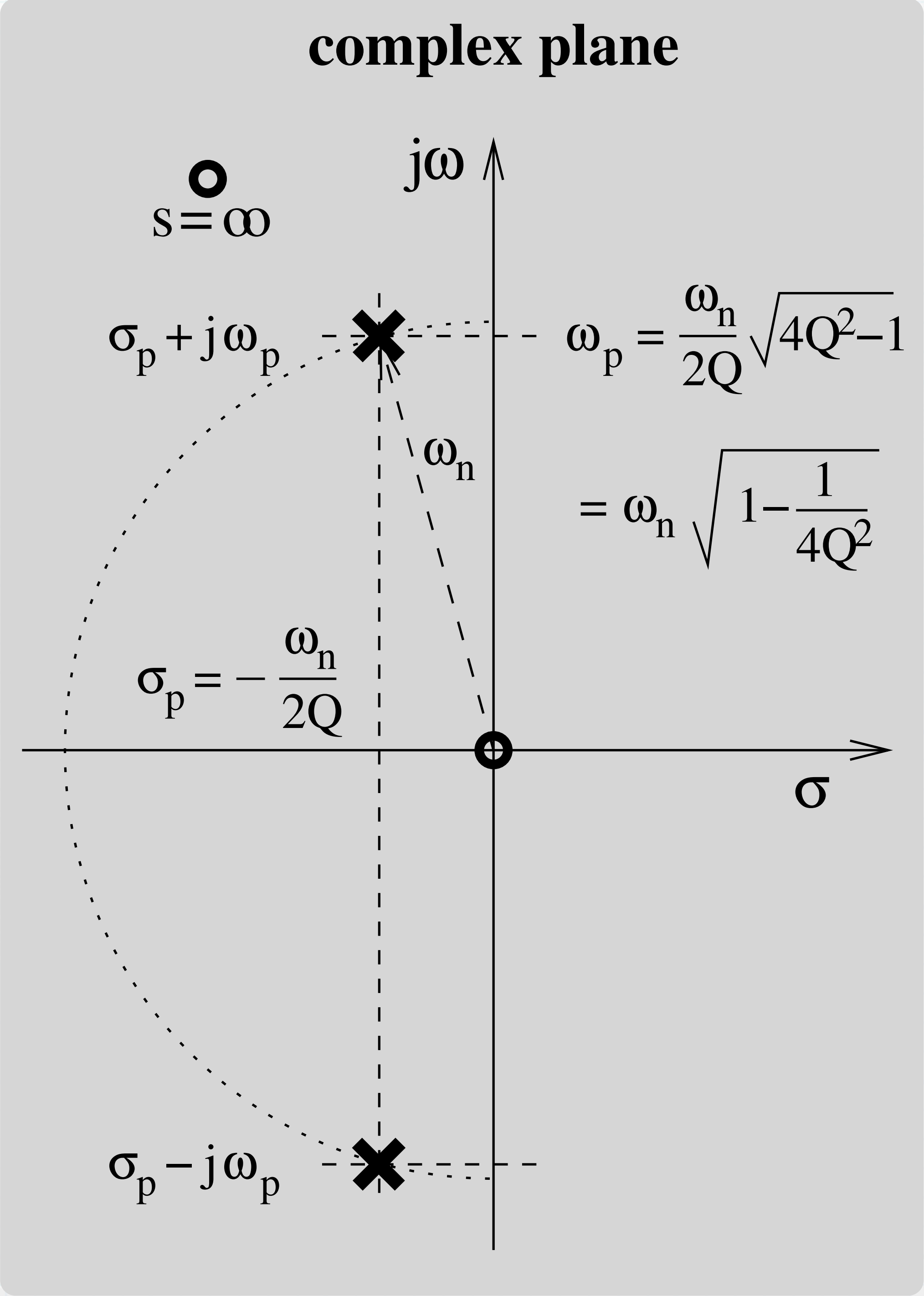




Resonator – frequency domain

$$\beta(s) = \frac{\omega_n}{Q} \frac{s}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2} \qquad \mathbf{s = \sigma + j\omega}$$





RESONATOR

B4

$$H(s) = \frac{V_o(s)}{V_i(s)}$$

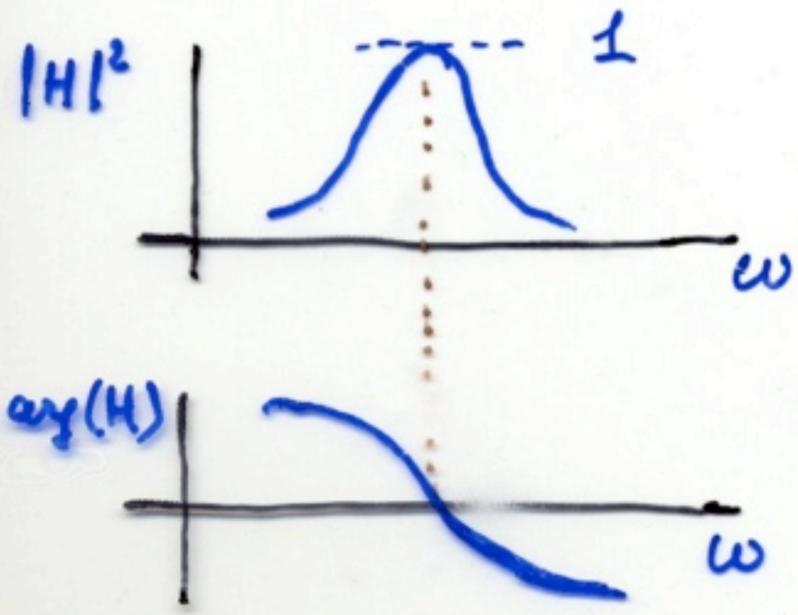
$s = \sigma + j\omega$

$$H(s) = \frac{\omega_0}{Q} \frac{s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

normalization for $H_{max} = 1$

define

$$x = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \xrightarrow{\omega \to \omega_0} 2\frac{\omega - \omega_0}{\omega}$$



$$H(j\omega) = \frac{1}{1 + jQx} = \frac{1 - jQx}{1 + Q^2x^2}$$

Real, Imag

$$R(\omega) = \frac{1}{1+Q^2x^2}$$

$$I(\omega) = \frac{-Qx}{1+Q^2x^2}$$

modulus, phase

$$M(\omega) = \frac{1}{\sqrt{1+Q^2x^2}}$$

$$\Phi(\omega) = -\arctan Qx$$

$$\chi = \frac{\omega}{\omega_n} - \frac{\omega_n}{\omega}$$

$$\beta = \frac{1}{1 + jQ\chi}$$

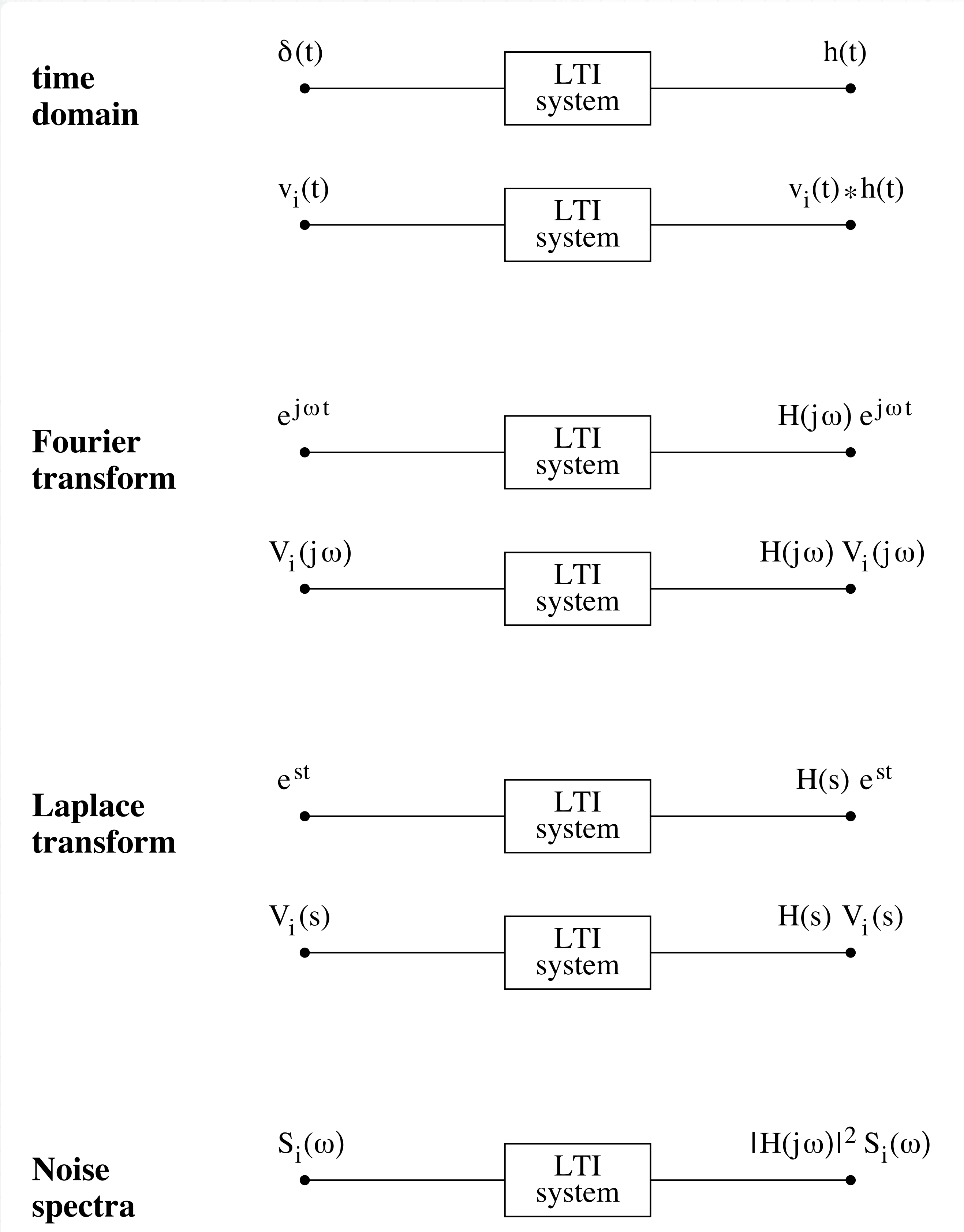
$$\Re\{\beta\} = \frac{1}{1 + Q^2\chi^2}$$

$$\Im\{\beta\} = \frac{-Q\chi}{1 + Q^2\chi^2}$$

$$|\beta|^2 = \frac{1}{1 + Q^2\chi^2}$$

$$\arg(\beta) = -\arctan(Q\chi)$$

Linear time-invariant (LTI) systems



impulse response

response to the generic signal $v_i(t)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

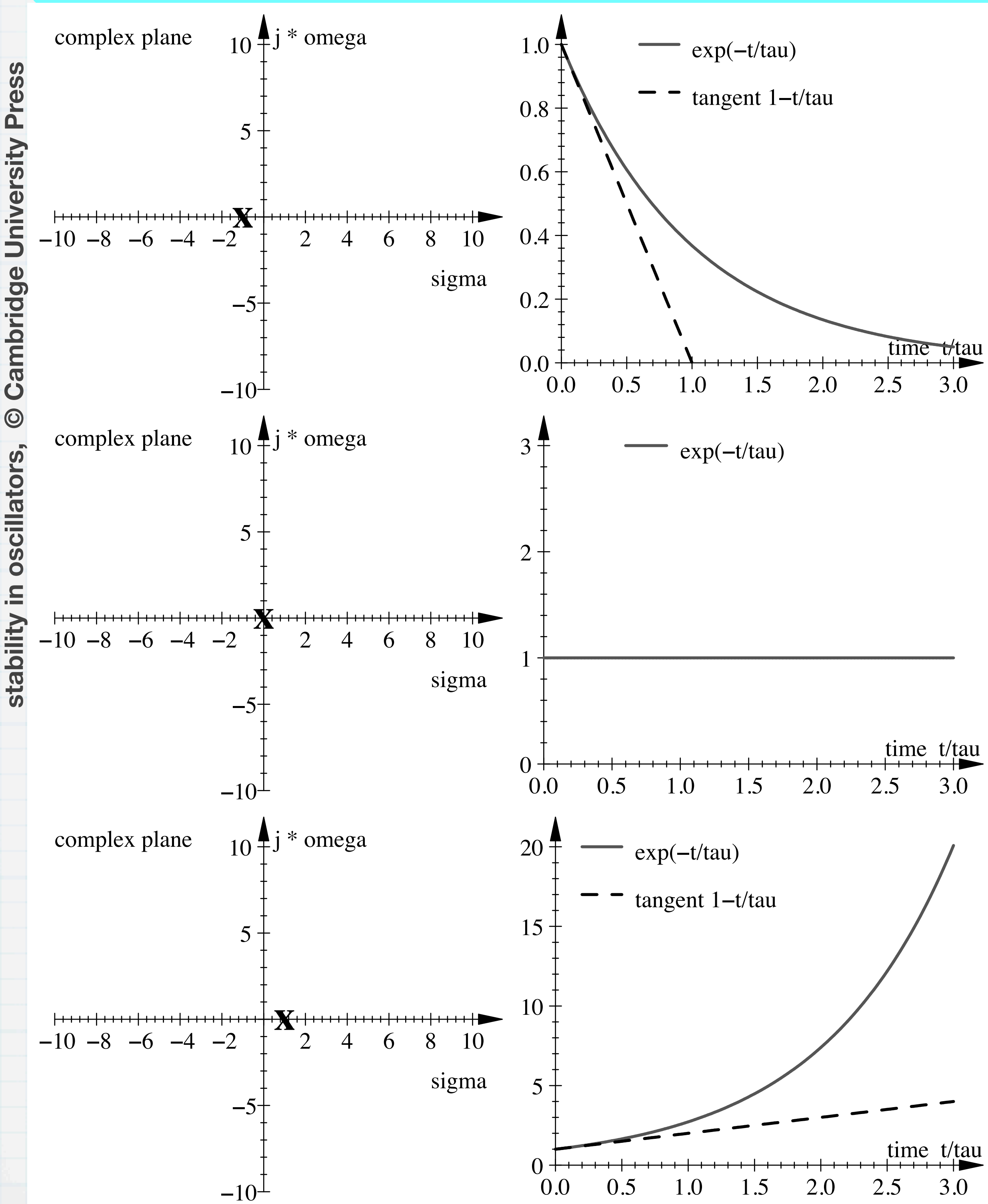
$$H(s) = \int_{0}^{\infty} h(t)\, e^{-st}\, dt$$

H(s), s=σ+jω, is the analytic continuation of H(ω) for causal system, where h(t)=0 for t<0

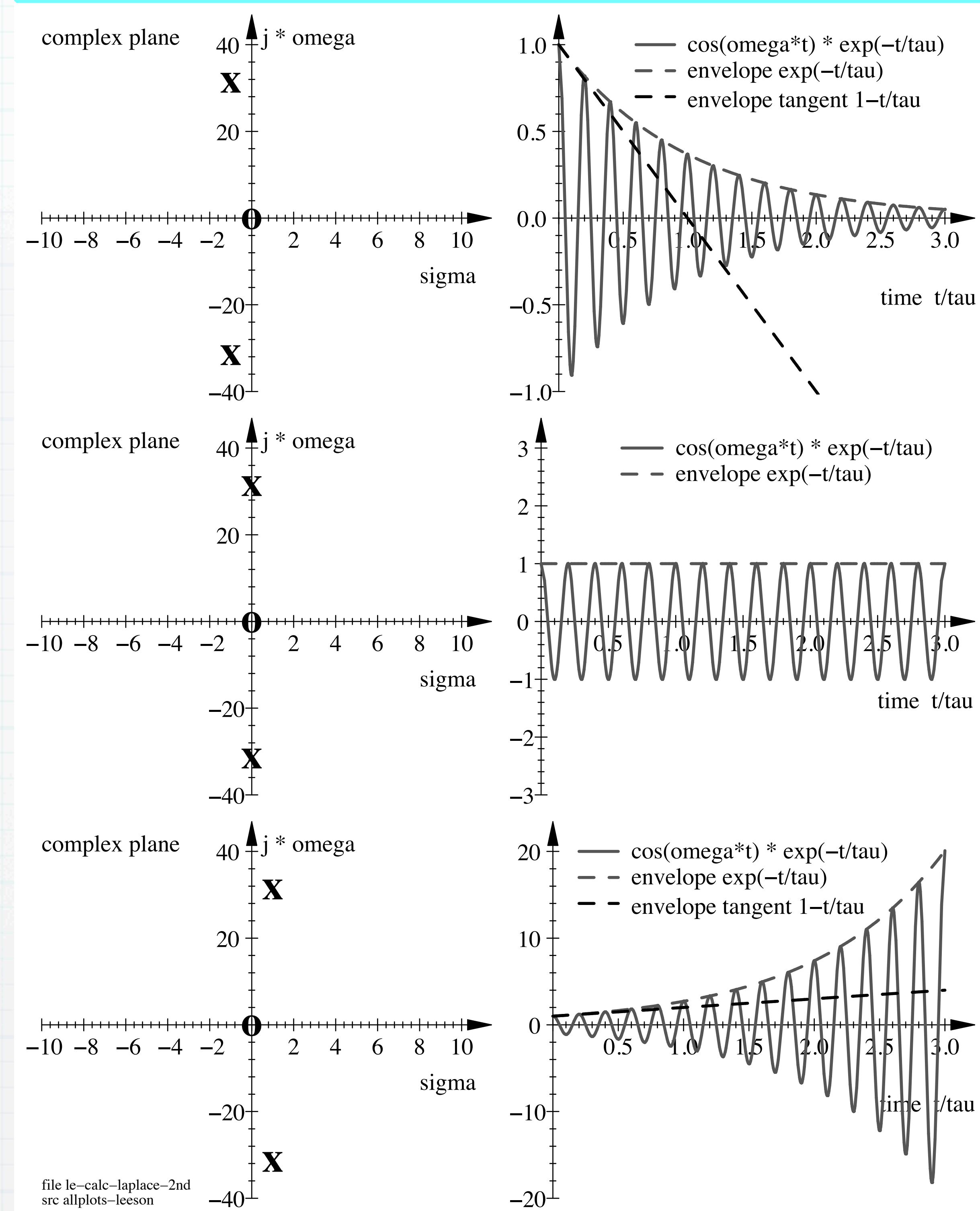
Laplace-transform patterns

Fundamental theorem of complex algebra: F(s) is completely determined by its roots

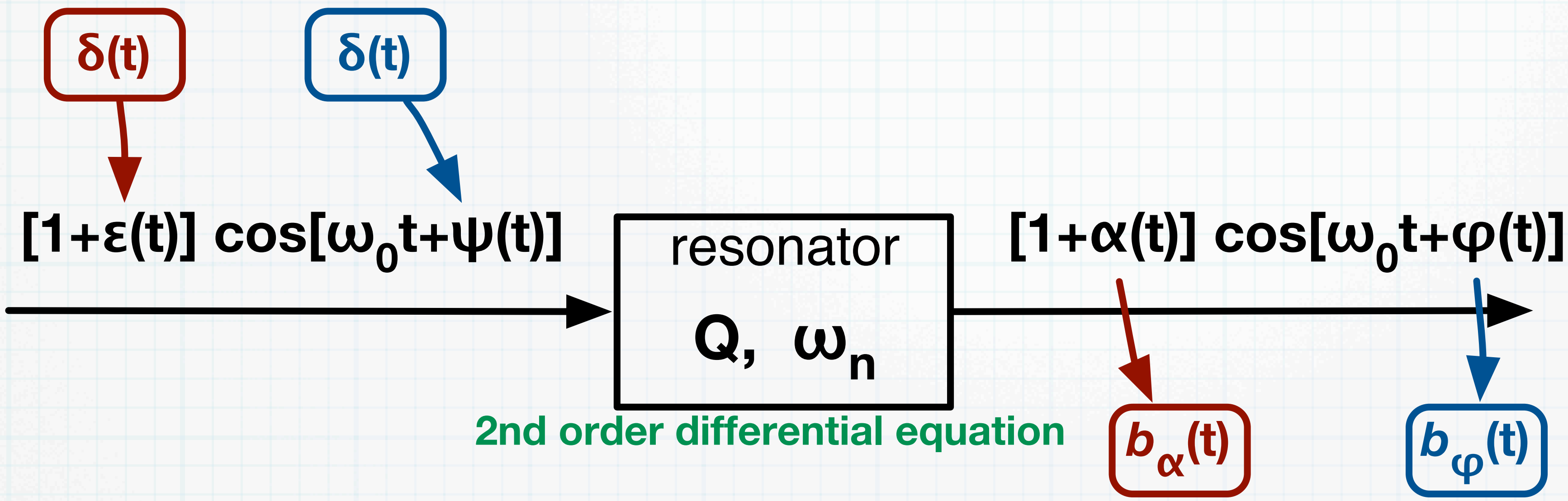
$$F(s) = \frac{1}{s + 1/\tau}$$



$$F(s) = \frac{s}{s^2 + 2s/\tau + \omega_n^2}$$

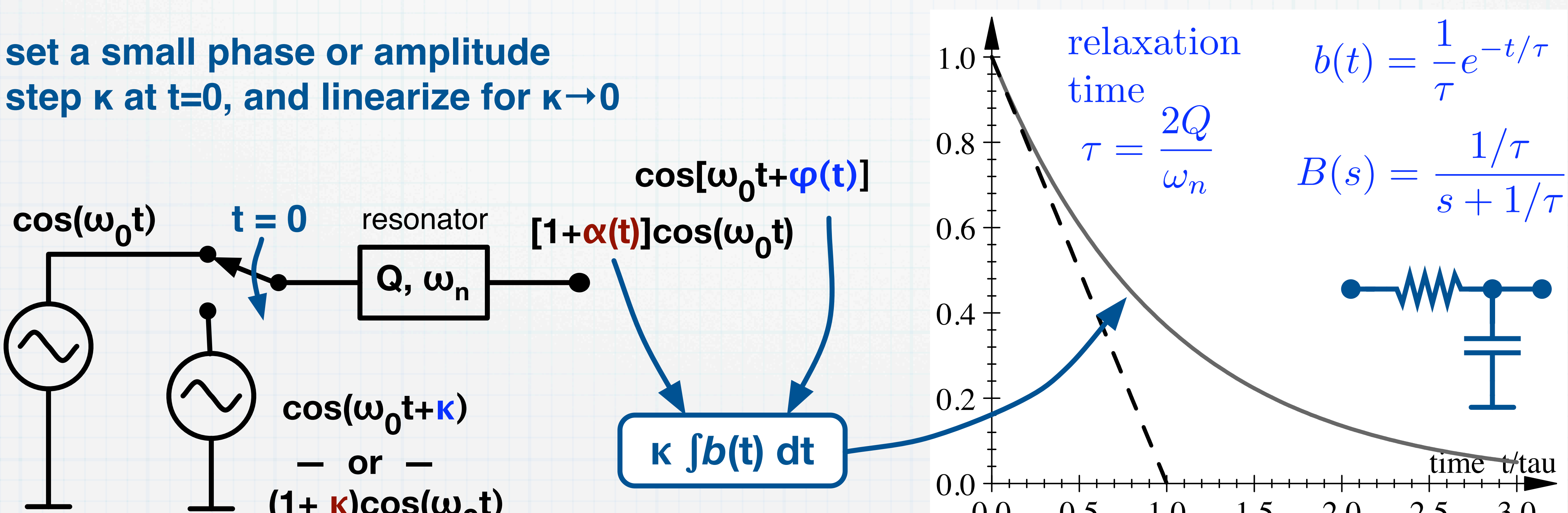


Resonator impulse response



Can't figure out a δ(t) of phase or amplitude? Use Heaviside (step) u(t) and differentiate

