The AM noise mechanism in oscillators

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Outline

- Introduction
- low-pass representation of the oscillator
- Generalization of the Leeson effect
- Analytical solutions and simulations

$$[1+\alpha(t)]\cos[\omega_0 t+\varphi(t)]$$

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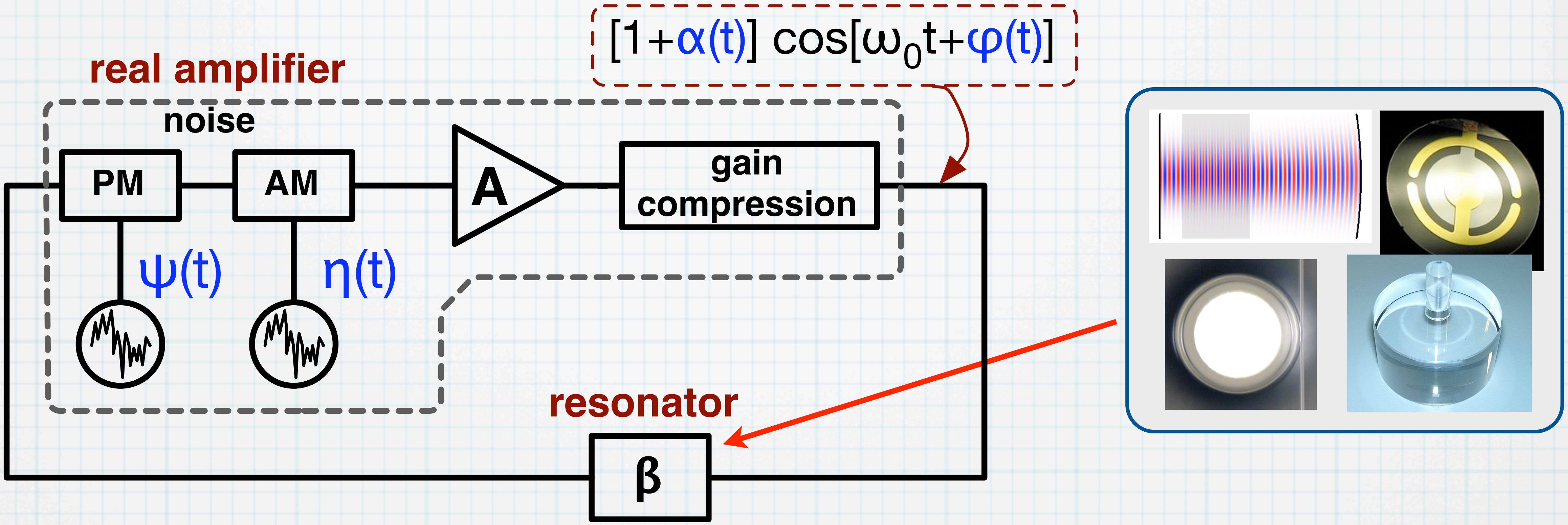
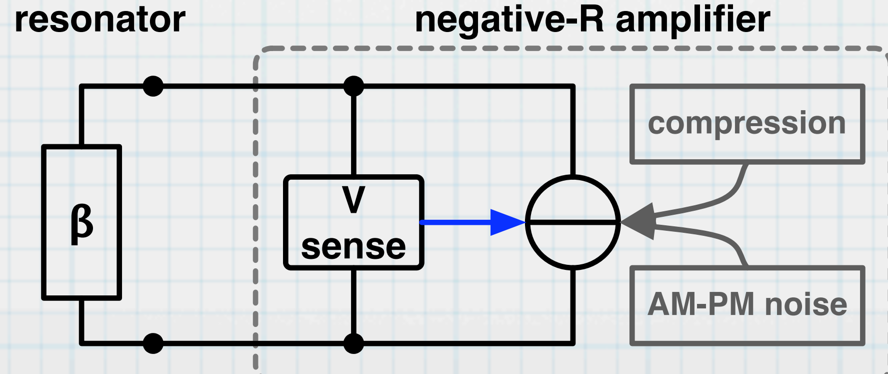
General oscillator model



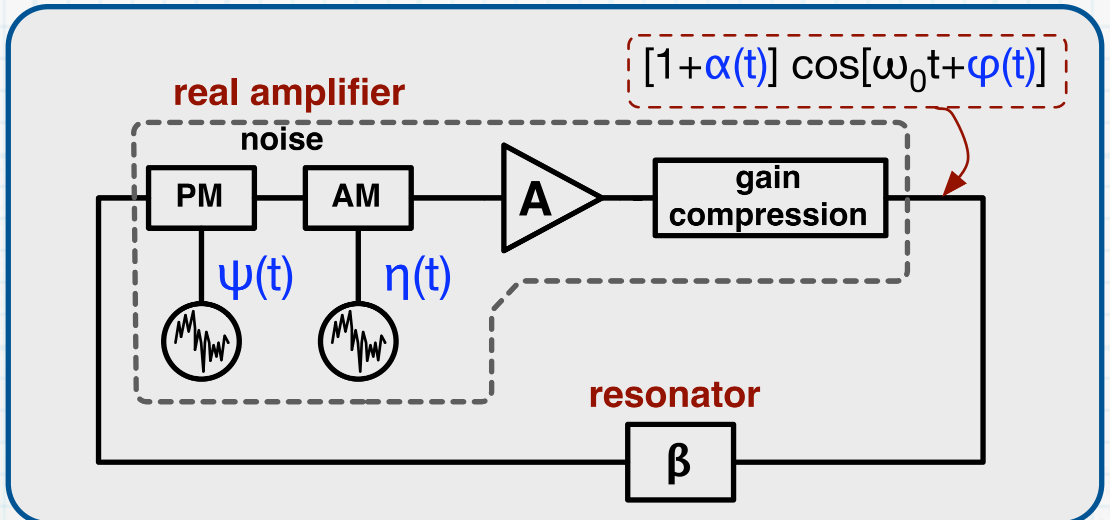
Barkhausen condition $A\beta = 1$ at ω_0