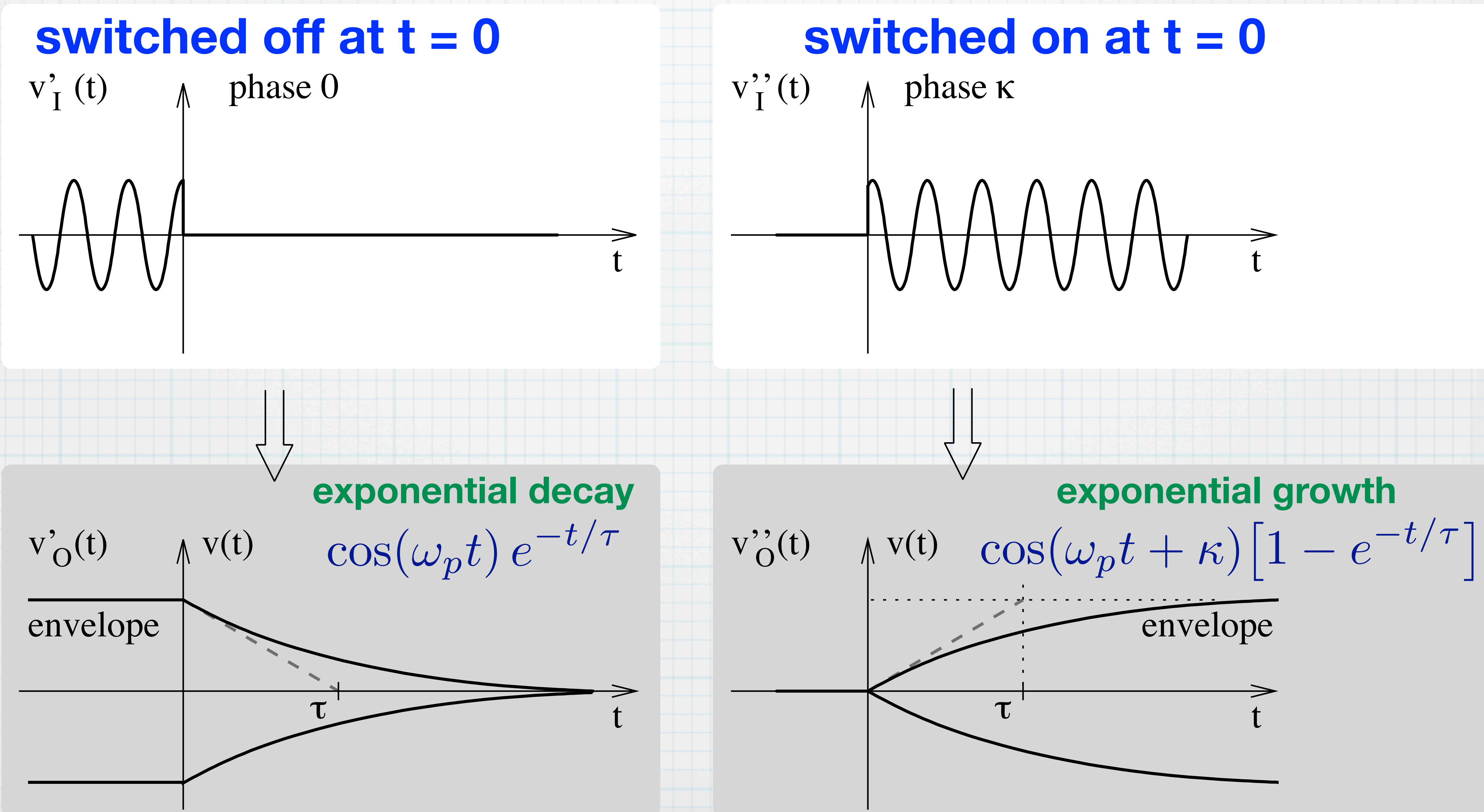
set a small phase or amplitude step κ at t=0, and linearize for κ→0



file: ele-resonator-delta-response

Response to a phase step κ

A phase step is equivalent to switching a sinusoid off at t = 0, and switching a shifted sinusoid on at t=0



Resonator impulse response ($\omega_0=\omega_n$)

$$v_i(t) = \underbrace{\cos(\omega_0 t)\,\mathfrak{u}(-t)}_{\text{switched off at } t=0} + \underbrace{\cos(\omega_0 t+\kappa)\,\mathfrak{u}(t)}_{\text{switched on at } t=0}$$

phase step κ at t=0

$$v_o(t) = \cos(\omega_p t)\, e^{-t/\tau} + \cos(\omega_p t+\kappa)\left[1-e^{-t/\tau}\right] \qquad t>0$$

output

$$v_o(t) = \cos(\omega_p t) - \kappa \sin(\omega_p t)\left[1-e^{-t/\tau}\right] \qquad \kappa \to 0$$

linearize

$$v_o(t) = \cos(\omega_0 t) - \kappa \sin(\omega_0 t)\left[1-e^{-t/\tau}\right] \qquad \omega_p \to \omega_0$$

high Q

$$\mathbf{V_o}(t) = \frac{1}{\sqrt{2}}\left\{1 + j\kappa\left[1-e^{-t/\tau}\right]\right\}$$

slow-varying phase vector

$$\arctan\left(\frac{\Im\{\mathbf{V_o}(t)\}}{\Re\{\mathbf{V_o}(t)\}}\right) \simeq \kappa\left[1-e^{-t/\tau}\right]$$

phasor angle

delete κ and differentiate

impulse response

$$\mathrm{b}(t) = \frac{1}{\tau} e^{-s\tau} \qquad \leftrightarrow \qquad \mathrm{B}(s) = \frac{1/\tau}{s+1/\tau}$$

Detuned resonator ($\omega_0 \neq \omega_n$)

amplitude
phase

$$\begin{bmatrix}\alpha\\ \varphi\end{bmatrix} = \begin{bmatrix}\mathrm{b}_{\alpha\alpha} & \mathrm{b}_{\alpha\varphi}\\ \mathrm{b}_{\varphi\alpha} & \mathrm{b}_{\varphi\varphi}\end{bmatrix} * \begin{bmatrix}\varepsilon\\ \psi\end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix}\mathcal{A}\\ \Phi\end{bmatrix} = \begin{bmatrix}\mathrm{B}_{\alpha\alpha} & \mathrm{B}_{\alpha\varphi}\\ \mathrm{B}_{\varphi\alpha} & \mathrm{B}_{\varphi\varphi}\end{bmatrix} \begin{bmatrix}\mathcal{E}\\ \Psi\end{bmatrix}$$

$$\Omega = \omega_0 - \omega_n$$ **detuning**

$$\beta_0 = |\beta(j\omega_0)|$$ **mudulus**

$$\theta = \arg(\beta(j\omega_0))$$ **phase**

$$v_i(t) = \underbrace{\frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t)}_{\text{switched off at } t=0} + \underbrace{\frac{1}{\beta_0}\cos(\omega_0 t - \theta + \kappa)\,\mathfrak{u}(t)}_{\text{switched on at } t=0}$$

phase step κ at t=0

$$= \frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t) + \frac{1}{\beta_0}\Big[\cos(\omega_0 t - \theta)\cos\kappa - \sin(\omega_0 t - \theta)\sin\kappa\Big]\,\mathfrak{u}(t)$$

$$\simeq \frac{1}{\beta_0}\cos(\omega_0 t - \theta)\,\mathfrak{u}(-t) + \frac{1}{\beta_0}\Big[\cos(\omega_0 t - \theta) - \kappa\sin(\omega_0 t - \theta)\Big]\,\mathfrak{u}(t) \qquad \kappa \ll 1.$$

Skip

Detuned resonator (cont.)

$$v_o(t) = \cos(\omega_0 t) - \kappa\sin(\omega_0 t) + \kappa\sin(\omega_n t)\, e^{-t/\tau}$$ **output, large Q (ω_p=ω_n)**

use $\Omega = \omega_0 - \omega_n$

$$v_o(t) = \cos(\omega_0 t)\left[1 - \kappa\sin(\Omega t)e^{-t/\tau}\right] - \kappa\sin(\omega_0 t)\left[1 - \cos(\Omega t)e^{-t/\tau}\right]$$

slow-varying phase vector

$$\mathbf{V_o}(t) = \frac{1}{\sqrt{2}}\left\{1 - \kappa\sin(\Omega t)e^{-t/\tau} + j\kappa\left[1 - \cos(\Omega t)e^{-t/\tau}\right]\right\} \qquad \kappa \ll 1$$

$$\arctan\frac{\Im\{\mathbf{V_o}(t)\}}{\Re\{\mathbf{V_o}(t)\}} = \kappa\left[1 - \cos(\Omega t)e^{-t/\tau}\right]$$ **angle**

$$|\mathbf{V_o}(t)| = |\mathbf{V_o}(0)| - \kappa\sin(\Omega t)e^{-t/\tau}$$ **amplitude**

delete κ and differentiate

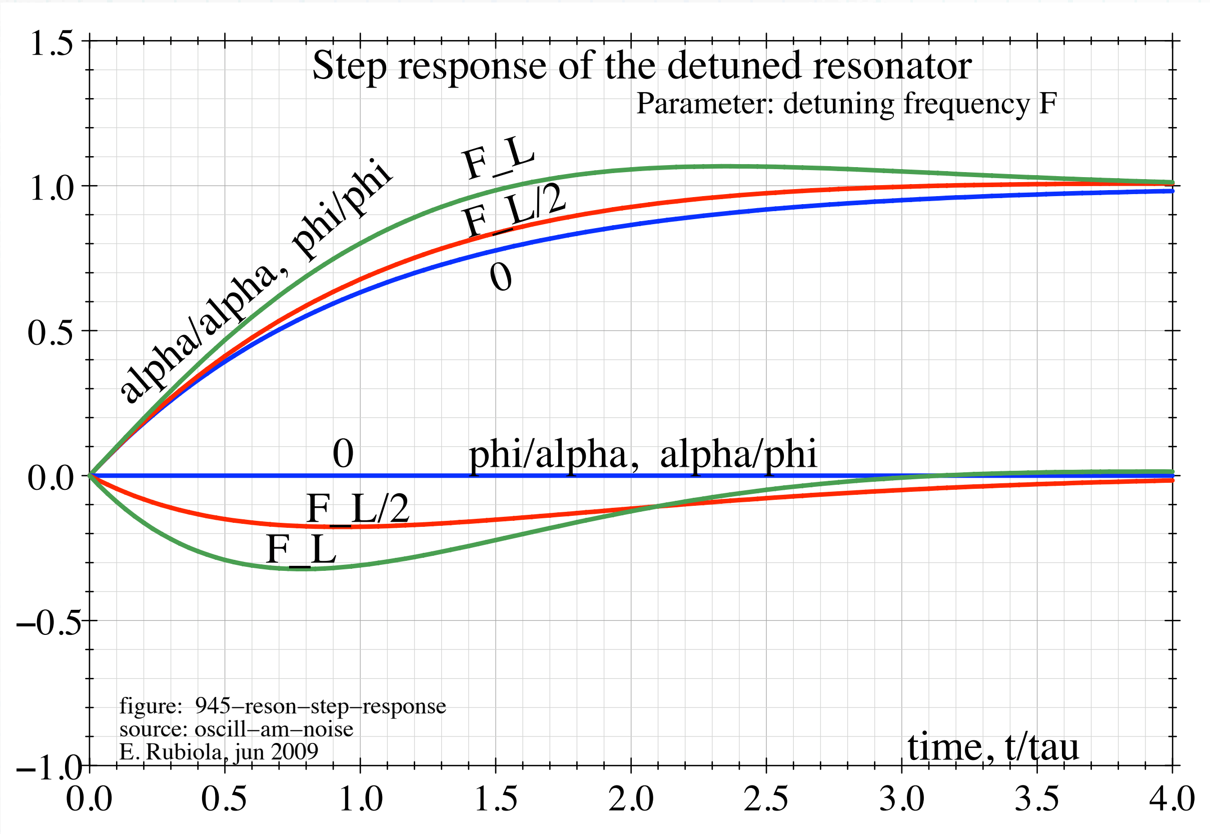
impulse response

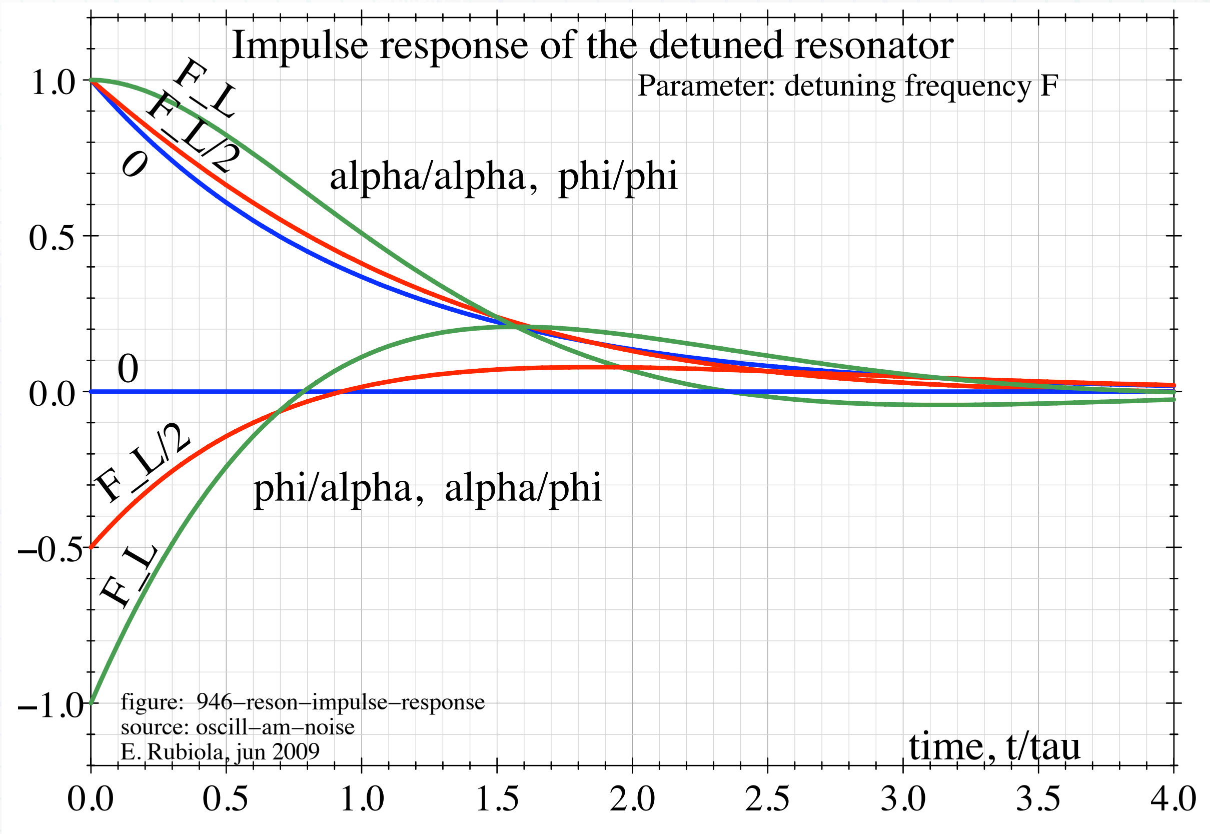
$$\mathrm{b}_{\varphi\varphi}(t) = \left[\Omega\sin(\Omega t) + \frac{1}{\tau}\cos(\Omega t)\right]e^{-t/\tau} \qquad \text{phase}$$

$$\mathrm{b}_{\alpha\varphi}(t) = \left[-\Omega\cos(\Omega t) + \frac{1}{\tau}\sin(\Omega t)\right]e^{-t/\tau} \qquad \text{amplitude}$$

Skip

Resonator step and impulse response



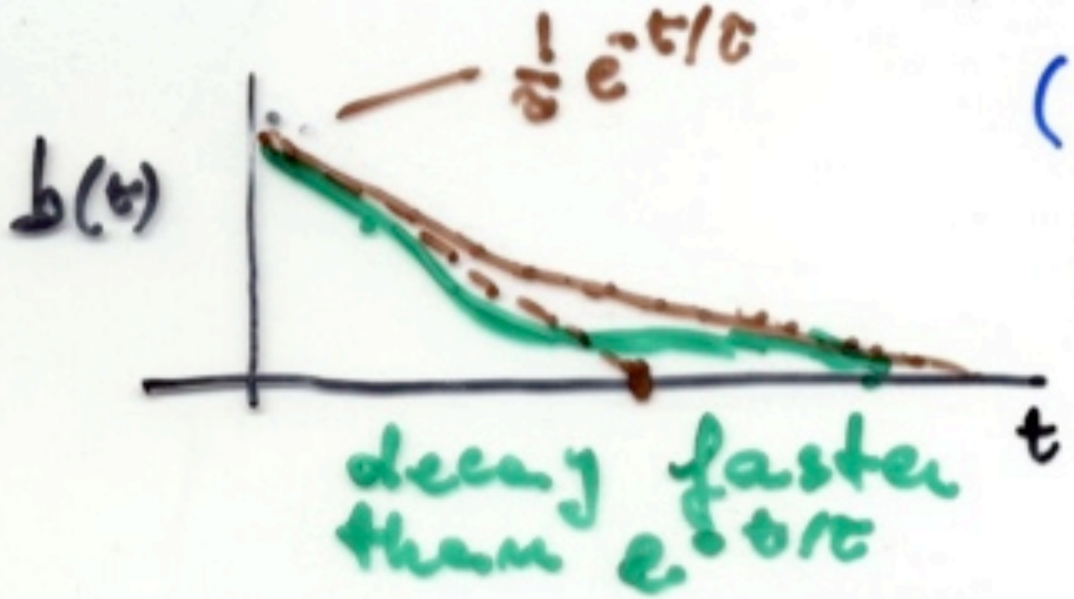


$$[\mathrm{b}](t) = \begin{bmatrix} \left(\Omega \sin \Omega t + \frac{1}{\tau} \cos \Omega t\right) e^{-t/\tau} & \left(-\Omega \cos \Omega t + \frac{1}{\tau} \sin \Omega t\right) e^{-t/\tau} \\ \left(-\Omega \cos \Omega t + \frac{1}{\tau} \sin \Omega t\right) e^{-t/\tau} & \left(\Omega \sin \Omega t + \frac{1}{\tau} \cos \Omega t\right) e^{-t/\tau} \end{bmatrix}$$

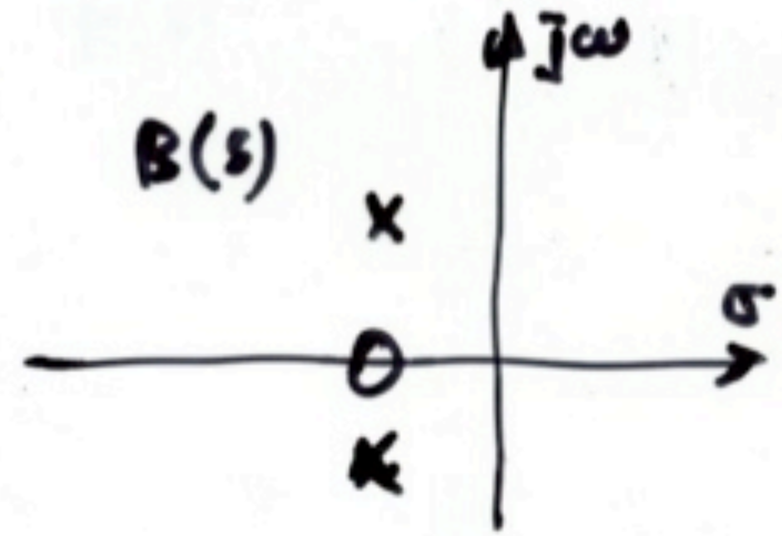
check on the sign of b_{21}

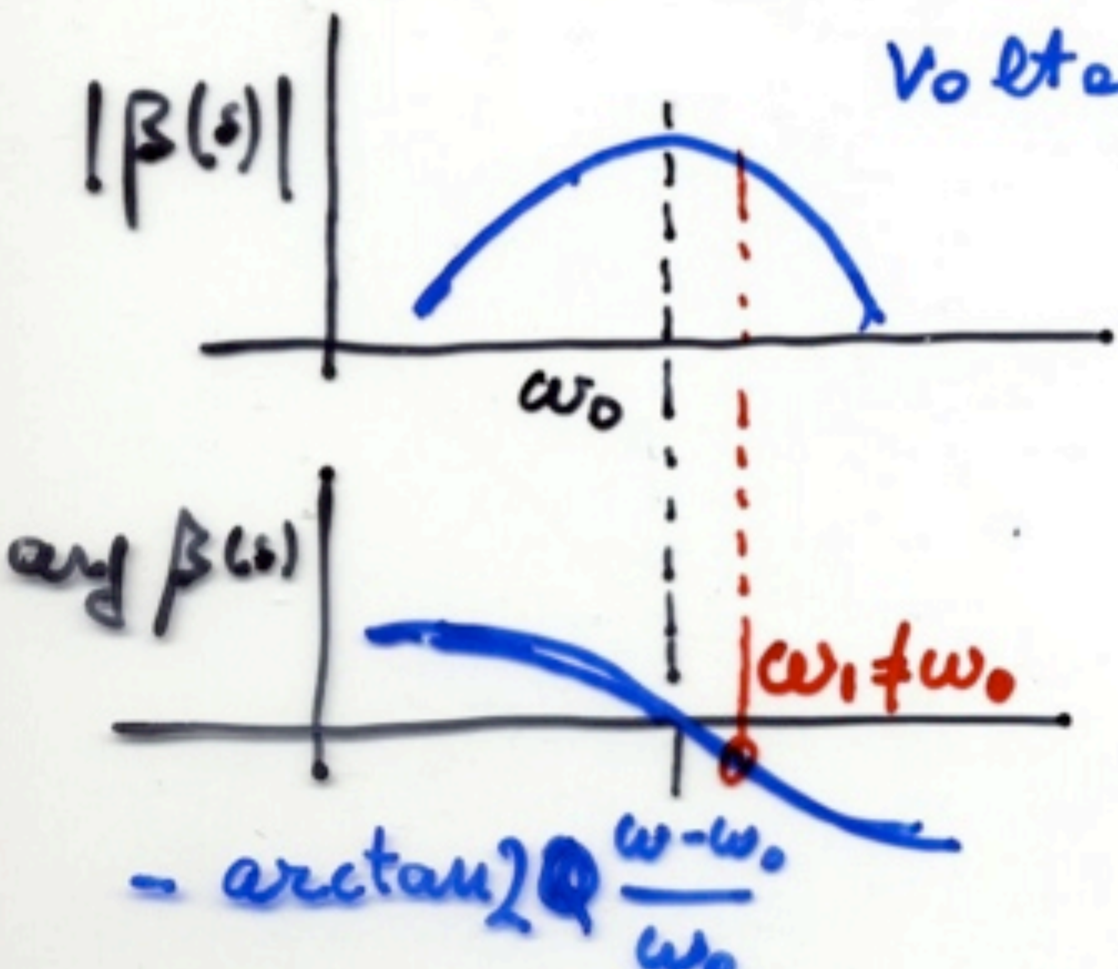
DETUNED RESONATOR

C4



(CONTROLS)
people use
2nd order
systems!



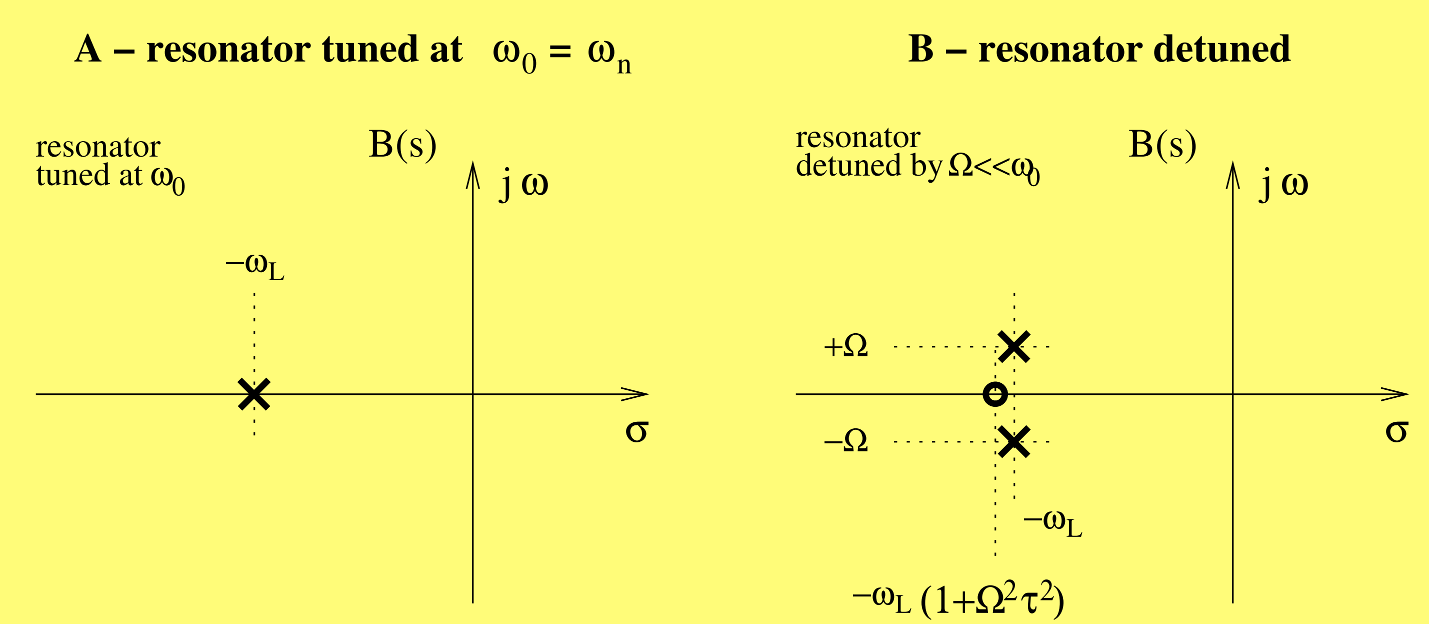


Voltage transfer function

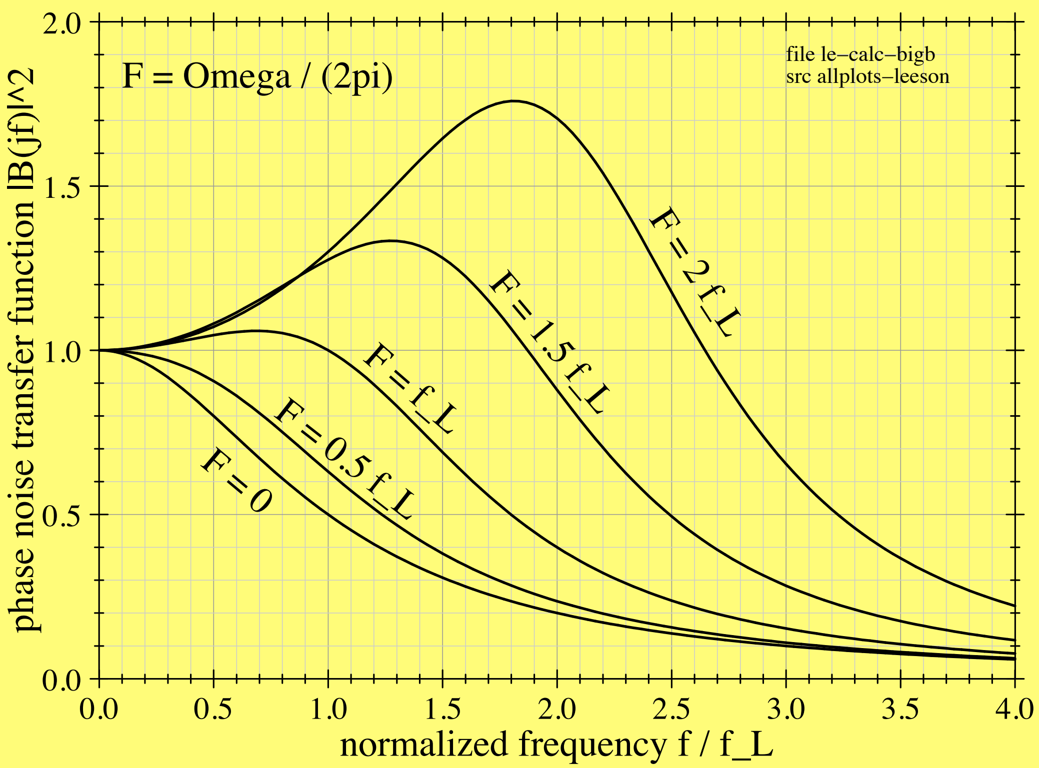
$\arg(\beta)$ has a lower slope at $\omega_1 \neq \omega_0$.