Let A fluctuate => $A \rightarrow (1+\eta)\exp(j\psi)$

The model also describes the negative-R oscillator



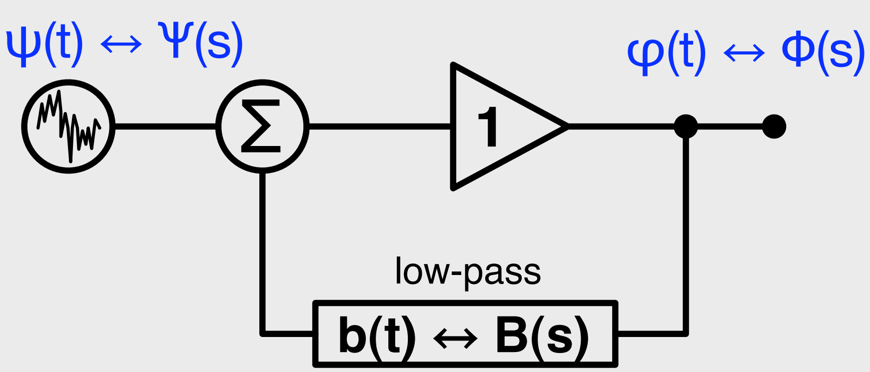
Low-pass representation of AM-PM noise



RF, µwaves or optics

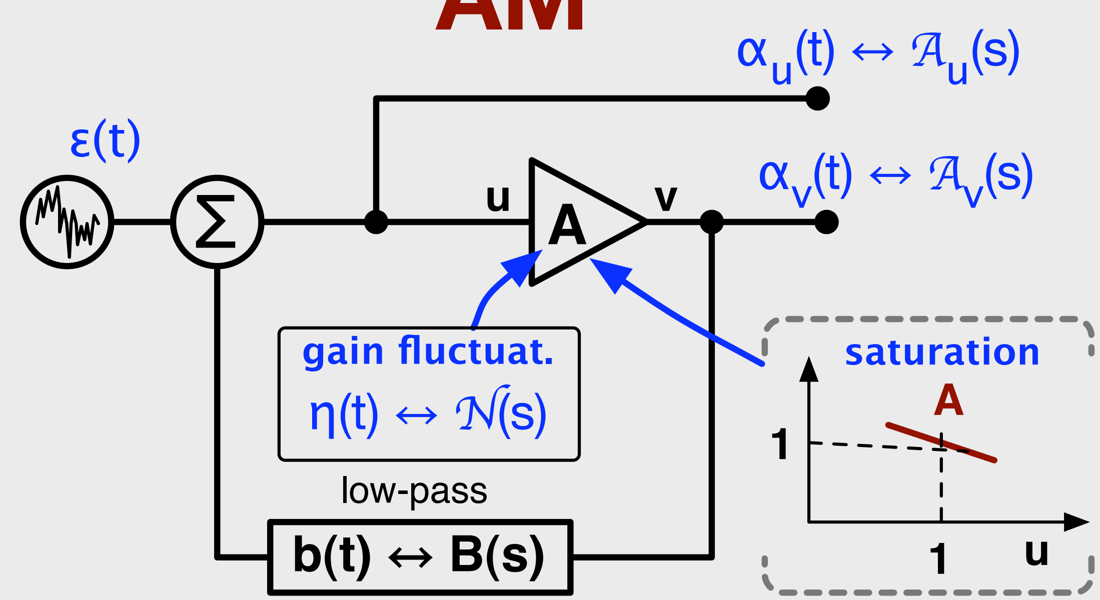
low-pass equivalent

PM



Leeson Effect

AM



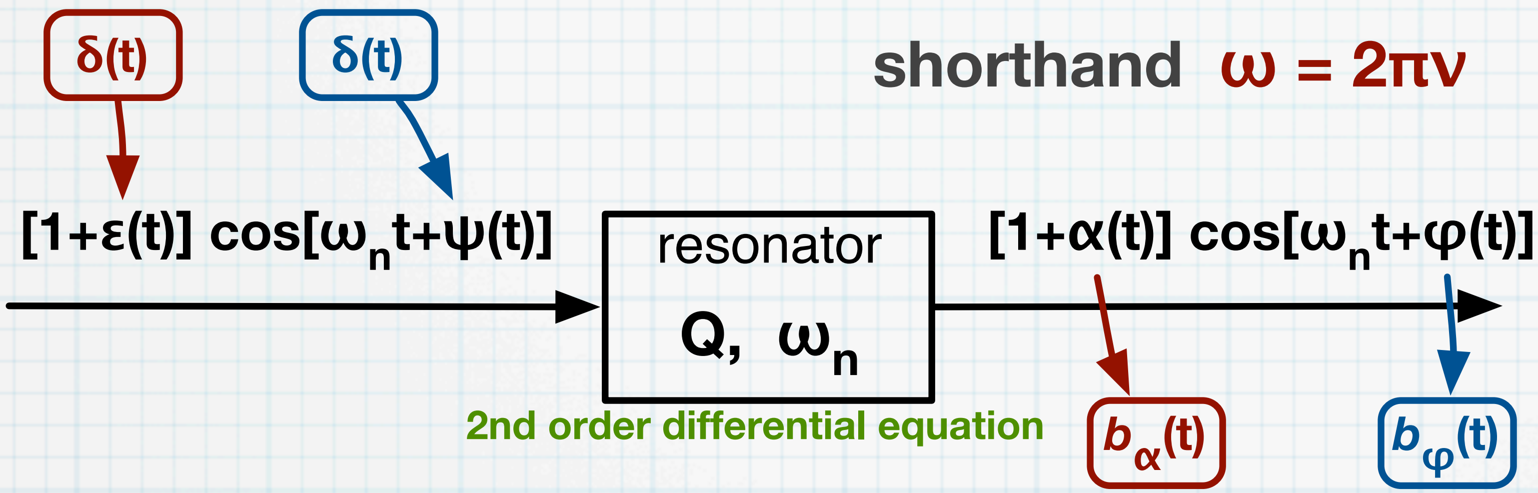