The local behavior (ω_1) is that of a lower-Q resonator.

Frequency response



diagonal terms



diagonal terms

$$
[\mathrm{B}](s) = \begin{bmatrix} \dfrac{1}{\tau}\;\dfrac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \dfrac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \\ \dfrac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \dfrac{1}{\tau}\;\dfrac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \end{bmatrix}
$$

check on the sign of B_{21}

Skip

Frequency response

Roots & complex plane

off-diagonal terms

Frequency response

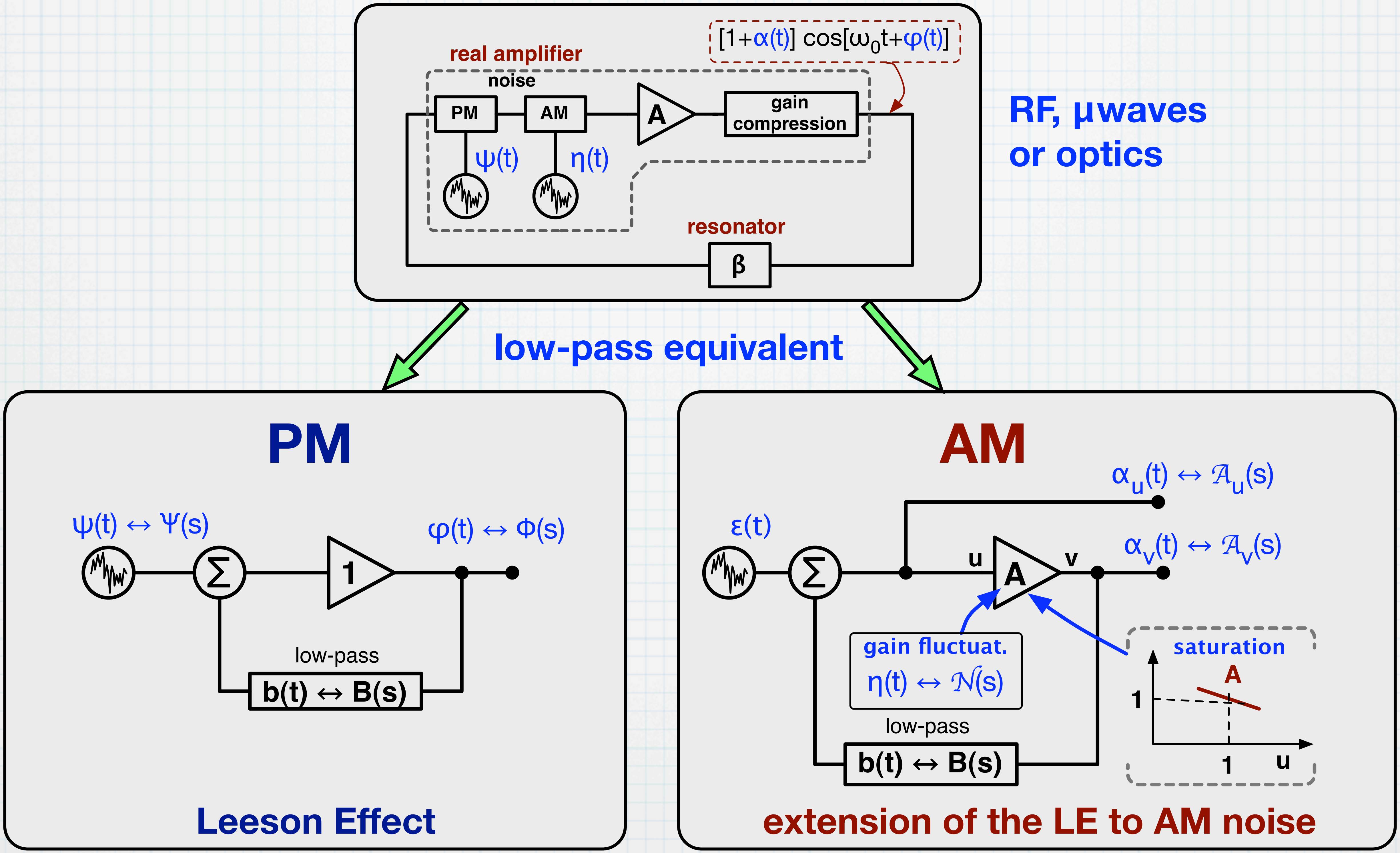
off-diagonal terms

$$[\mathrm{B}](s) = \begin{bmatrix} \frac{1}{\tau}\,\frac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \frac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \\ \frac{-\Omega s}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} & \frac{1}{\tau}\,\frac{s+\frac{1}{\tau}+\Omega^2\tau}{s^2+\frac{2}{\tau}s+\frac{1}{\tau^2}+\Omega^2} \end{bmatrix}$$

check on the sign of B_{21}

Formal proof for the Leeson effect

Low-pass representation of AM-PM noise

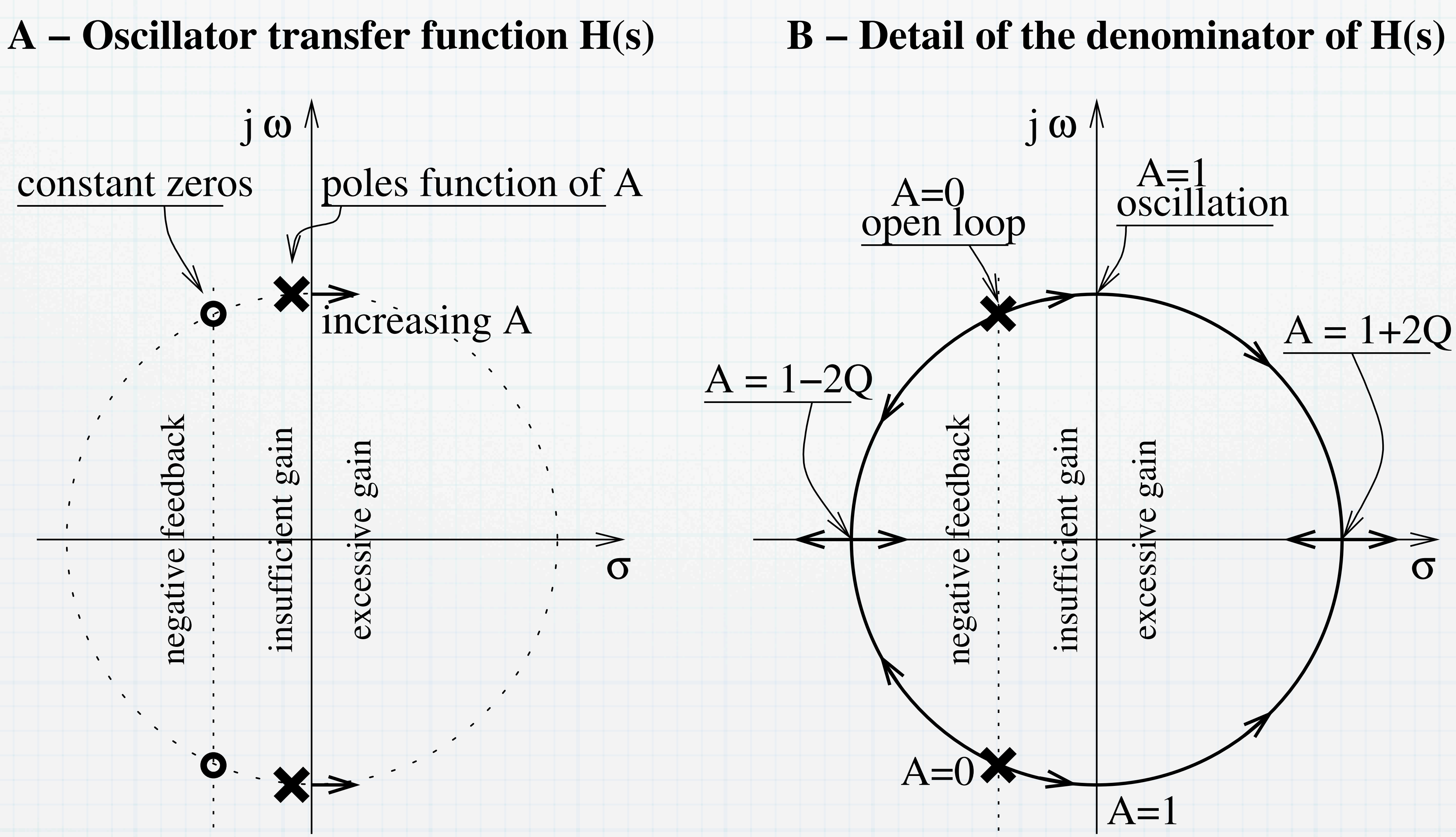


The amplifier
– “copies” the input phase to the out
– adds phase noise

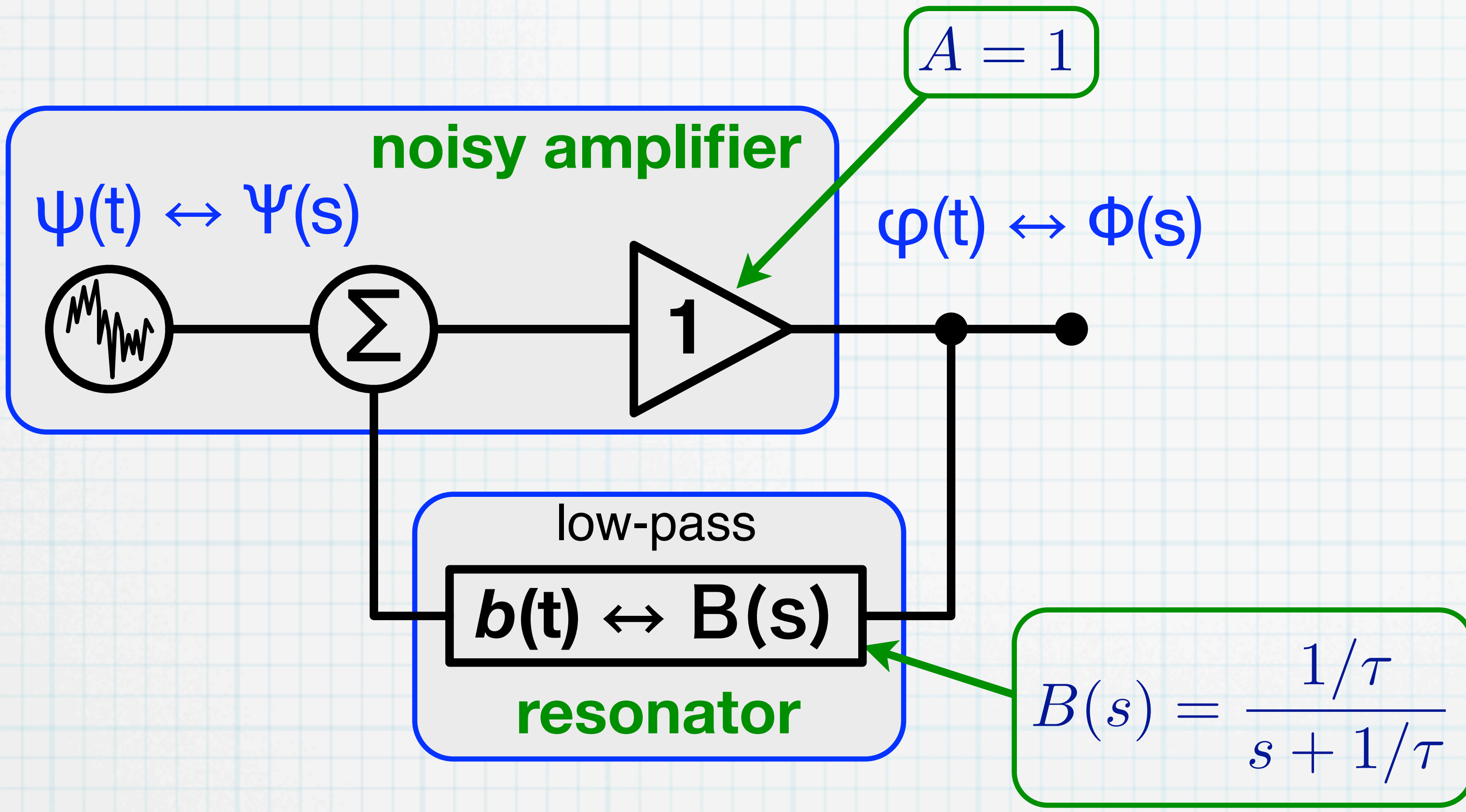
The amplifier
– compresses the amplitude
– adds amplitude noise

Effect of feedback

Oscillator transfer function (RF)



Leeson effect

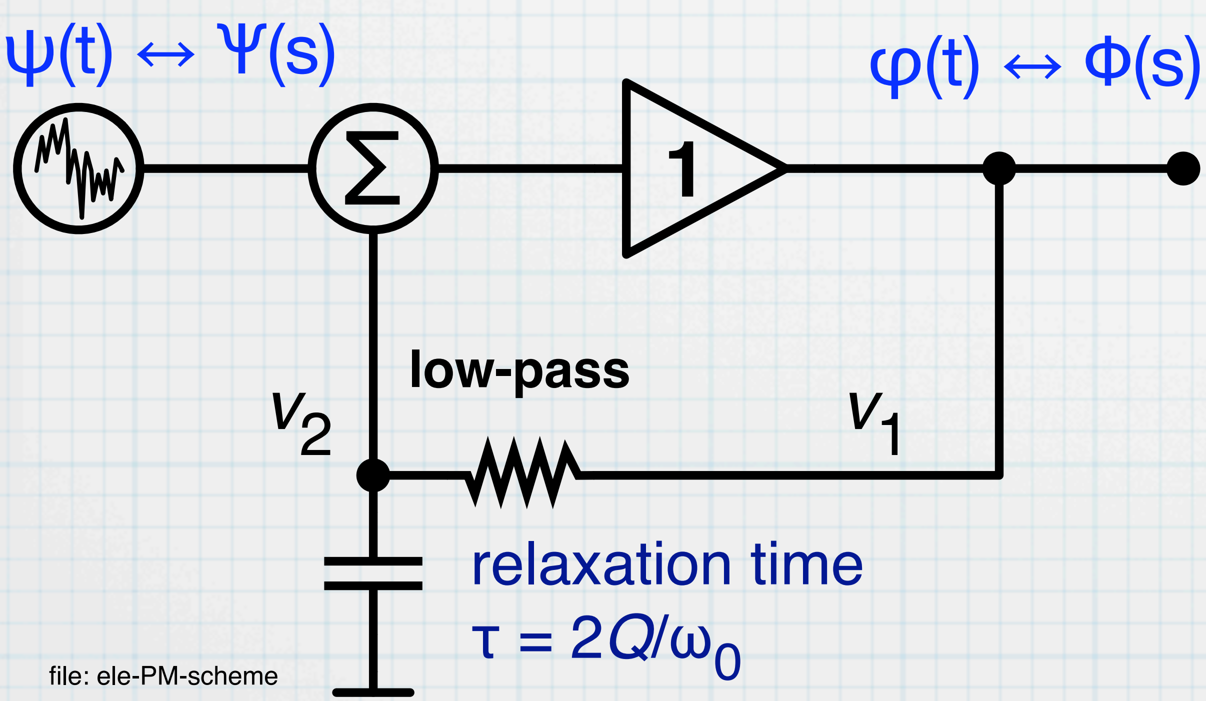


phase-noise transfer function

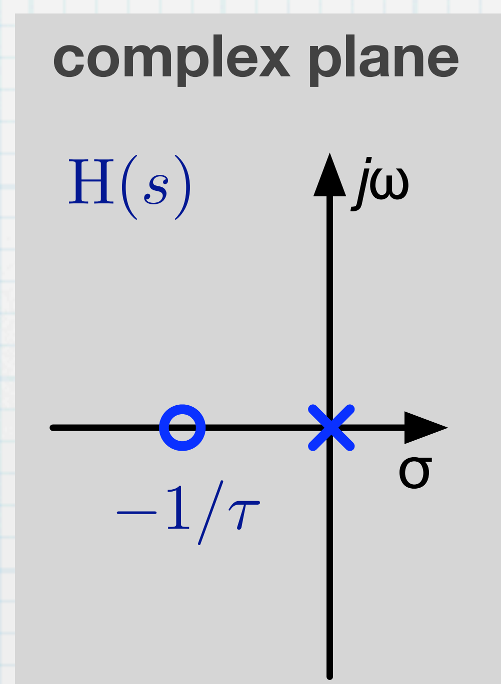
$$\mathrm{H}(s) = \frac{\Phi(s)}{\Psi(s)}$$ **definition**

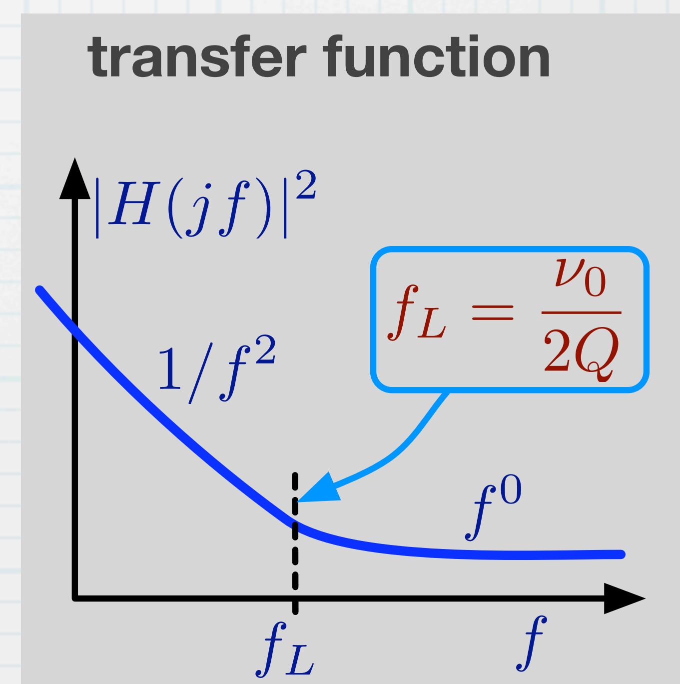
$$\mathrm{H}(s) = \frac{1}{1 + AB(s)}$$ **general feedback theory**

$$\mathrm{H}(s) = \frac{1 + s\tau}{s\tau}$$ **Leeson effect**

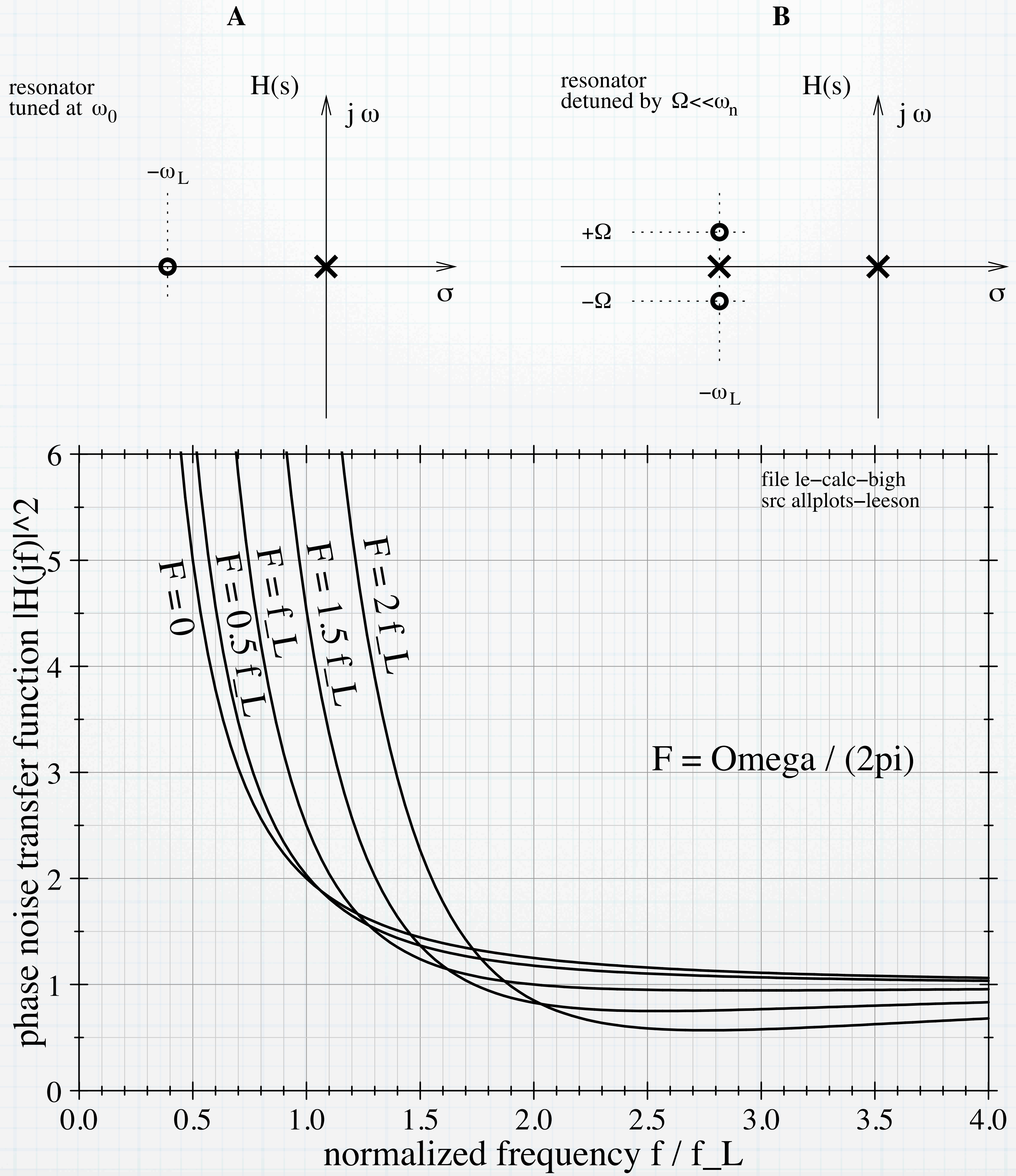


file: ele-PM-scheme



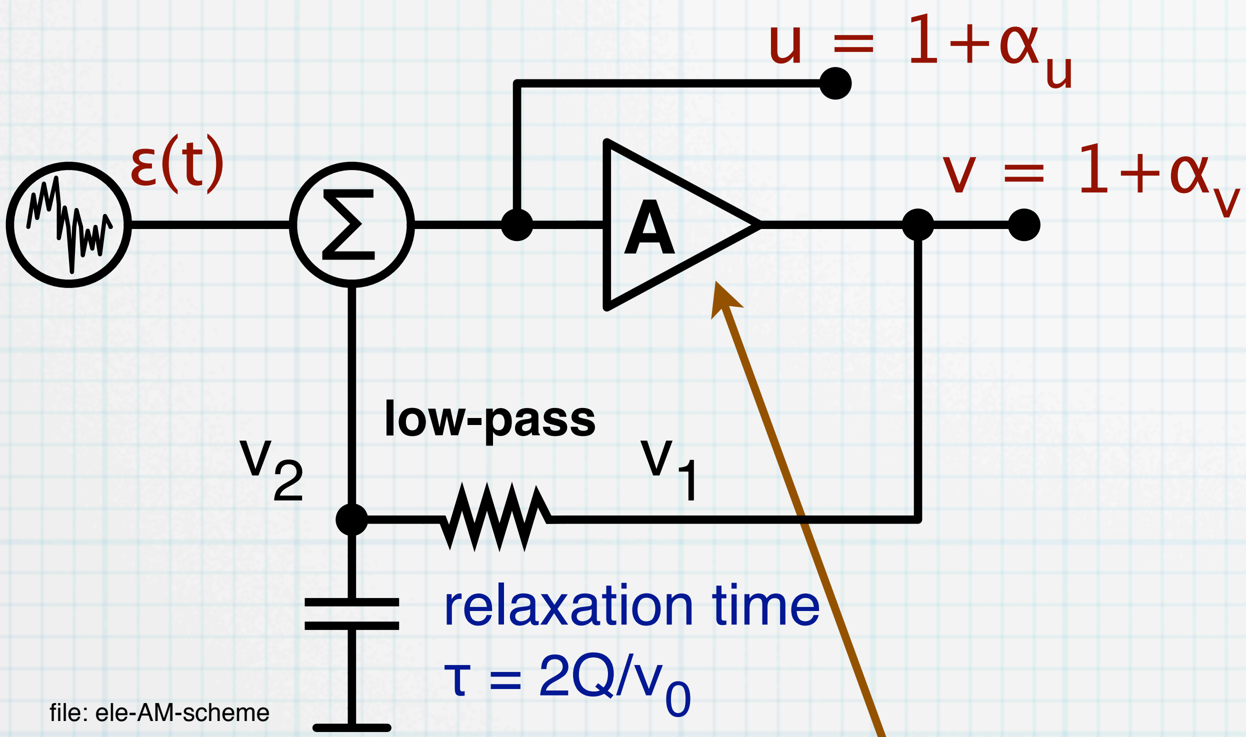


Oscillator with detuned resonator



Low-pass model of amplitude (1)

First we need to relate the system restoring time τ_r to the relaxation time τ



simple feedback theory

$$u = \epsilon + v_2$$

$$v_2 = \frac{1}{\tau}\int (v_1 - v_2)\,dt$$

$$v_2 = u - \epsilon$$

$$v_1 = v = Au$$

$$u = \epsilon + \frac{1}{\tau}\int (A-1)u + \epsilon\,dt$$

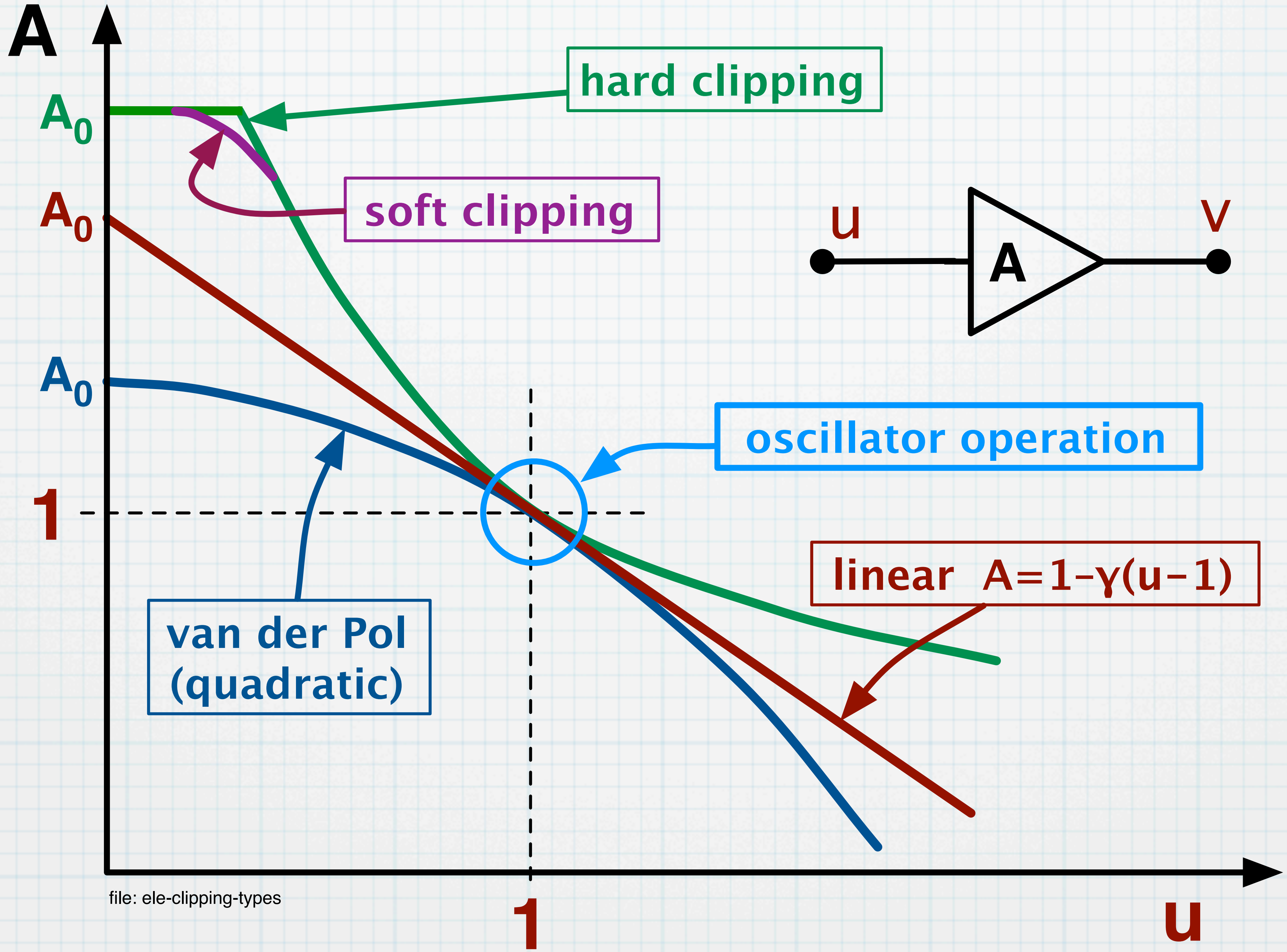
differential equation

$$\dot{u} - \frac{1}{\tau}\,(A-1)\,u = \frac{1}{\tau}\epsilon + \dot{\epsilon}$$

Gain compression is necessary for the oscillation amplitude to be stable

The Laplace / Heaviside formalism cannot be used because the amplifier is non-linear

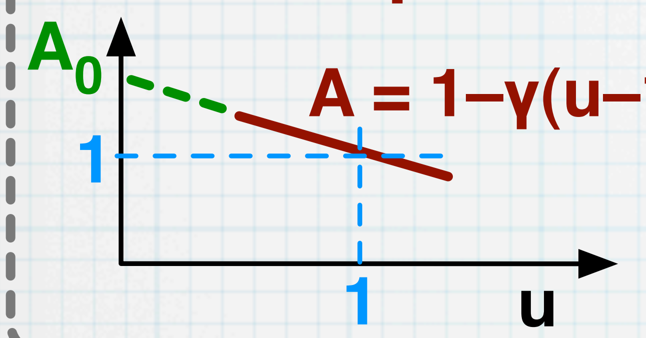
Common types of gain saturation



Gain compression is necessary for the oscillation amplitude to be stable

Low-pass model of amplitude (2)

Gain compression

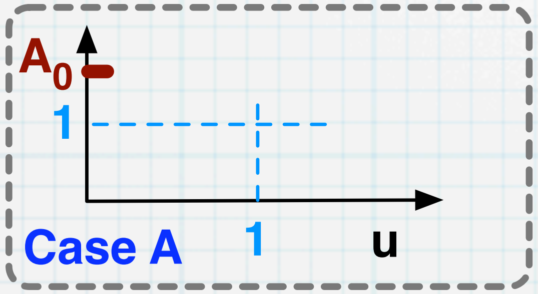


homogeneous differential equation

$$\dot{u} - \frac{1}{\tau}\,(A-1)\,u = 0$$

Three asymptotic cases

At low RF amplitude, let the gain be an arbitrary value denoted with A_0

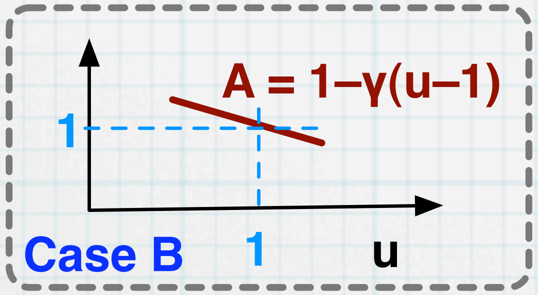


Startup: $u \to 0$, $A \to A_0 > 1$

$$\dot{u} - \frac{1}{\tau}\,(A_0-1)\,u = 0 \quad \Rightarrow \quad u = C_1 e^{(A_0-1)\,t/\tau}$$

rising exponential

For small fluctuation of the stationary RF amplitude, the gain amplitude varies linearly with V

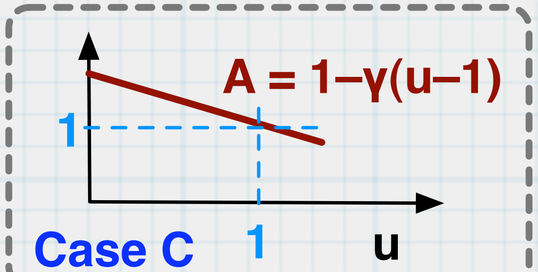


Regime: $u \to 1$, $A = 1 - \gamma\,(u-1)$

$$\dot{u} + \frac{\gamma}{\tau}\,(u-1)\,u = 0 \quad \Rightarrow \quad u = C_2 e^{-\gamma\,t/\tau}$$

restoring time constant $\tau_r = \tau/\gamma$

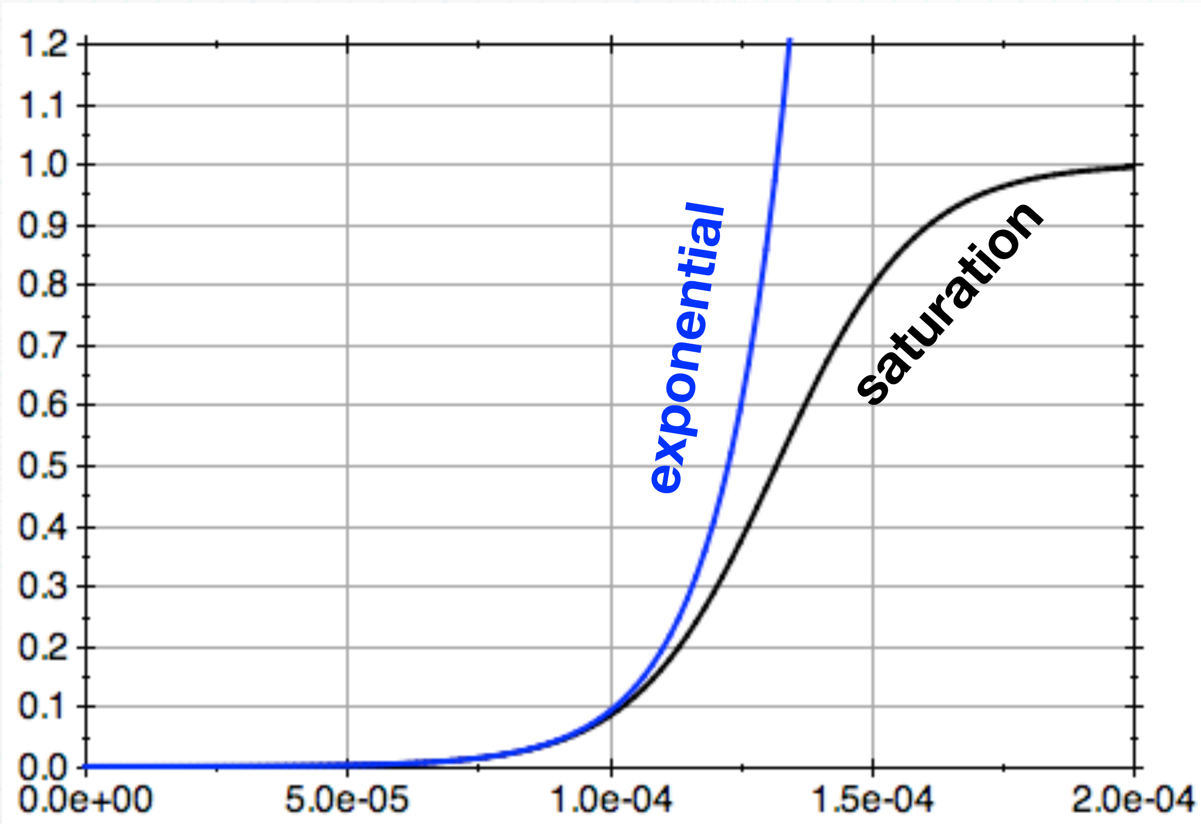
Simplification: the gain varies linearly with V in all the input range



Linear gain: $A = 1 - \gamma\,(u-1)$

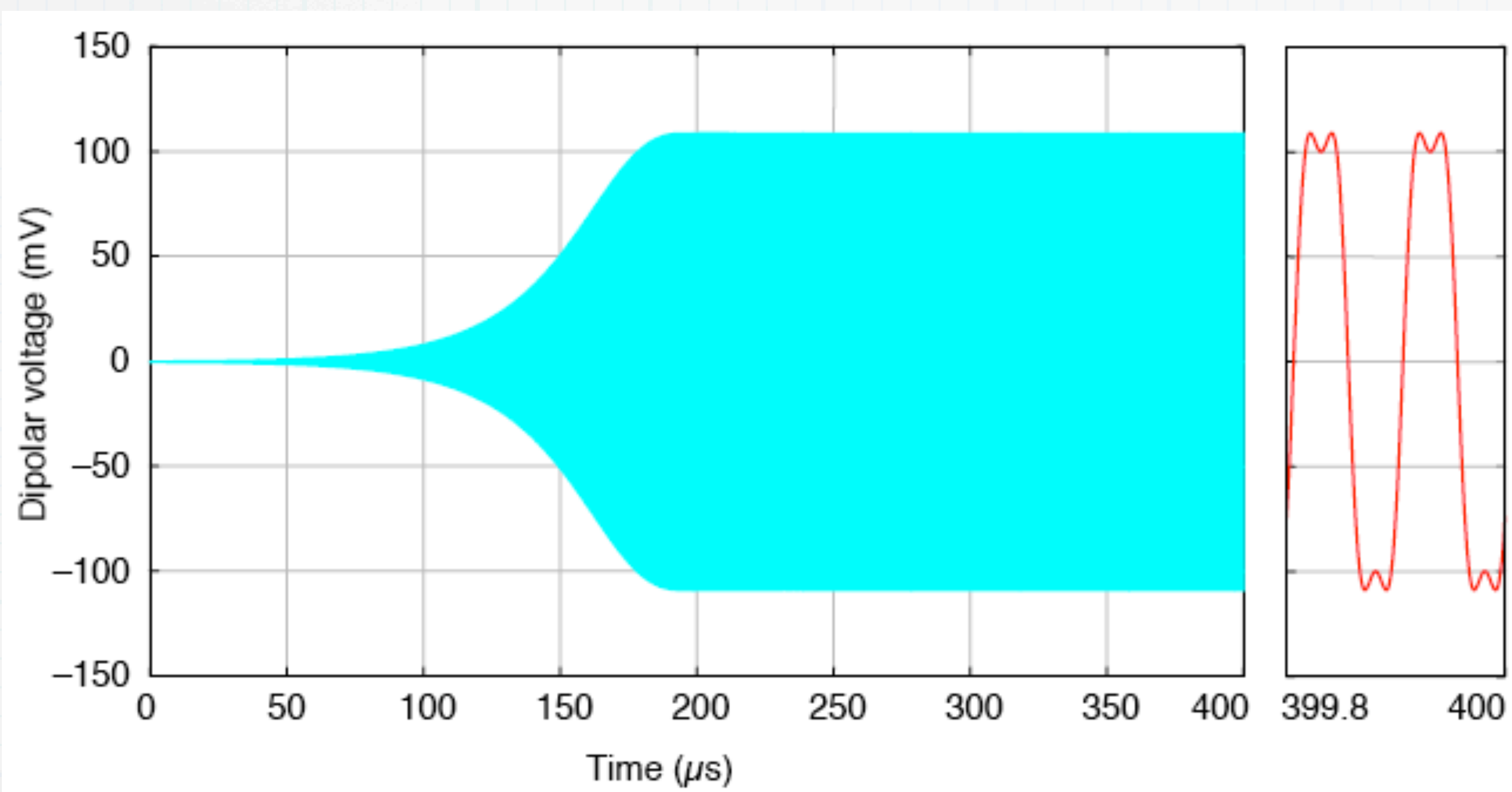
$$u = \frac{1}{\left(\frac{1}{u(0)} - 1\right) e^{-\gamma t/\tau} + 1}$$

Startup – analysis vs. simulation



analytical solution,
A = 1–γ(u–1)

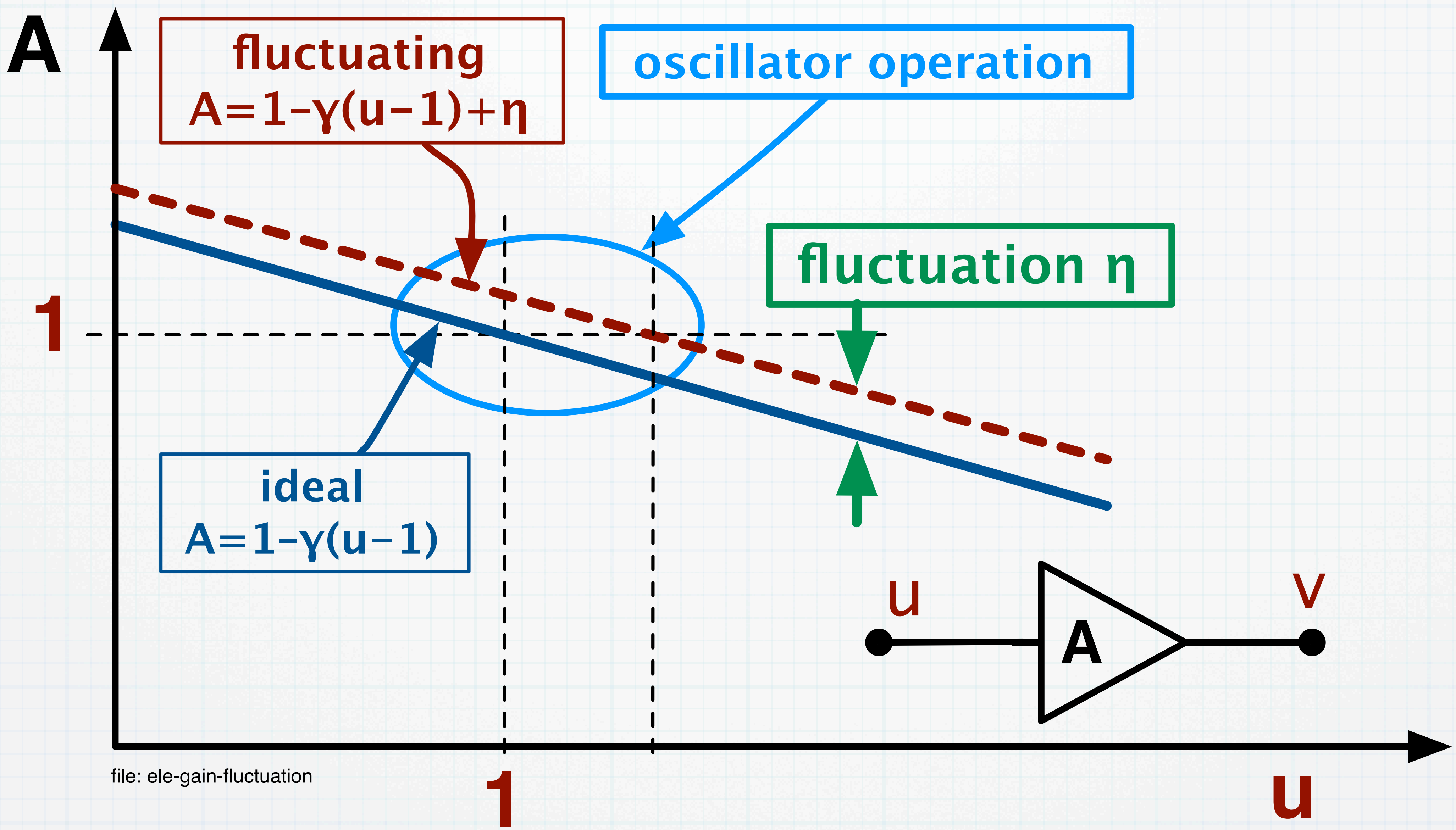
10 MHz oscillator
L = 1 mH
R = 125 Ω
Q ~ 503



van der Pol oscillator
simulated by RB

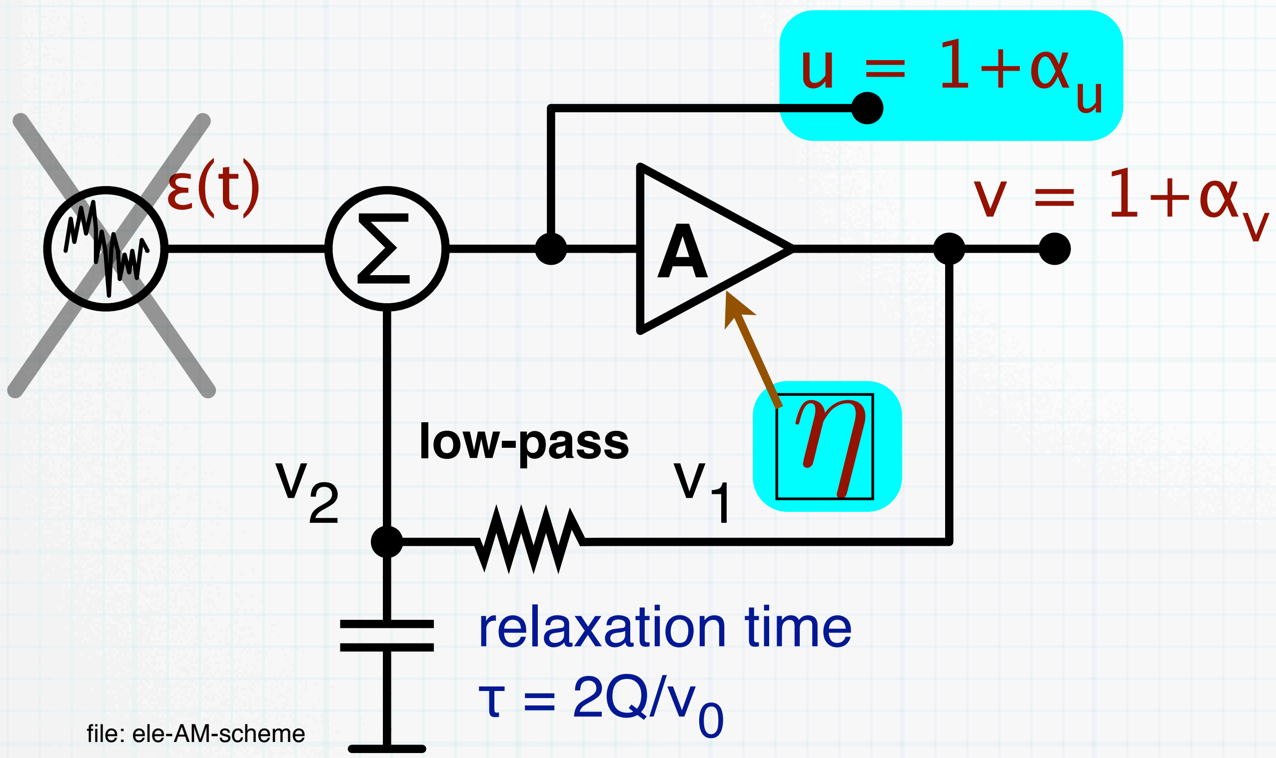
Rising exponential.
We find the same time constant –τ/γ

Gain fluctuations – definition



Gain compression is necessary for the oscillation amplitude to be stable

Gain fluctuations – output is u



$$\dot{u} = \frac{1}{\tau}(A-1)u$$ **non-linear equation**

$$A = 1 - \gamma(u-1) + \eta$$

$$\dot{u} + \frac{\gamma}{\tau}(u-1)u = \frac{\eta}{\tau}u$$ **linearization for low noise**

$\dot{u} \to \dot{\alpha}_u$, $(u-1) \to \alpha_u$, $u \to 1$, $u \to 1$

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \frac{1}{\tau}\eta$$ **linearized equation**

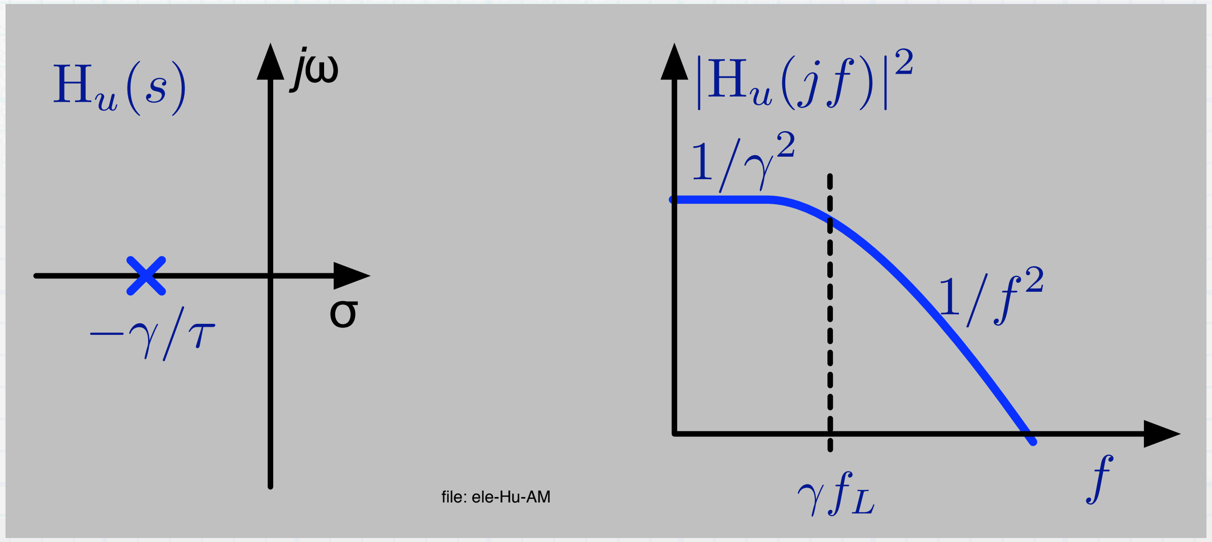
$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \frac{1}{\tau}\mathcal{N}(s)$$ **Laplace transform**