extension of the LE to AM noise

The amplifier
- **"copies" the input phase to the out**
- **adds phase noise**

The amplifier
- **compresses the amplitude**
- **adds amplitude noise**

Resonator impulse response

shorthand $\omega = 2\pi\nu$

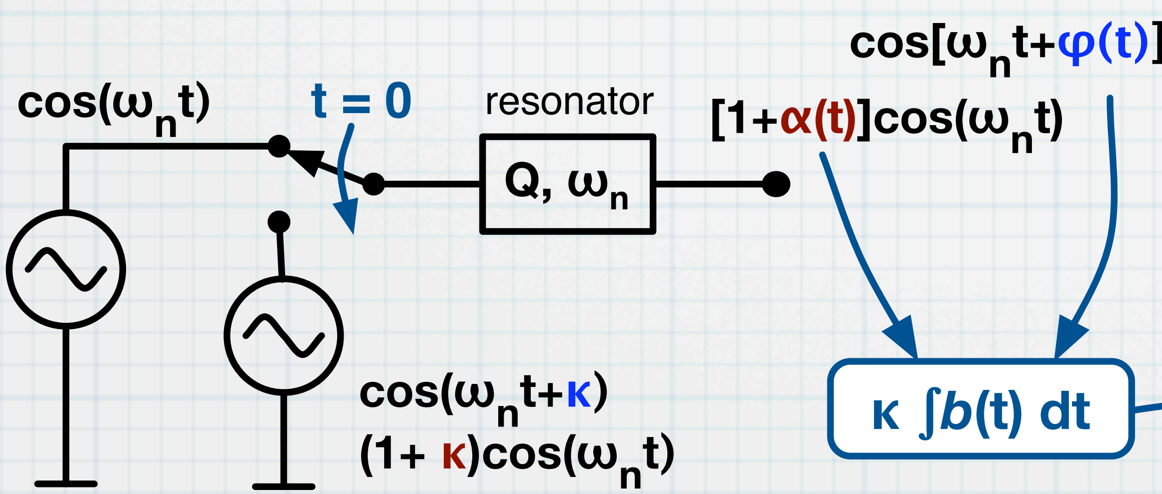


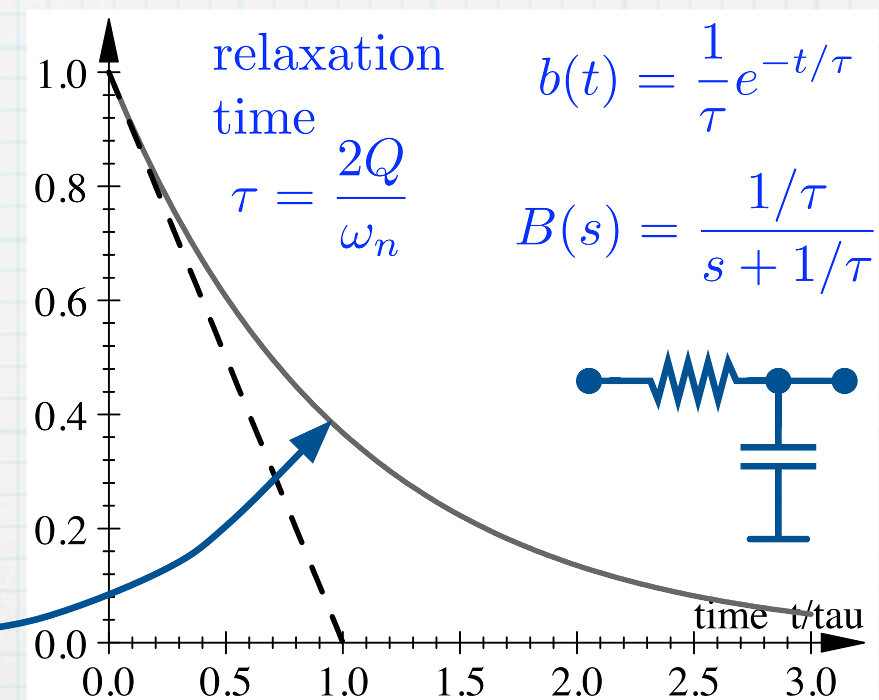
$\omega_n = \omega_0$