Linearize for low noise and use the Laplace transforms

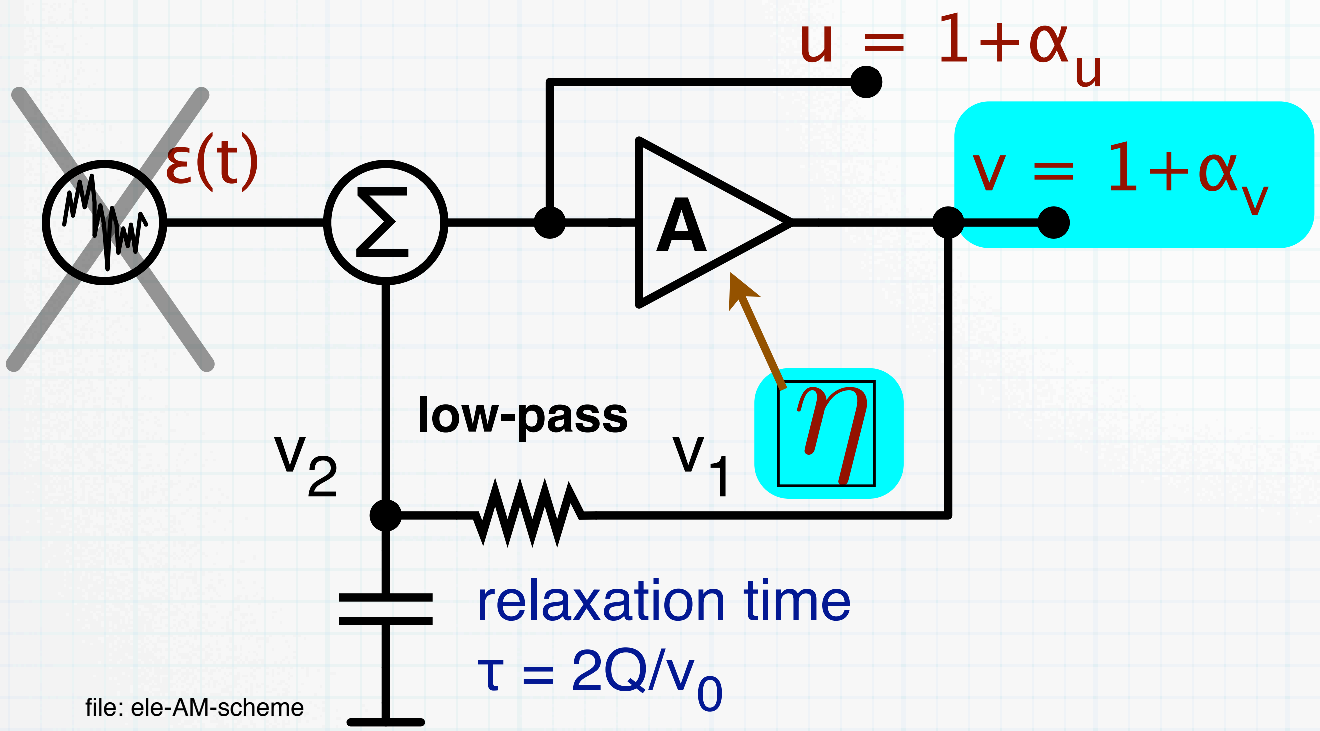
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \eta(t) \leftrightarrow \mathcal{N}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}_u(s) = \frac{1/\tau}{s + \gamma/\tau}$$ **result**



Gain fluctuations – output is v



$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_u(s)=\frac{1}{\tau}\mathcal{N}(s)$$ **starting equation**

$$\mathcal{A}_u(s)=\frac{\mathcal{A}_v(s)-\mathcal{N}(s)}{1-\gamma}$$

$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_v(s)=\left(s+\frac{1}{\tau}\right)\mathcal{N}(s)$$

$$\mathrm{H}(s)=\frac{\mathcal{A}_v(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}(s)=\frac{s+1/\tau}{s+\gamma/\tau}$$ **result**

boring algebra relates α_v to α_u

$$v=Au$$

$$A=-\gamma(u-1)+1+\eta$$

$$v=[-\gamma(u-1)+1+\eta]\,u$$

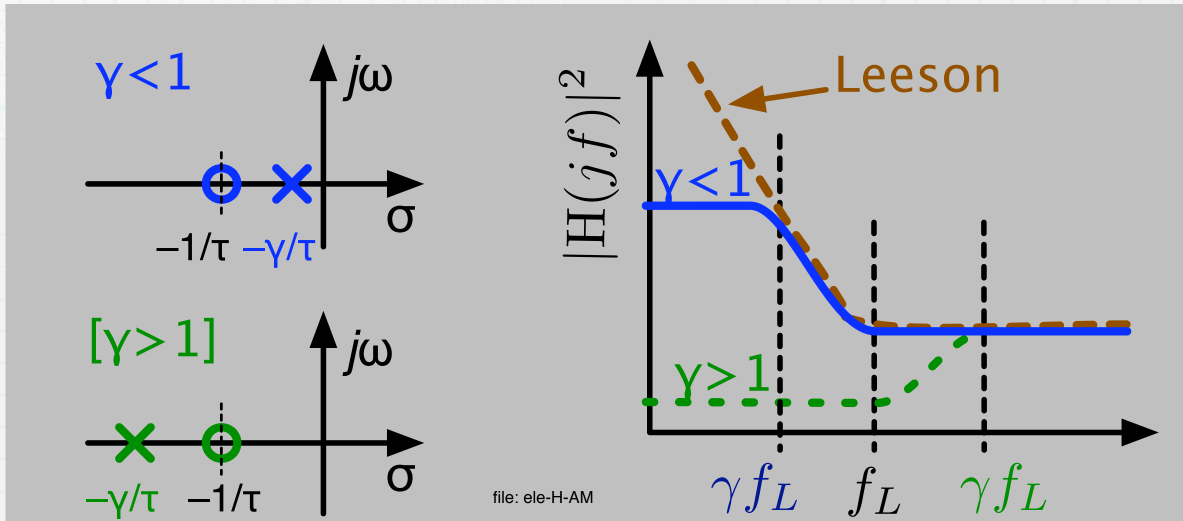
$$v=[-\gamma\alpha_u+1+\eta]\;[1+\alpha_u]$$

$$\cancel{1}+\alpha_v=\cancel{1}+\eta-\gamma\alpha_u+\alpha_u-\cancel{\alpha_u\eta}-\cancel{\gamma\alpha_u^2}$$

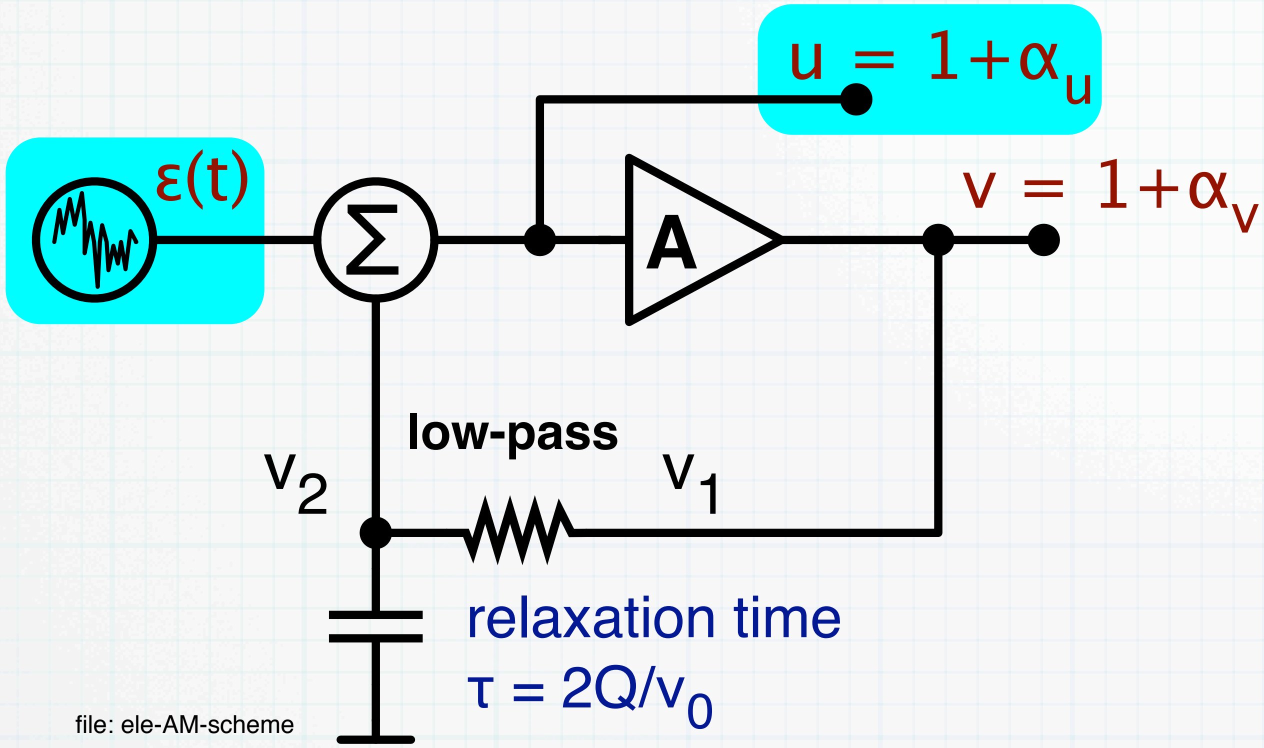
linearization for low noise

$$\alpha_v=(1-\gamma)\alpha_u+\eta$$

$$\alpha_u=\frac{\alpha_v-\eta}{1-\gamma}$$



Additive noise – output is u



$$\dot u = \frac{1}{\tau}(A-1)u + \dot\epsilon + \frac{1}{\tau}\epsilon$$

$$A = 1-\gamma(u-1)$$

non-linear equation

$$\underbrace{\dot u}_{\dot\alpha_u} + \frac{\gamma}{\tau}\underbrace{(u-1)}_{\alpha_u}\underbrace{u}_{1} = \dot\epsilon + \frac{1}{\tau}\epsilon$$

lineariz. for low noise

$$\dot\alpha_u + \frac{\gamma}{\tau}\alpha_u = \dot\epsilon + \frac{1}{\tau}\epsilon$$

linearized equation

$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \left(s+\frac{1}{\tau}\right)\mathcal{E}(s)$$

Laplace transform

Linearize for low noise and use the Laplace transforms

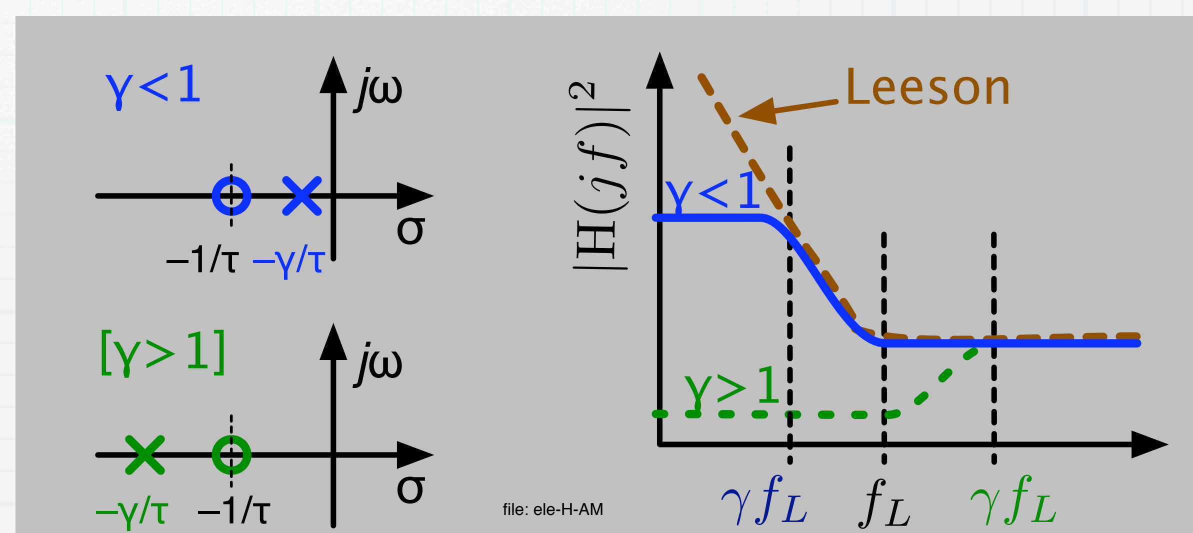
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \epsilon(t) \leftrightarrow \mathcal{E}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{E}(s)}$$

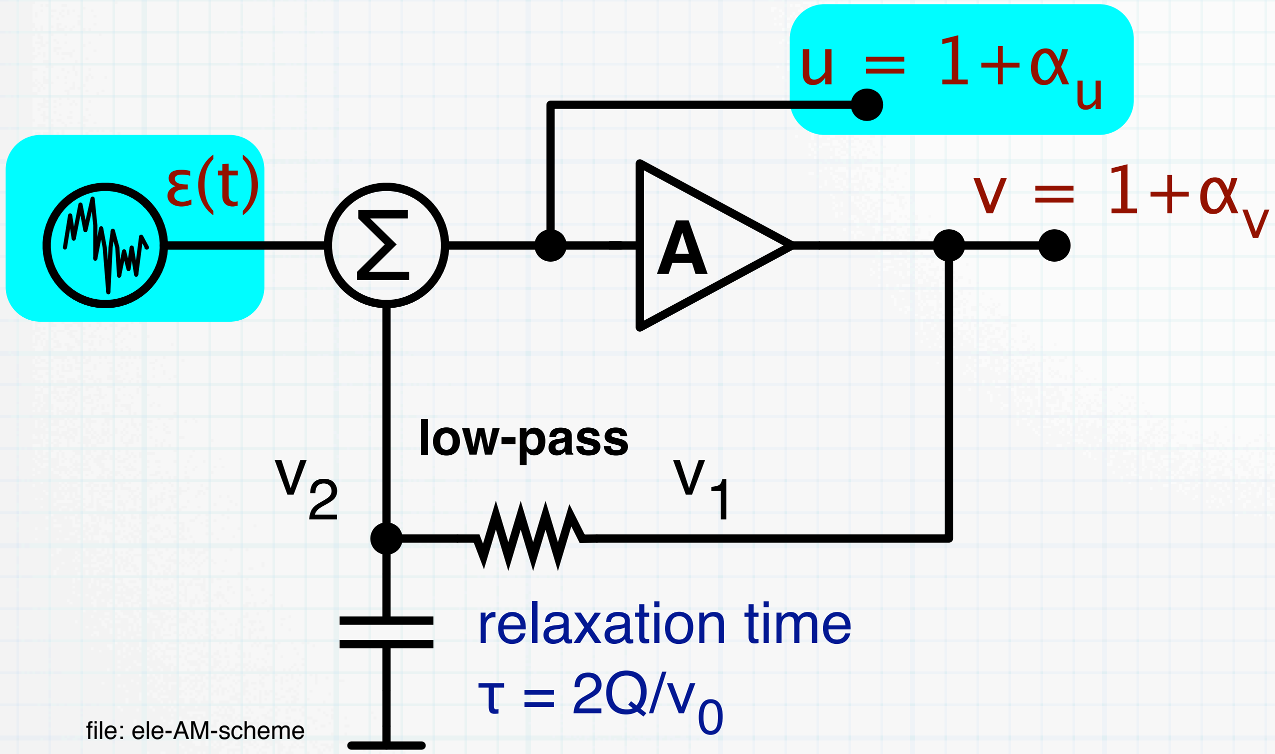
definition

$$\mathrm{H}_u(s) = \frac{s+1/\tau}{s+\gamma/\tau}$$

result



Additive noise – output is v



$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$ **linearized equation**

$$\alpha_u = \alpha_v/(1-\gamma)$$

$$\frac{1}{1-\gamma}\left(\dot{\alpha}_v + \frac{\gamma}{\tau}\alpha_v\right) = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

$$\frac{1}{1-\gamma}\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_v(s) = \left(s + \frac{1}{\tau}\right)\mathcal{E}(s)$$ **Laplace transform**

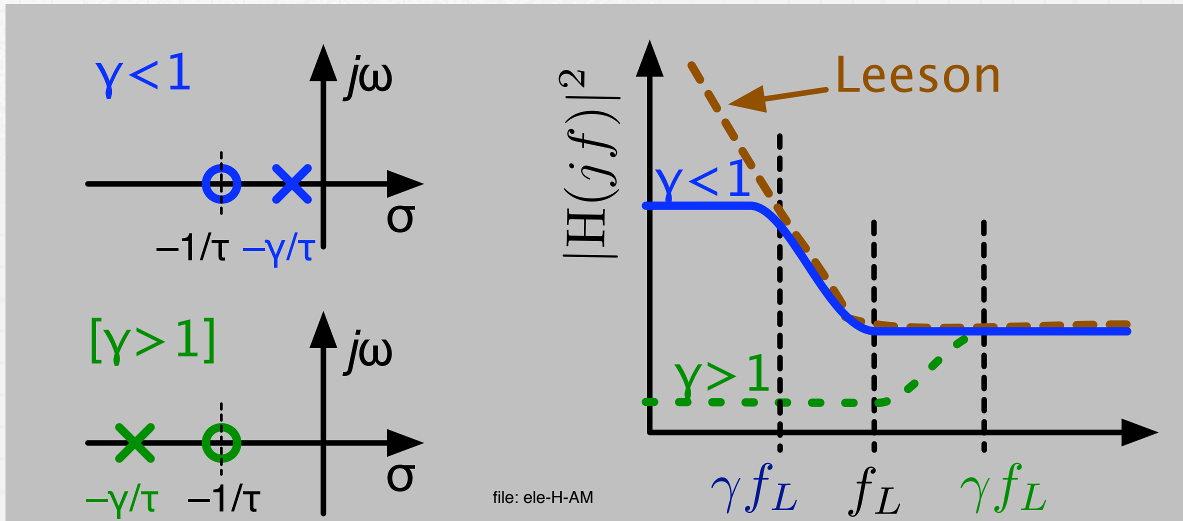
$$\mathrm{H}(s) = \frac{\mathcal{A}_v(s)}{\mathcal{E}(s)}$$ **definition**

$$\mathrm{H}(s) = (1-\gamma)\,\frac{s+1/\tau}{s+\gamma/\tau}$$ **result**

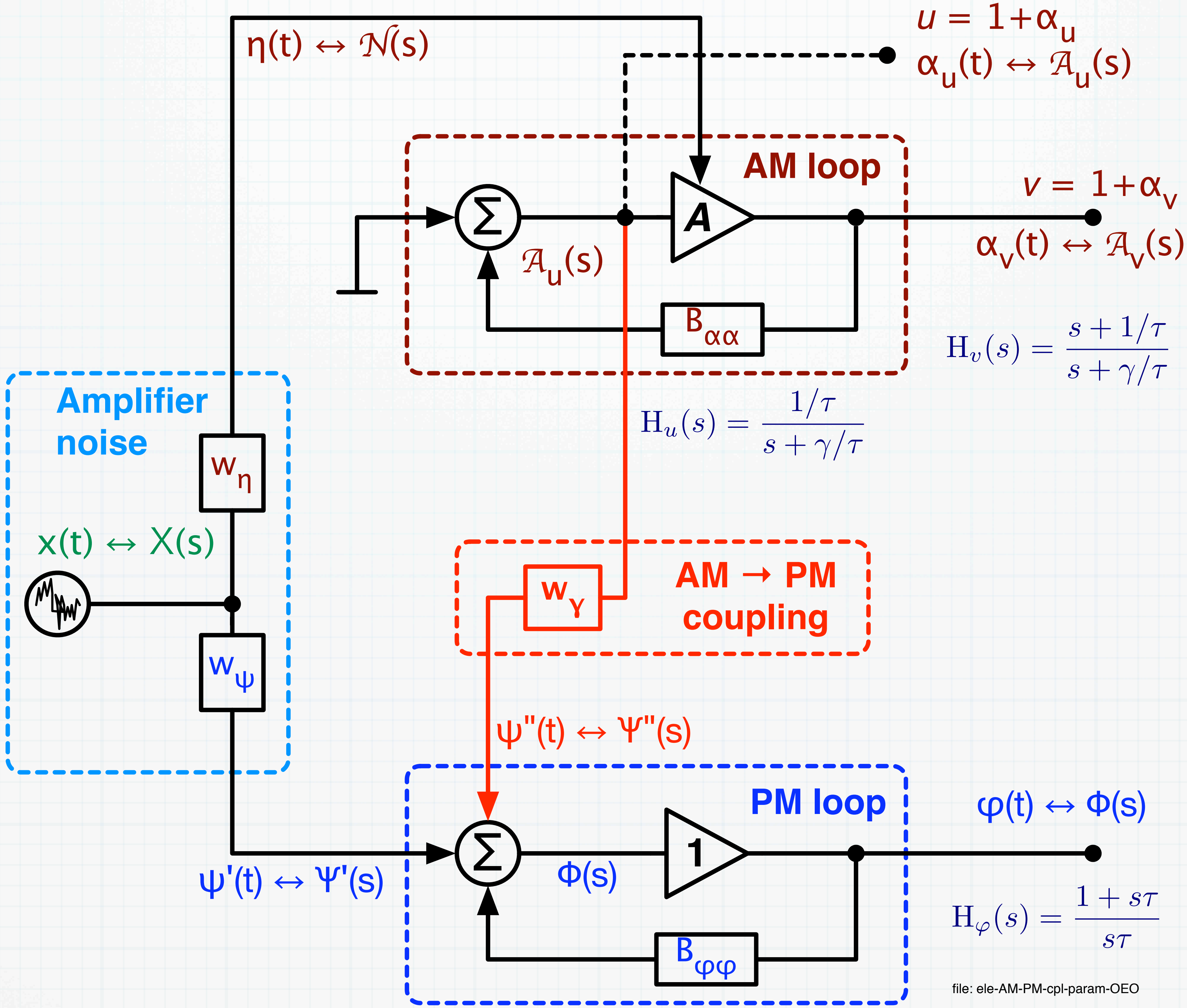
boring algebra relates α' to α

$$v = Au$$
$$A = 1 - \gamma(u-1)$$
$$v = [1-\gamma(u-1)]\,u$$
$$1+\alpha_v = [1-\gamma\alpha_u]\,[1+\alpha_u]$$
$$\cancel{1}+\alpha_v = \cancel{1}+\alpha_u - \gamma\alpha_u - \cancel{\gamma\alpha_u^2}$$
$$\alpha_v = (1-\gamma)\alpha_u$$
$$\alpha_u = \frac{\alpha_v}{1-\gamma}$$