ω_n = natural frequency

ω_0 = oscillation frequency

set a small phase or amplitude step κ at t=0, and linearize for κ→0



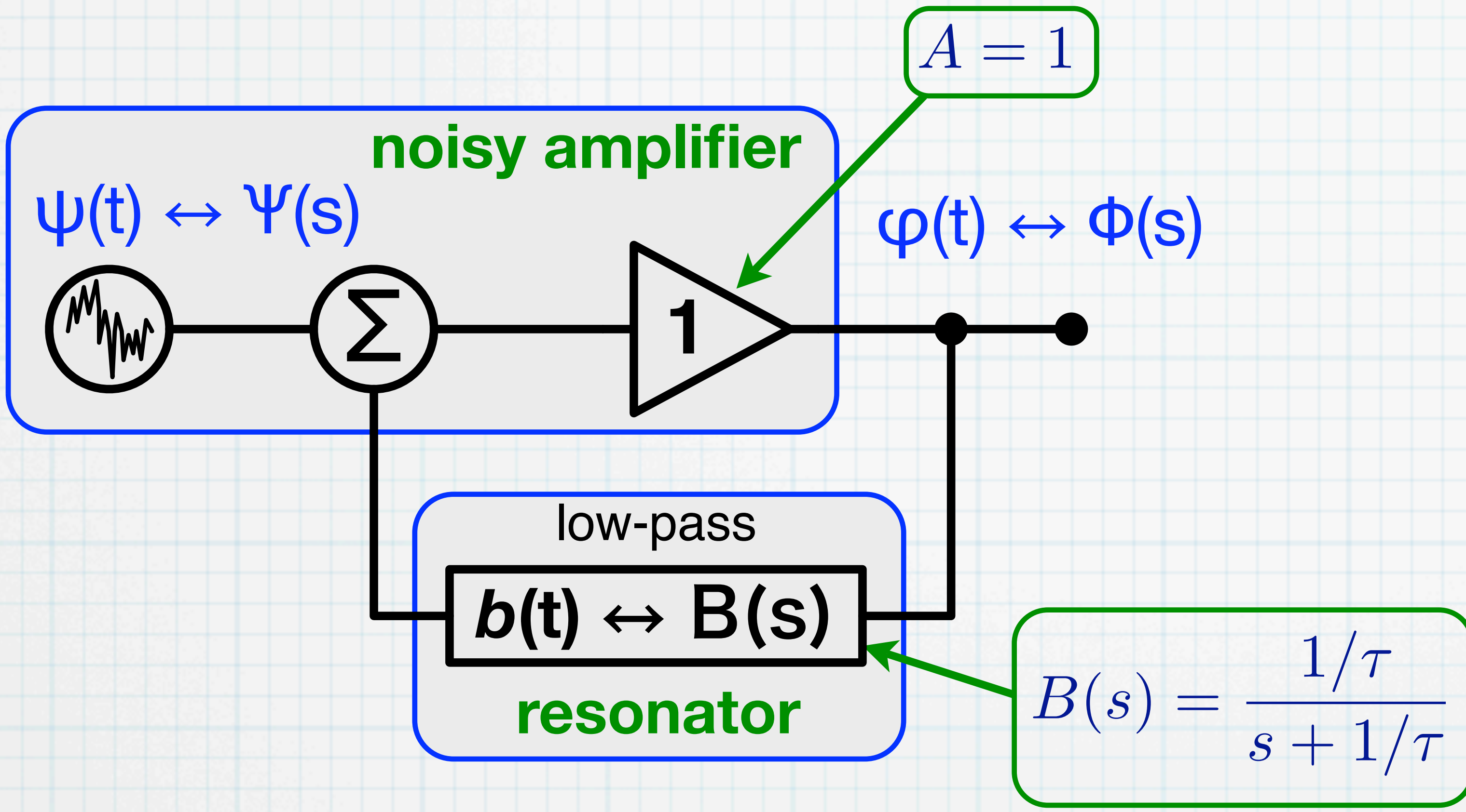


— boring mathematics omitted —

The low-pass equivalent of a resonator is a 1st order low-pass filter

The time constant is equal to the resonator relaxation time τ

Leeson effect