linearization for low noise

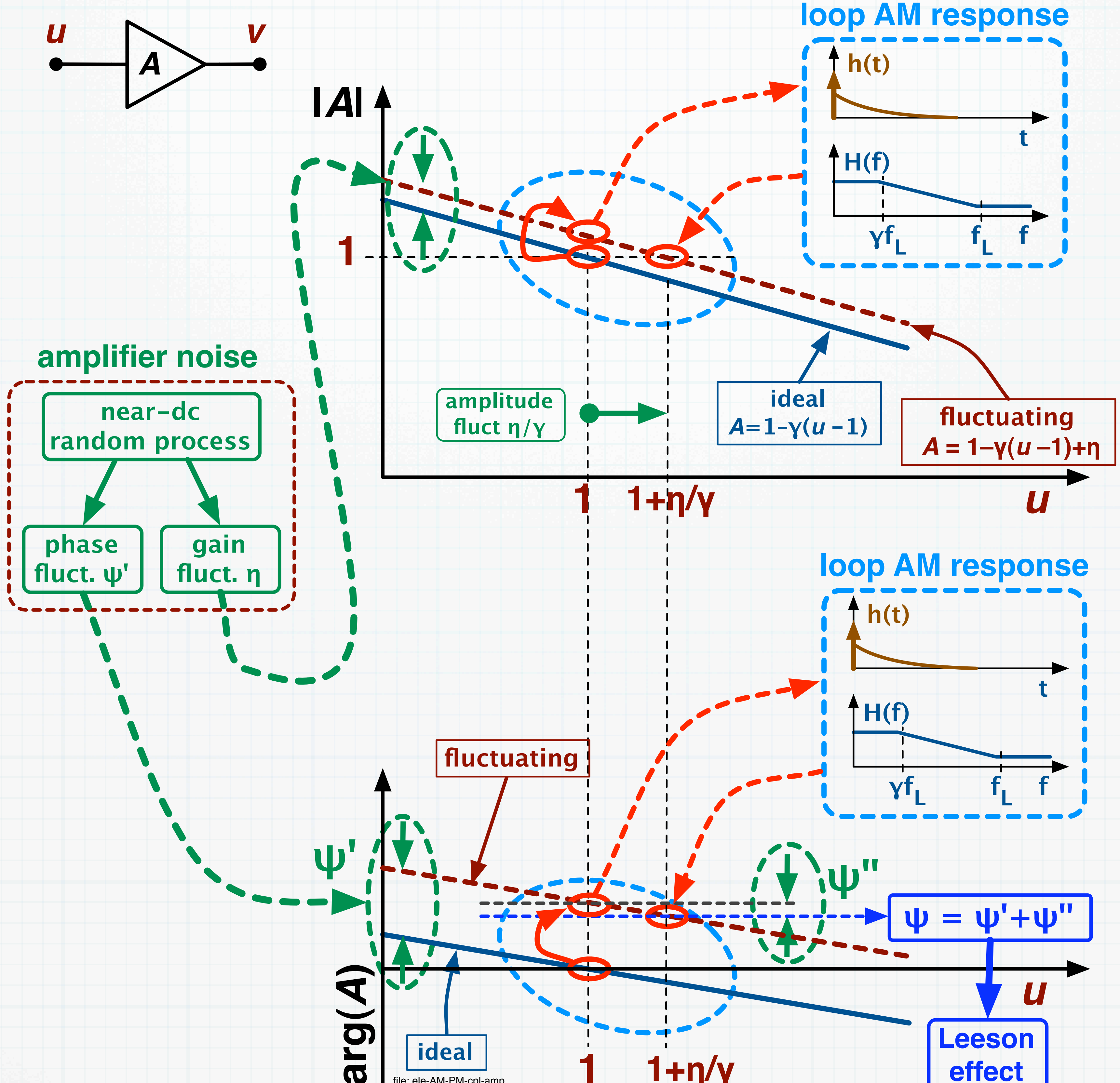


Parametric noise & AM-PM noise coupling



E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

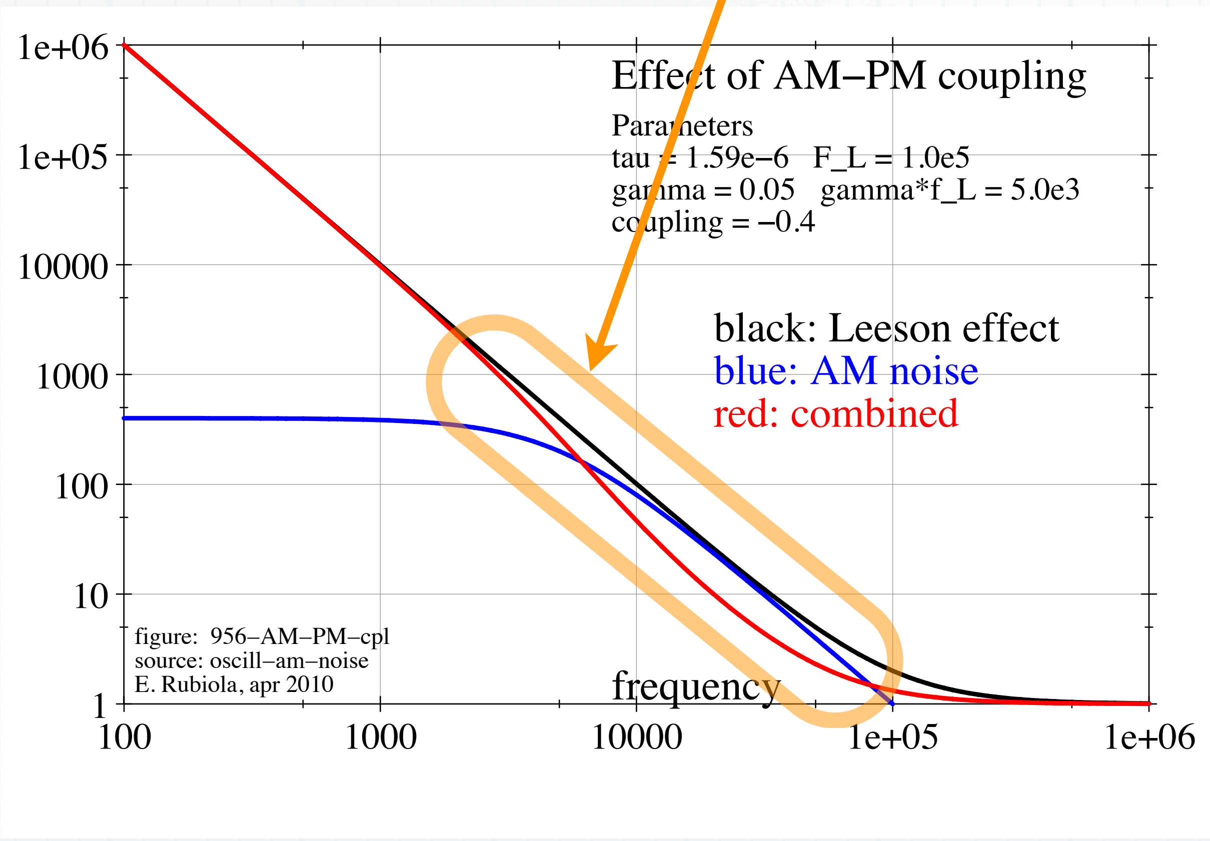
Effect of AM-PM noise coupling

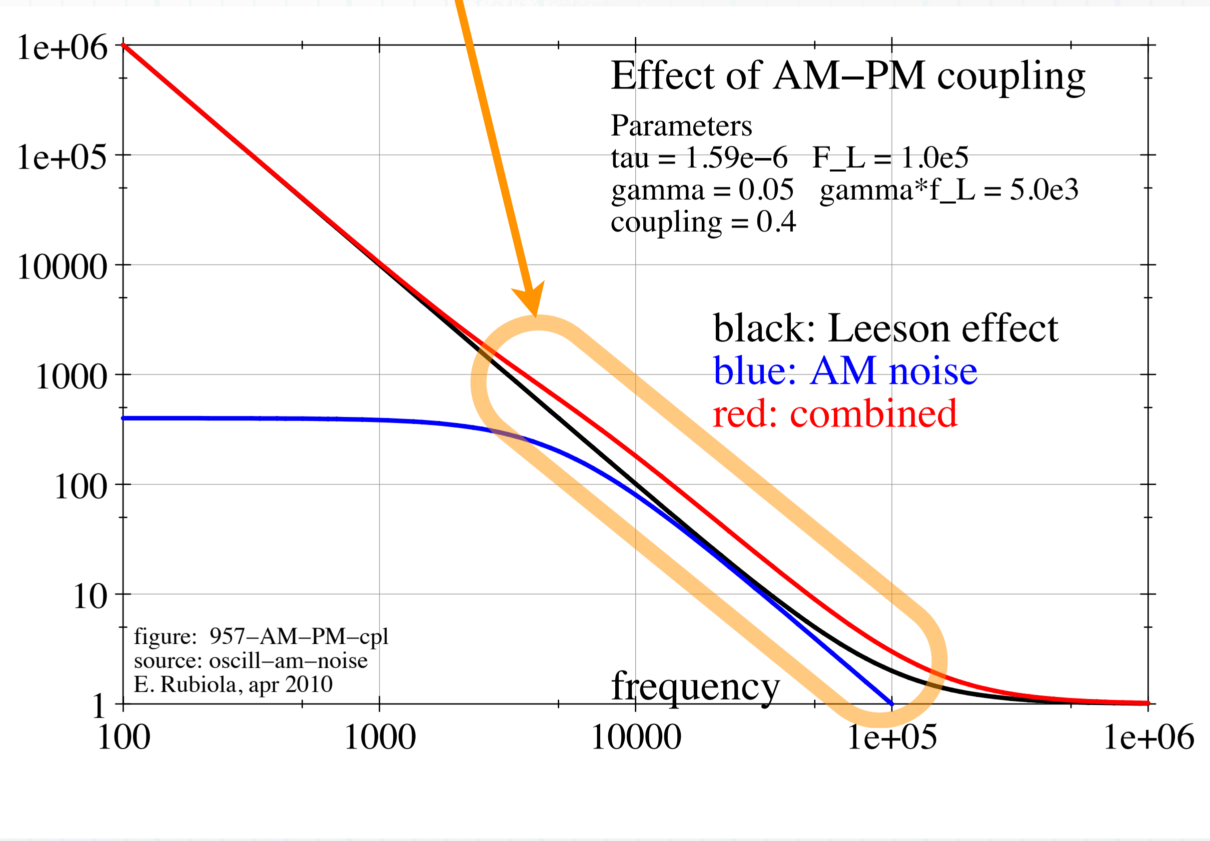


E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Noise transfer function, and spectra

AM-PM coupling shows up here



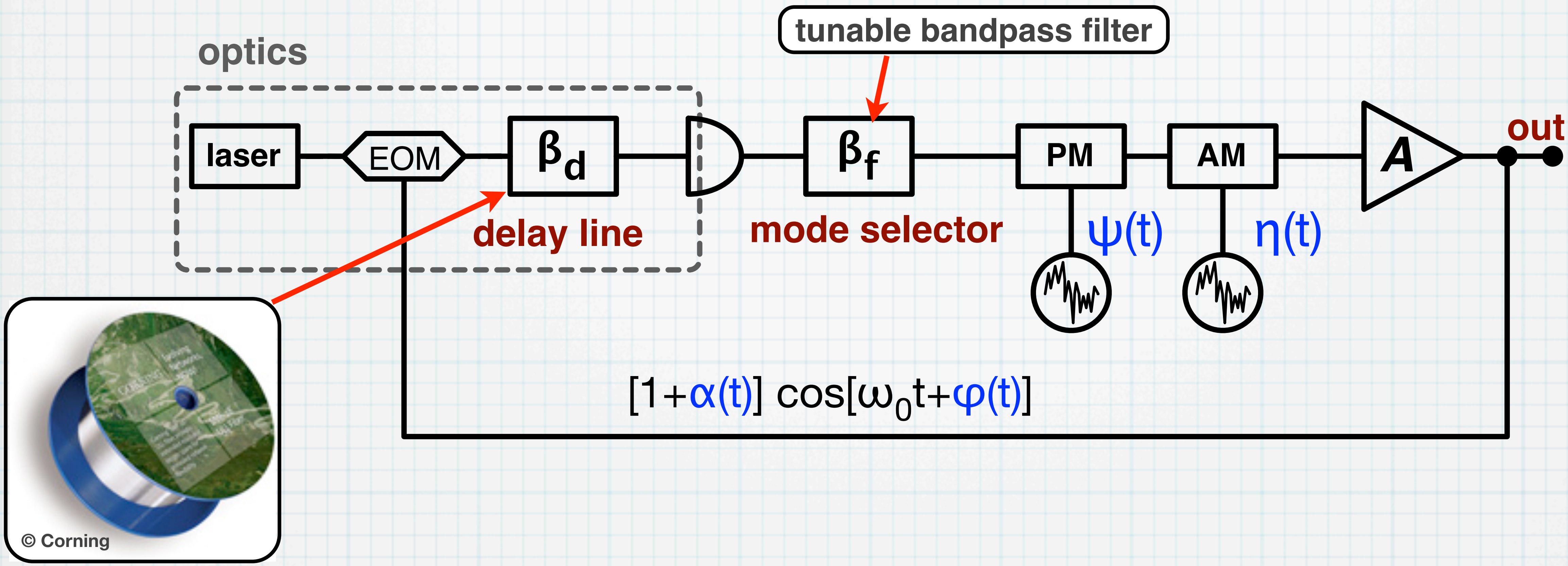


Notice that the AM-PM coupling can increase or decrease the PM noise

In a real oscillator, flicker noise shows up below some 10 kHz
In the flicker region, all plots are multiplied by 1/f

Leeson effect in the delay-line oscillator

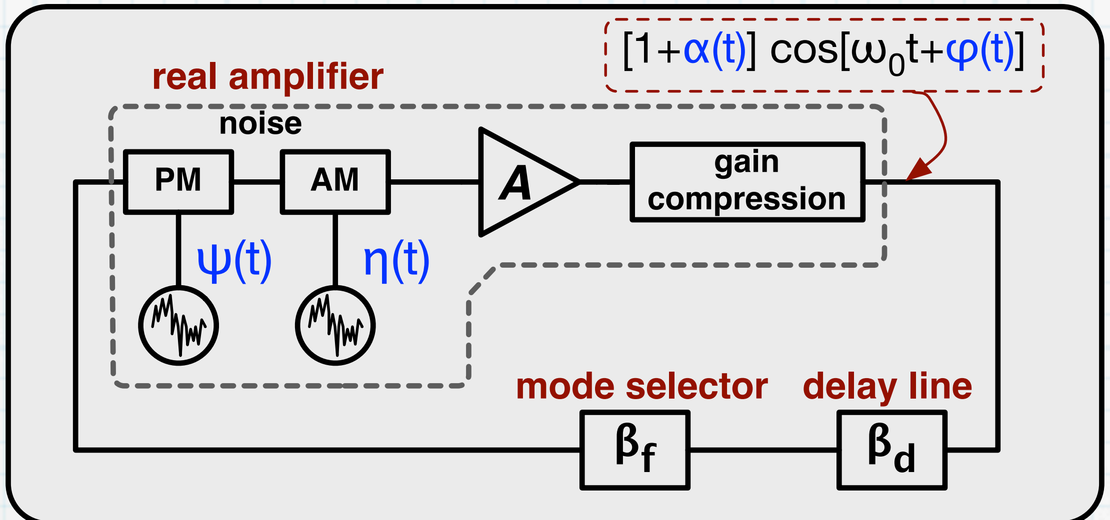
Motivations



- **Potential for very-low phase noise in the 100 Hz – 1 MHz range**
- **Invented at JPL, X. S. Yao & L. Maleki, JOSAB 13(8) 1725–1735, Aug 1996**
- **Early attempt of noise modeling, S. Römisch & al., IEEE T UFFC 47(5) 1159–1165, Sep 2000**
- **PM-noise analysis, E. Rubiola, *Phase noise and frequency stability in oscillators*, Cambridge 2008 [Chapter 5]**

- **Since, no progress in the analysis of noise at system level**
- **Nobody reported on the consequences of AM noise**

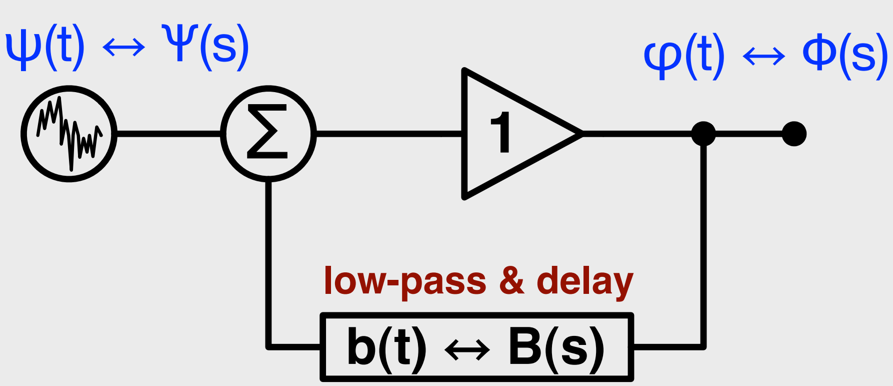
Low-pass representation of AM-PM noise



µwave

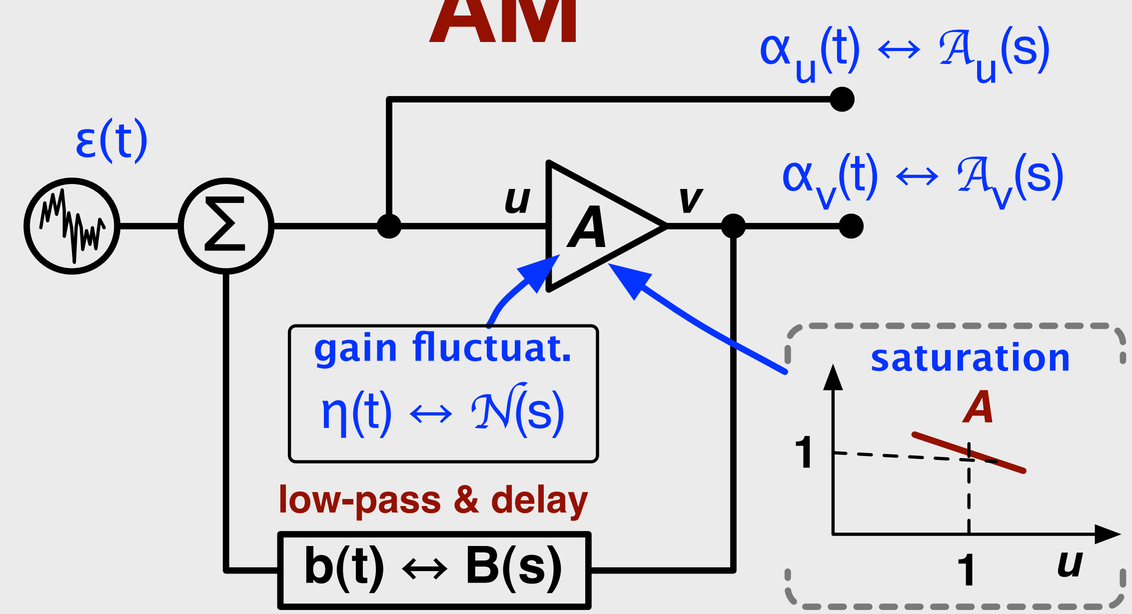
low-pass equivalent

PM



Leeson Effect

AM

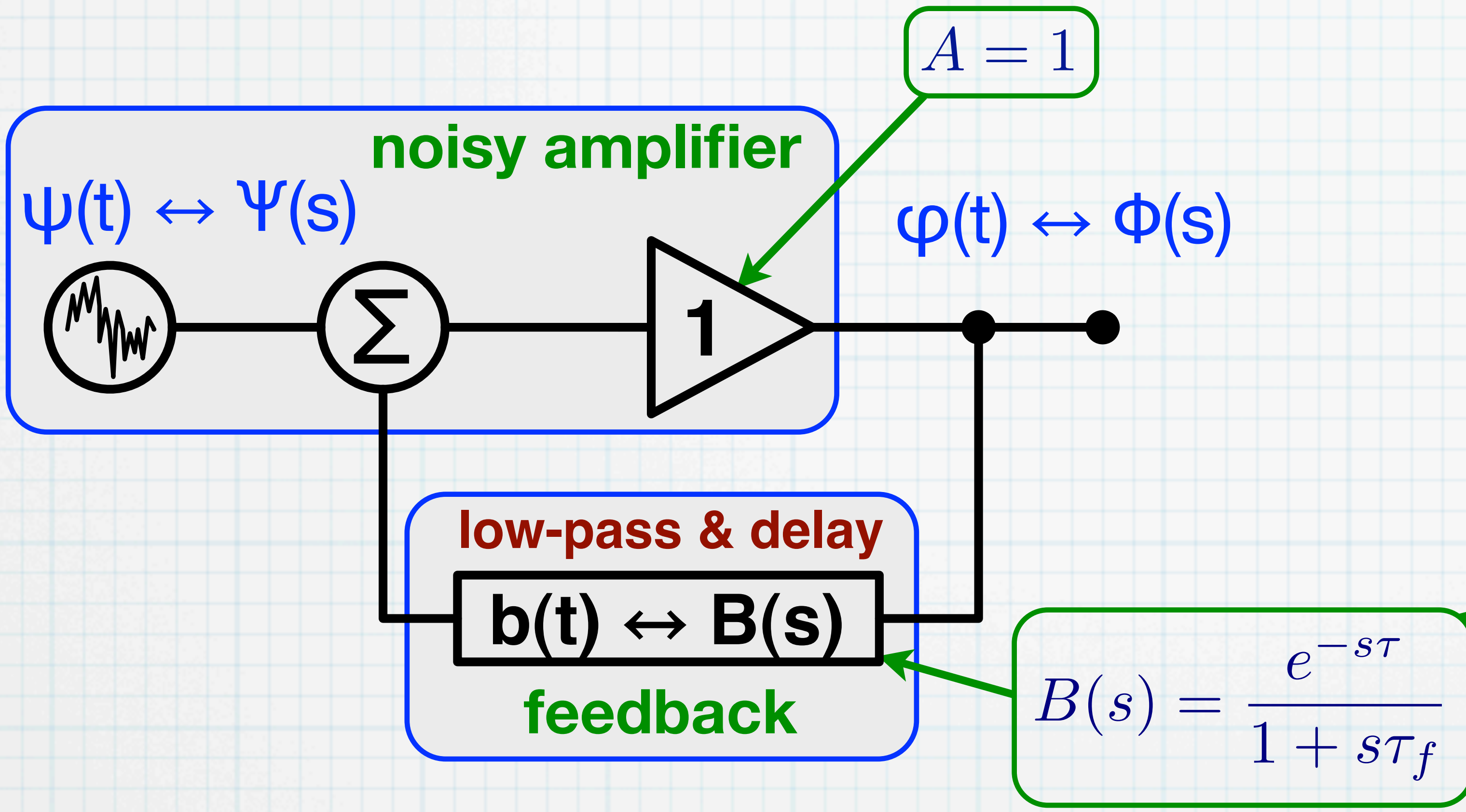


extension of the LE to AM noise

The amplifier
– "copies" the input phase to the out
– adds phase noise

The amplifier
– compresses the amplitude
– adds amplitude noise

Leeson effect



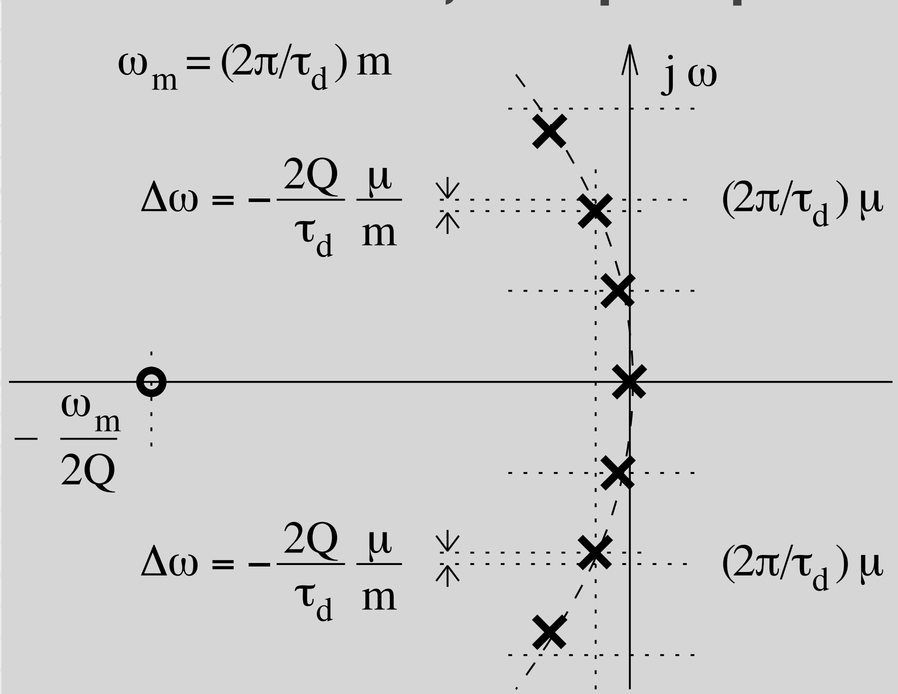
phase-noise transfer function

$$\mathrm{H}(s) = \frac{\Phi(s)}{\Psi(s)}$$ **definition**

$$\mathrm{H}(s) = \frac{1}{1+AB(s)}$$ **general feedback theory**

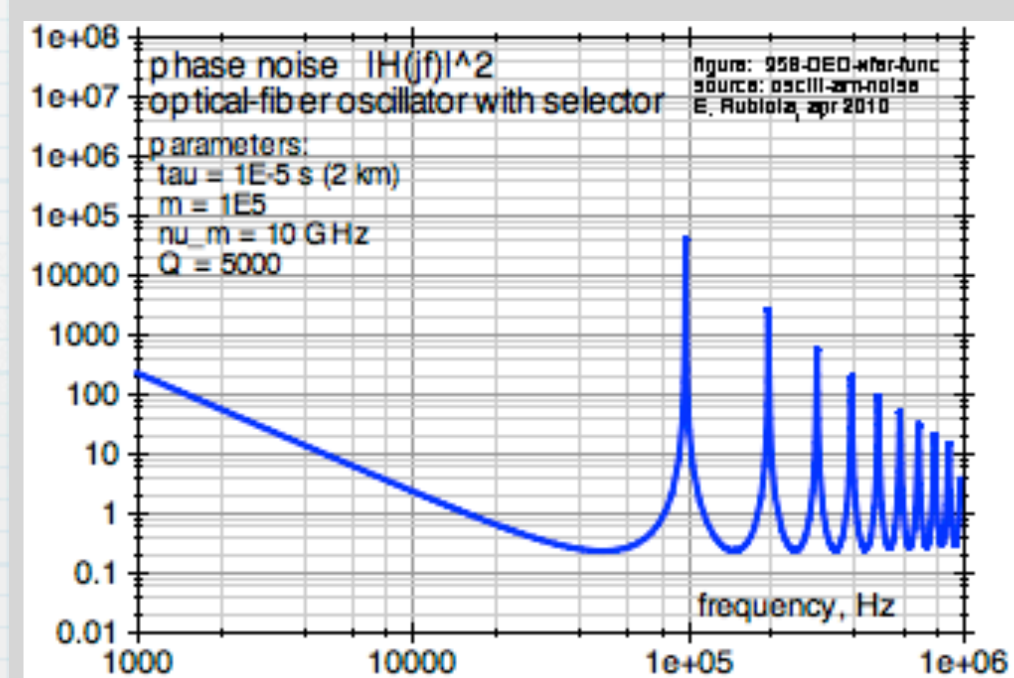
$$\mathrm{H}(s) = \frac{1+s\tau_f}{1+s\tau_f - e^{-s\tau}}$$ **Leeson effect**

H, complex plane



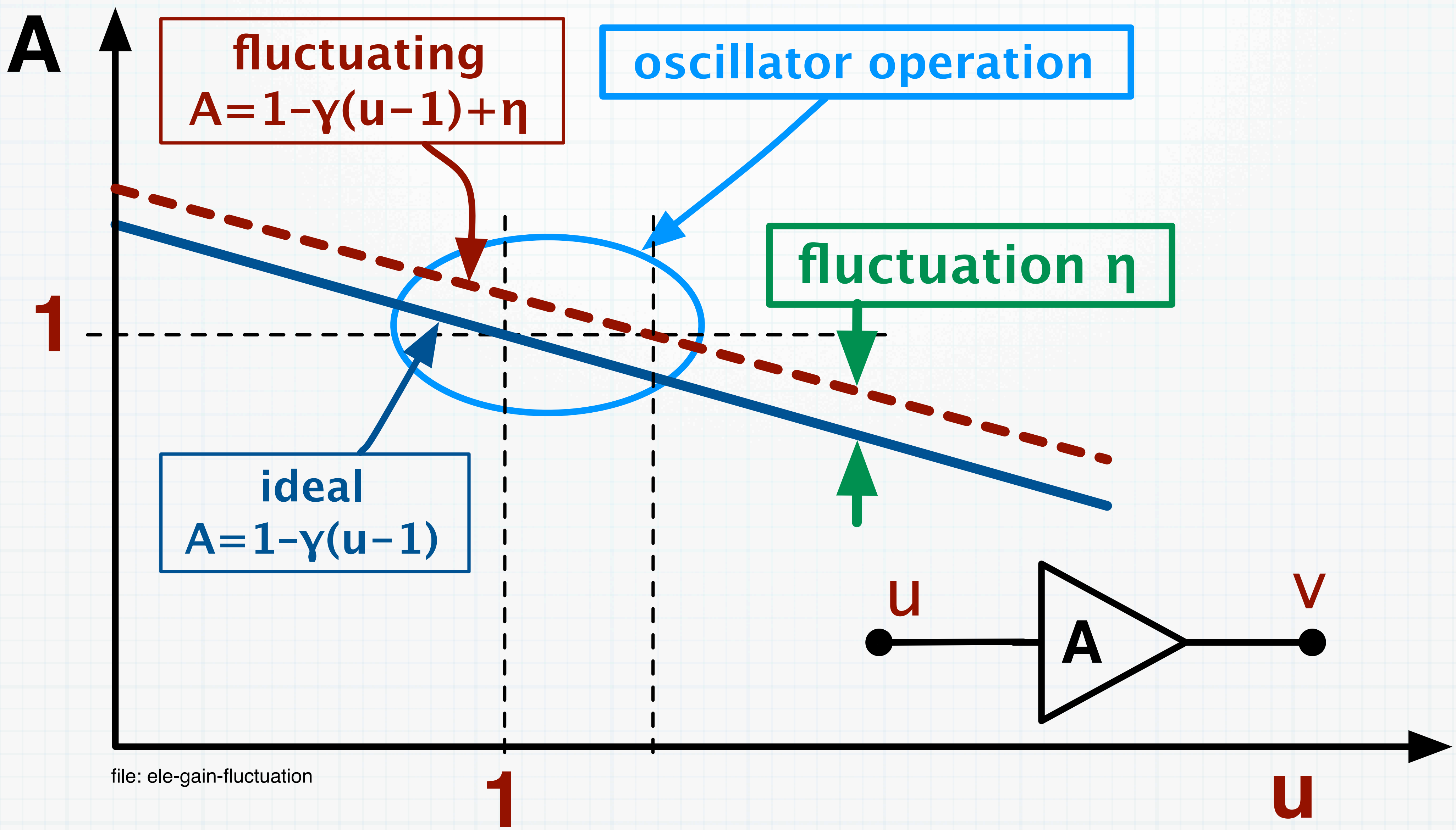
This figure is from E. Rubiola, Phase noise and frequency stability in oscillators, © Cambridge University Press

transfer function $|H|^2$



E. Rubiola, Phase noise and frequency stability in oscillators, Cambridge 2008

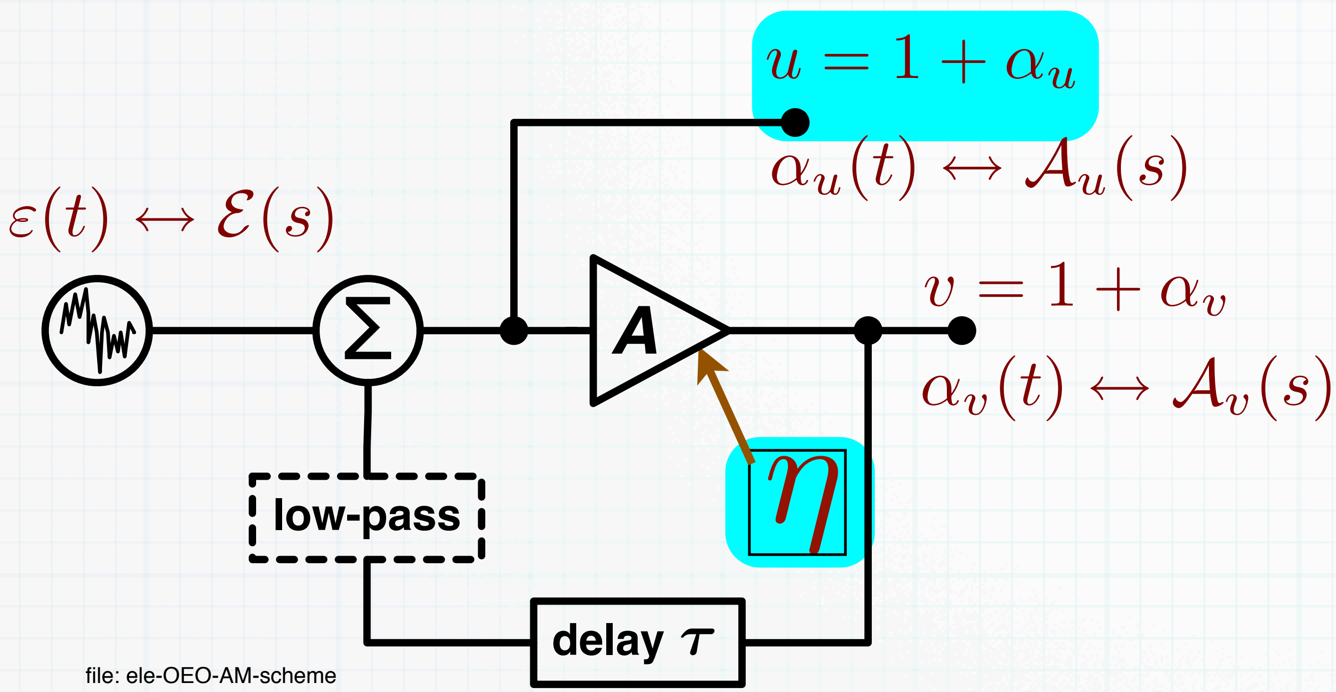
Gain fluctuations – definition



Gain compression is necessary for the oscillation amplitude to be stable

E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Gain fluctuations – output is *u*(*t*)



The low-pass has only 2nd order effect on AM

non-linear equation

$$u = A(t-\tau)\,u(t-\tau)$$

$$A = 1 - \gamma(u-1) + \eta$$

use u=α+1, expand and linearize for low noise

$$\alpha(t) = (1-\gamma)\alpha(t-\tau) - \cancel{\gamma\alpha^2(t-\tau)} \to 0$$
$$+\,\eta(t-\tau) + \cancel{\eta(t-\tau)\alpha(t-\tau)} \to 0$$

linearized equation

$$\alpha(t) = (1-\gamma)\alpha(t-\tau) + \eta(t-\tau)$$

Laplace transform

$$\mathcal{A}_u(s) = \left[1-(1-\gamma)e^{-s\tau}\right] = \mathcal{N}(s)$$

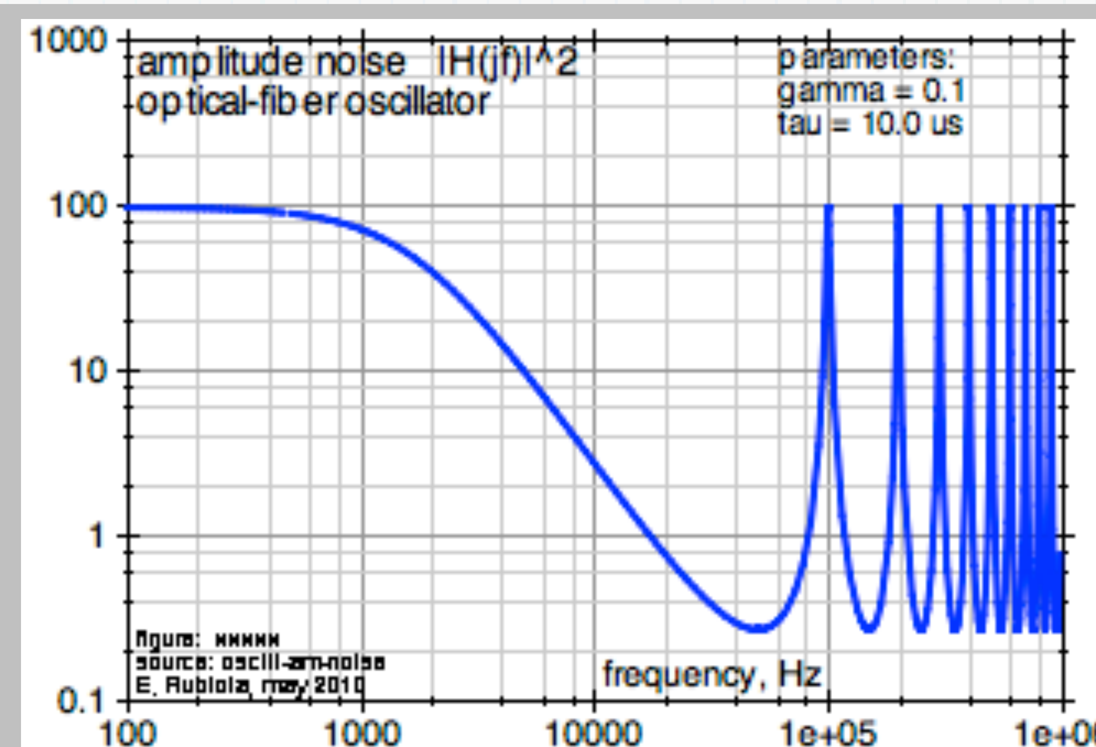
Linearize for low noise and use the Laplace transform

$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \eta(t) \leftrightarrow \mathcal{N}(s)$$

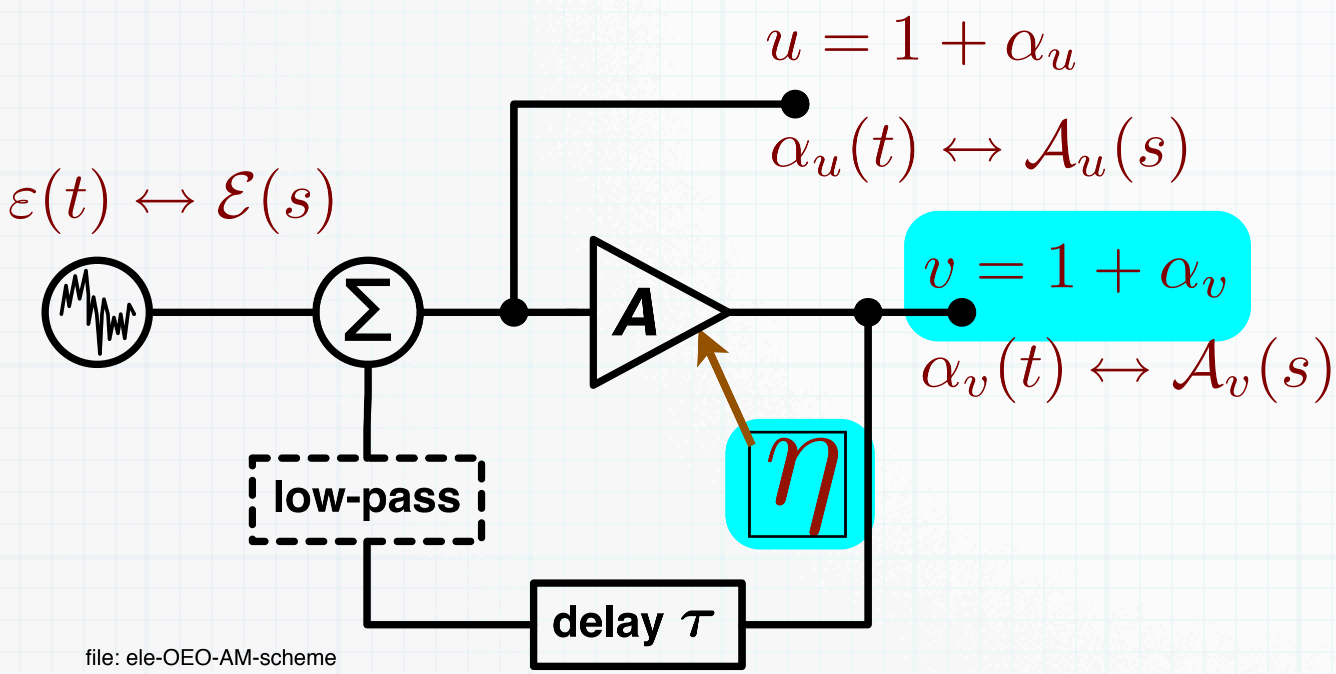
$$\mathrm{H}(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)}$$ **definition**

result

$$\mathrm{H}(s) = \frac{1}{1-(1-\gamma)e^{-s\tau}}$$



Gain fluctuations – output is *v*(*t*)



The low-pass has only 2nd order effect on AM

boring algebra relates α_v to α_u

$$v = Au$$

$$A = -\gamma(u-1) + 1 + \eta$$

$$v = [-\gamma(u-1) + 1 + \eta]\, u$$

use u=α+1

$$v = [-\gamma\alpha_u + 1 + \eta]\;[1 + \alpha_u]$$

$$\cancel{1} + \alpha_v = \cancel{1} + \eta - \gamma\alpha_u + \alpha_u - \cancel{\alpha_u\eta} - \cancel{\gamma\alpha_u^2}$$

linearization for low noise

$$\alpha_v = (1-\gamma)\alpha_u + \eta$$

$$\alpha_u = \frac{\alpha_v - \eta}{1-\gamma}$$

$$\mathcal{A}_u(s)\left[1 - (1-\gamma)e^{-\imath\omega\tau}\right] = \mathcal{N}(s)$$

starting equation

$$\mathcal{A}_u(s) = \frac{\mathcal{A}_v(s) - \mathcal{N}(s)}{1-\gamma}$$

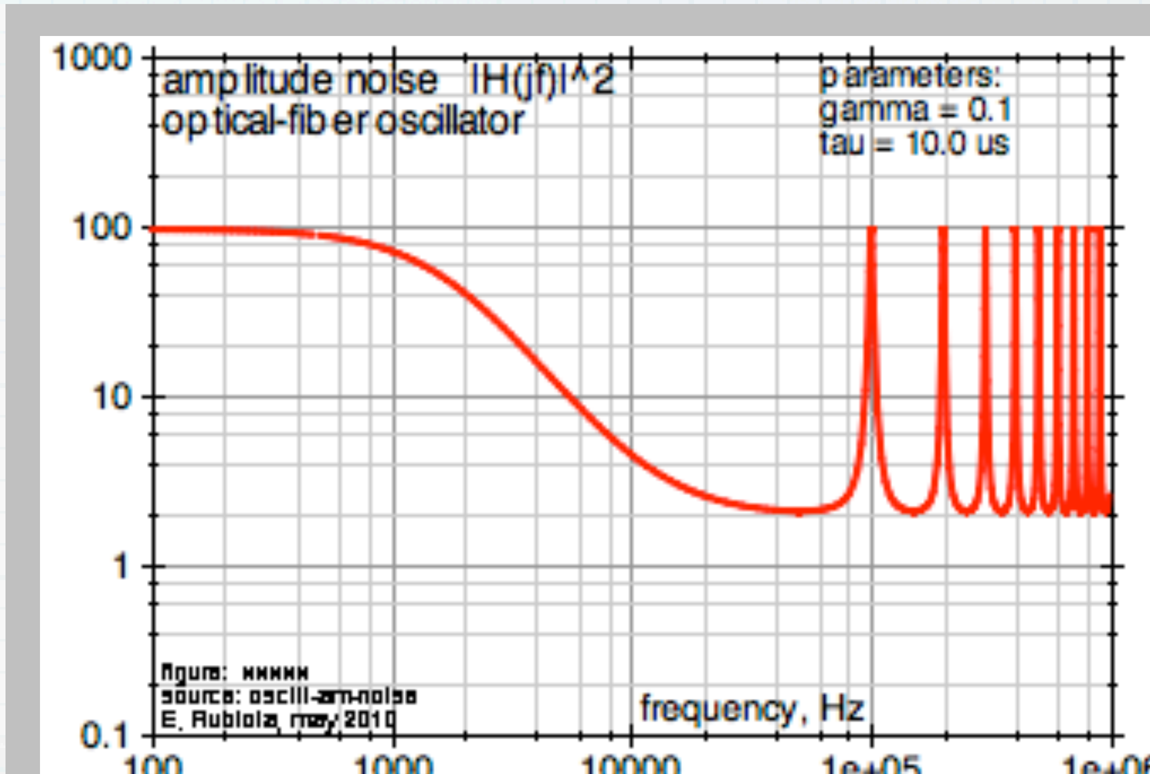
$$\left[1 + (1-\gamma)\left(1 - e^{-s\tau}\right)\right]\mathcal{A}_v(s) = \left[1 - (1-\gamma)e^{-s\tau}\right]\mathcal{N}(s)$$

$$\mathrm{H}(s) = \frac{\mathcal{A}_v(s)}{\mathcal{N}(s)}$$

definition

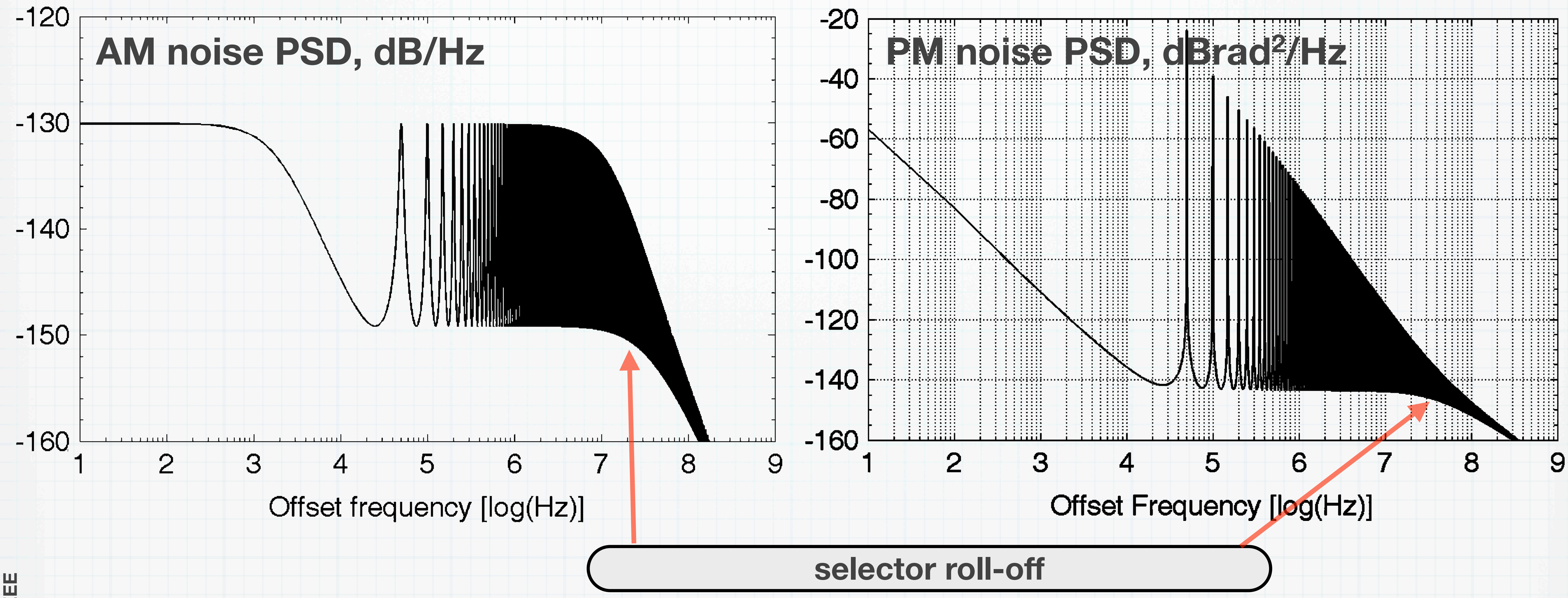
$$\mathrm{H}(s) = \frac{1 + (1-\gamma)\left(1 - e^{-s\tau}\right)}{1 - (1-\gamma)e^{-s\tau}}$$

result



Skip

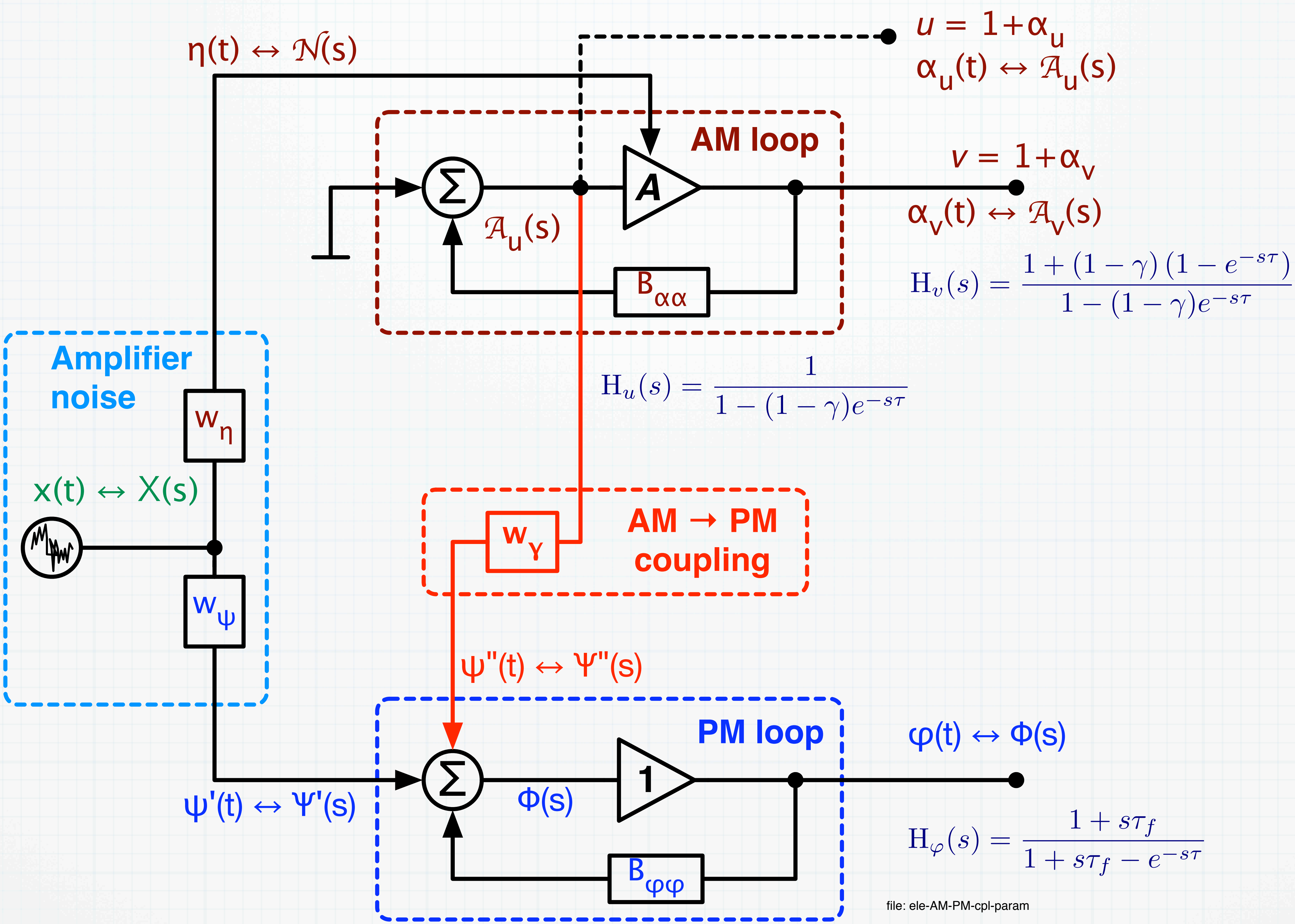
AM & PM spectra were anticipated



- Prediction is based on the stochastic diffusion (Langevin) theory
- However complex, the Langevin theory provides an independent check

Y.K. Chembo & al., IEEE J. Quant. Electron. 45(2) p.178-186, Feb 2009

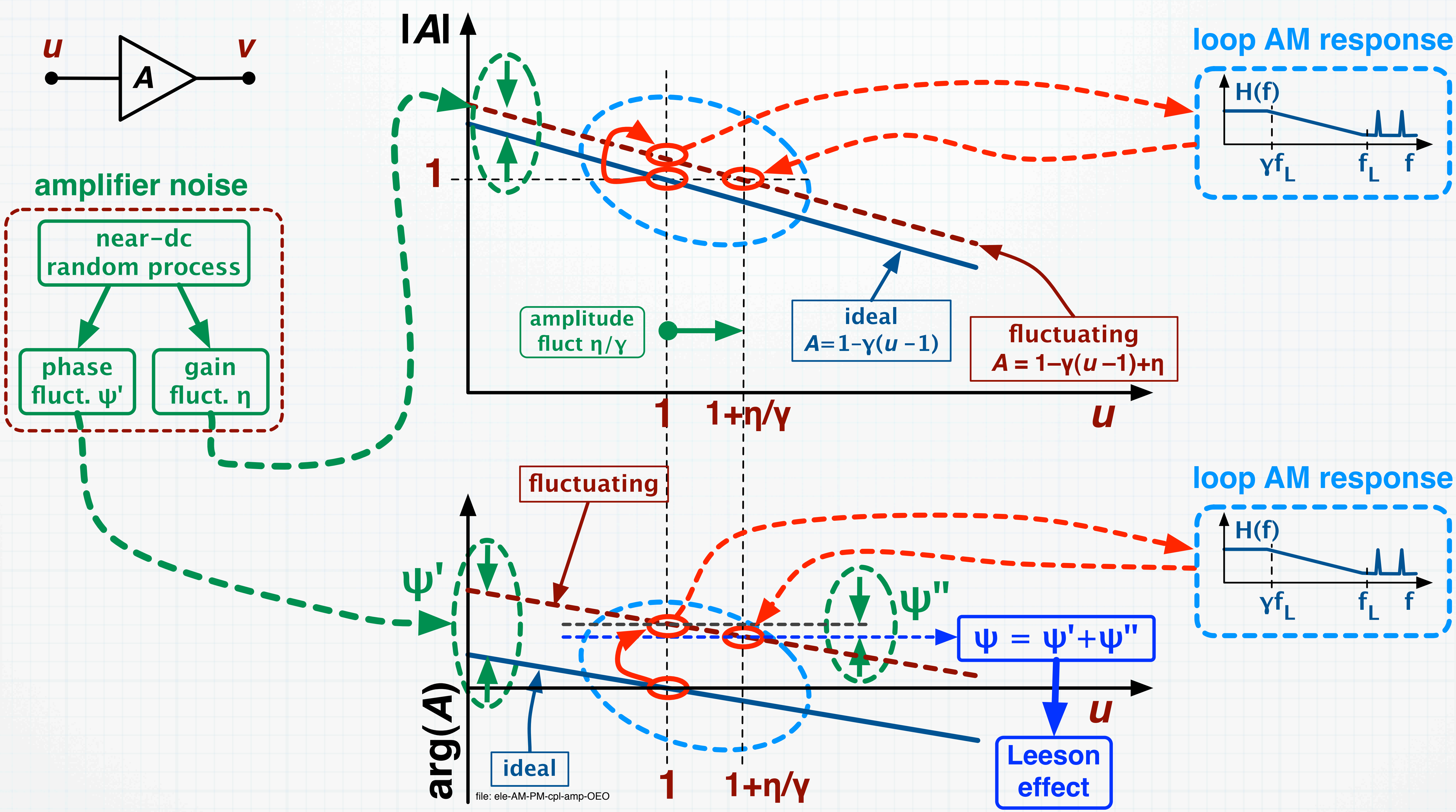
Parametric noise & AM-PM noise coupling



E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Similar to the oscillator based on a simple resonator

Effect of AM-PM noise coupling

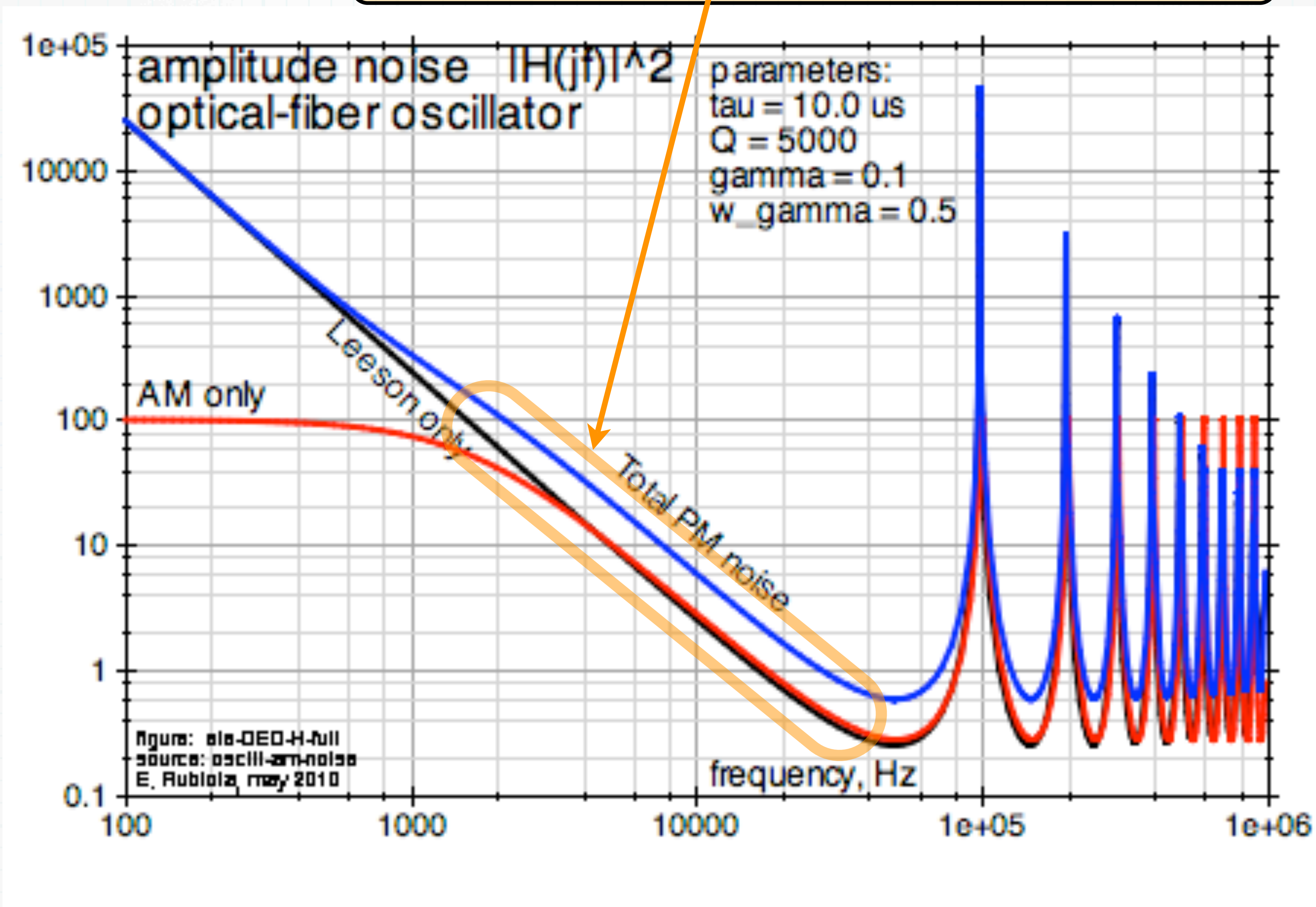


E. Rubiola & R. Brendel, arXiv:1004.5539v1 [physics.ins-det]

Similar to the oscillator based on a simple resonator

Noise transfer function and spectra

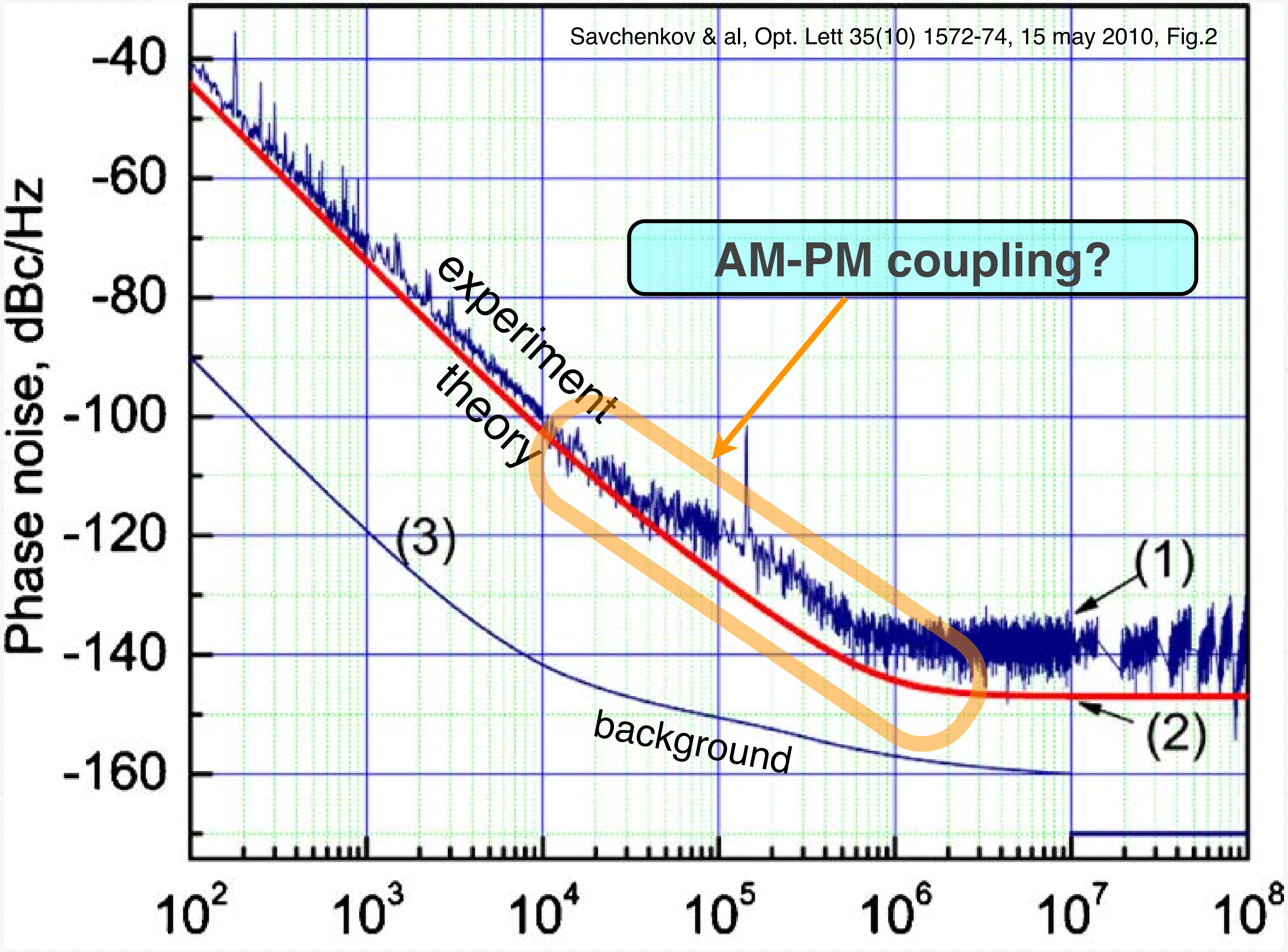
AM-PM coupling shows up here



Notice that the AM-PM coupling can increase or decrease the PM noise

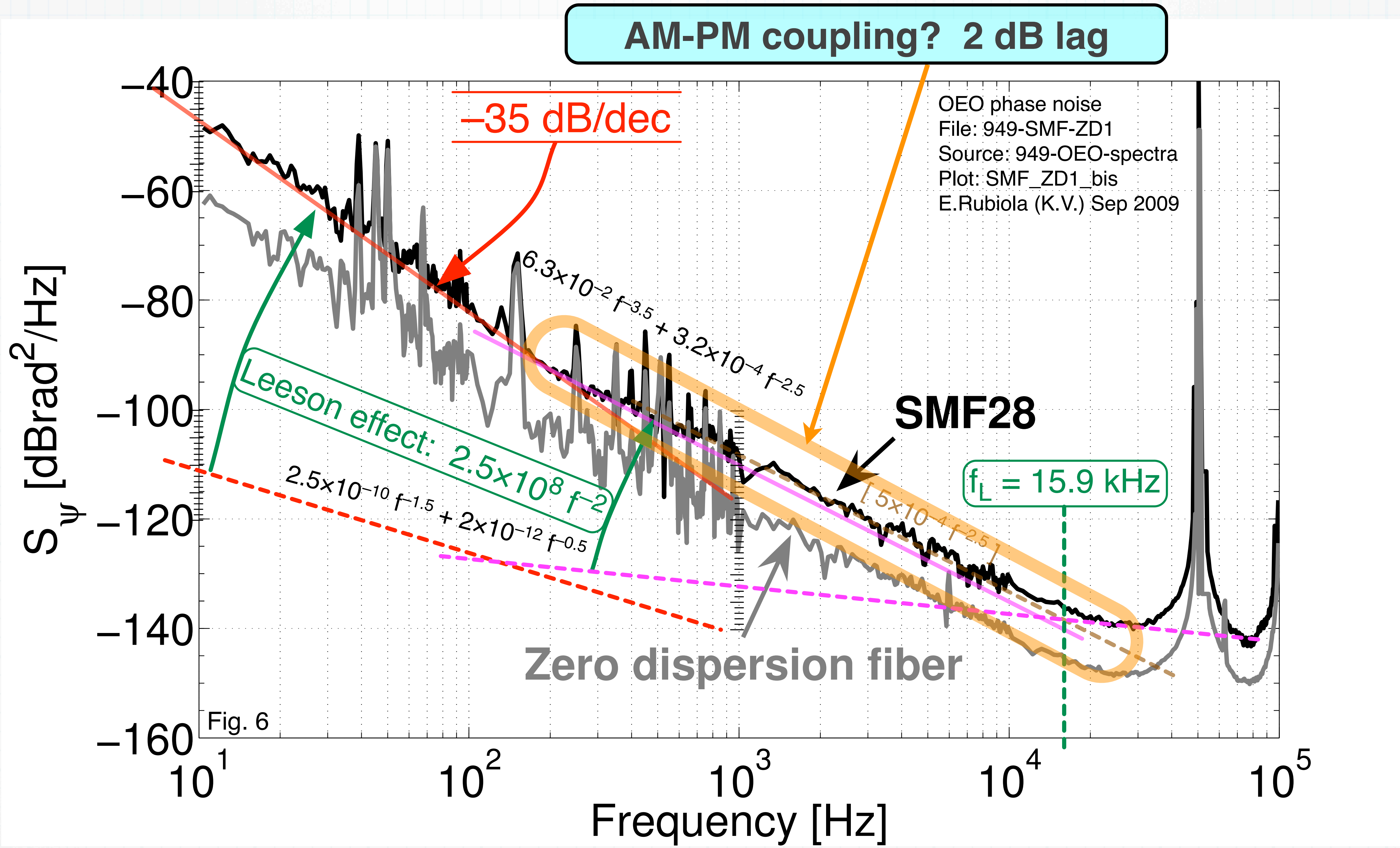
In a real oscillator, flicker noise shows up below some 10 kHz
In the flicker region, all plots are multiplied by 1/f

Noise spectra



A. Savchenkov & al, Opt. Lett 35(10) 1572-74, 15 may 2010, Fig.2

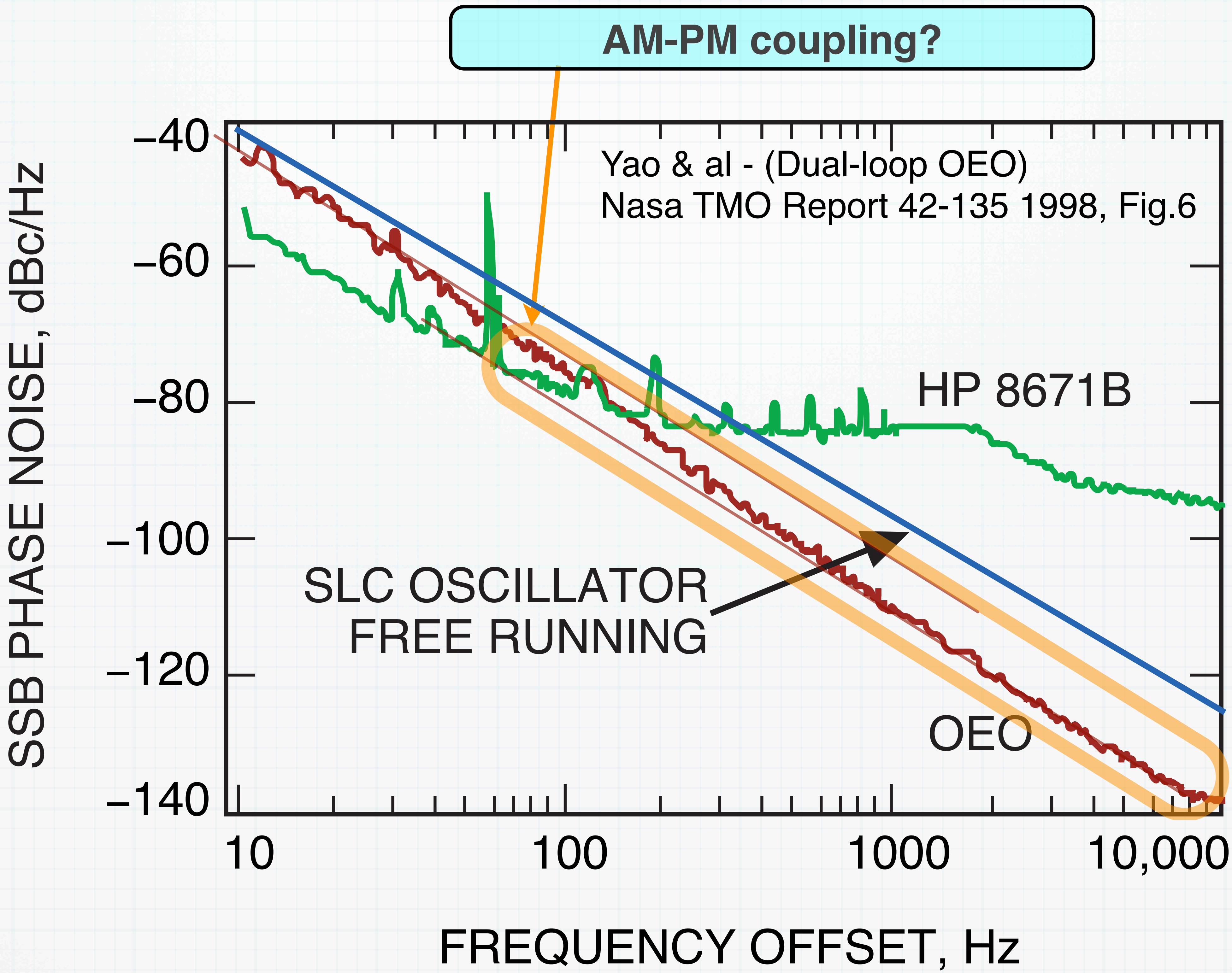
Noise spectra



Unfortunately, the awareness of this model come after the end of the experiments

Spectrum from K. Volyanskiy & al., IEEE JLT (Submitted, Apr. 2010)

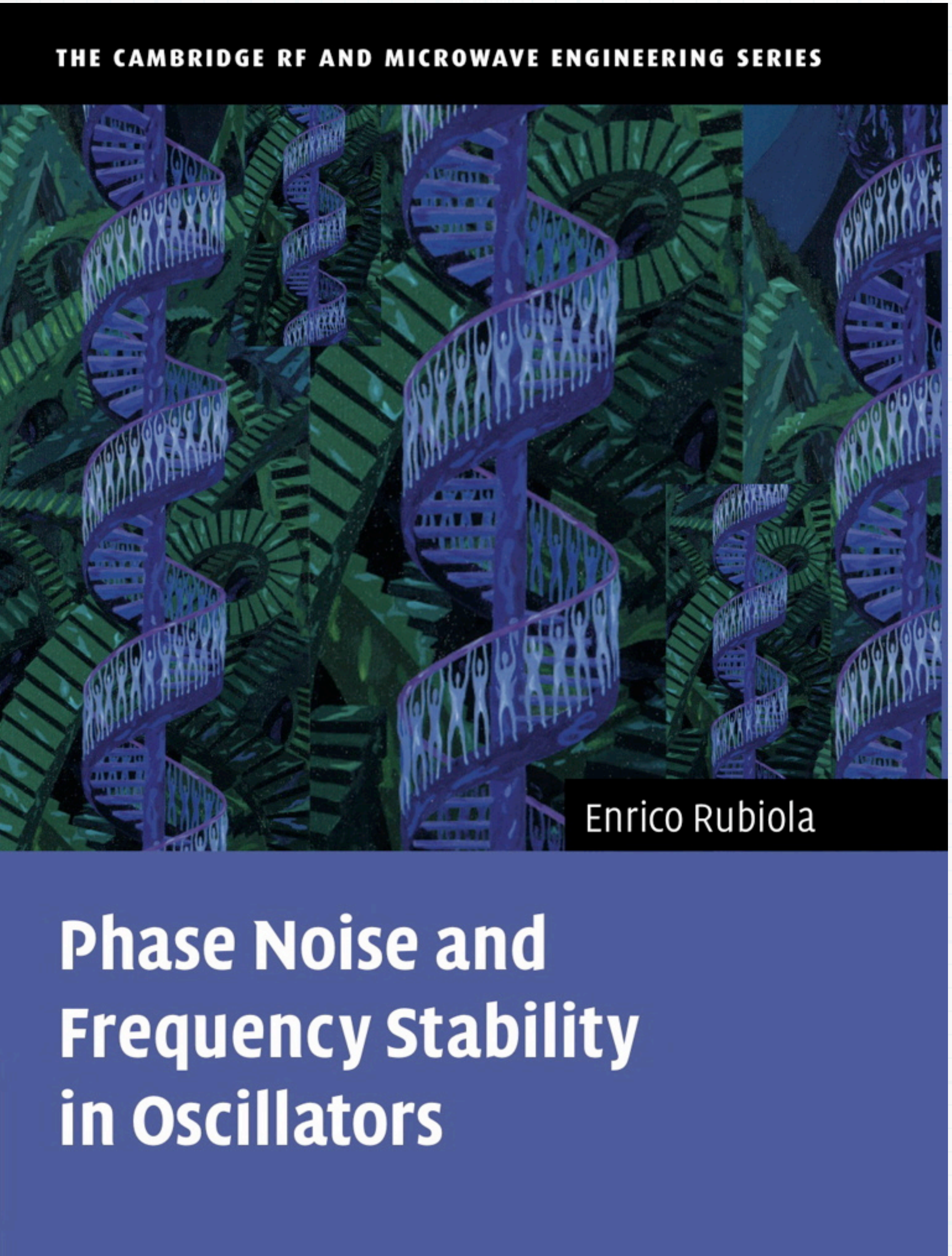
Noise spectra



X.S.Yao & al., NASA TMO Report 42-135 (1998), Fig. 6

Conclusions

Phase noise and frequency stability in oscillators



Cambridge University Press,
November 2008

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Front cover: The Wind Machines

Artist view of the AM and PM noise

Courtesy of Roberto Bergonzo, http://robertobergonzo.com

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This presentation is based on

E.Rubiola, *Phase noise and frequency stability in oscillators*, Cambridge 2008,

and on the complementary material

E. Rubiola, R. Brendel, A generalization of the Leeson effect,
arXiv:1004.5539 [physics.ins-det]

Please visit my home page http://rubiola.org

Summary of relevant points

- The Leeson effect consists in a phase-to-frequency conversion
  - fully explained as a phase (noise) integration
  - takes place below $f_L = \nu_0/2Q$
- The step response provides analytical solutions and physical inside. (Same formalism introduced by O. Heaviside in network theory)
- Buffer noise and resonator instability add to the Leeson effect
- Amplifier phase noise
  - white noise: S_φ scales down as the carrier power P_0
  - flicker noise: S_φ is independent of P_0
- Numerous oscillator spectra can be interpreted successfully
- The amplitude-noise response is similar to phase noise, but gain compression provides stabilization at low frequencies
- The theory indicates that amplitude-phase coupling results in a deviation from the polynomial law
- Unified AM/PM noise that applies to resonator-oscillators and to delay-line oscillators, including optical oscillators

A bunch of free material is available on the author's home page

http://rubiola.org

David and Enrico at the end of the tutorial

Photo by Barbara Leeson, David's wife