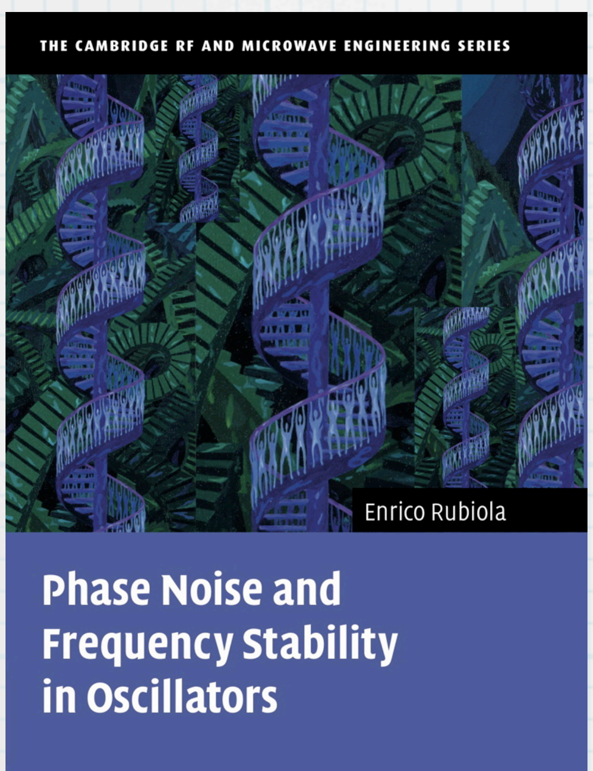


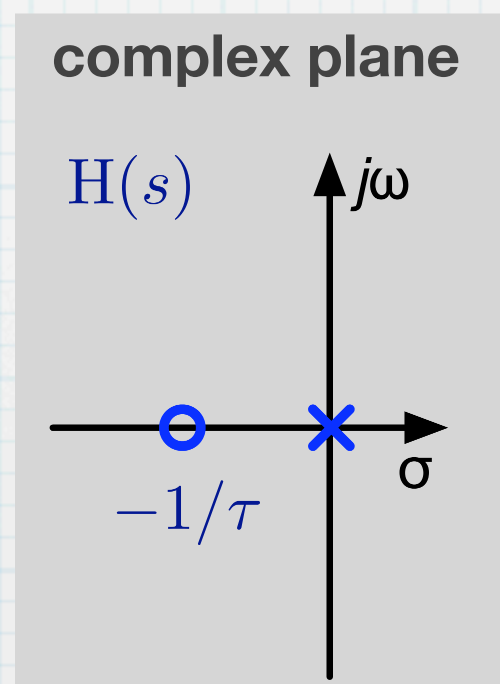
phase-noise transfer function

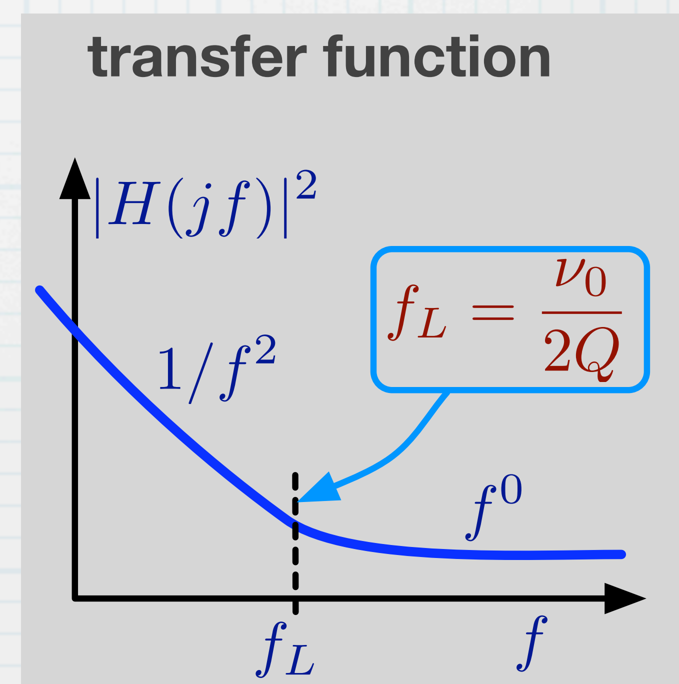
$$\mathrm{H}(s) = \frac{\Phi(s)}{\Psi(s)}$$ **definition**

$$\mathrm{H}(s) = \frac{1}{1 + AB(s)}$$ **general feedback theory**

$$\mathrm{H}(s) = \frac{1 + s\tau}{s\tau}$$ **Leeson effect**



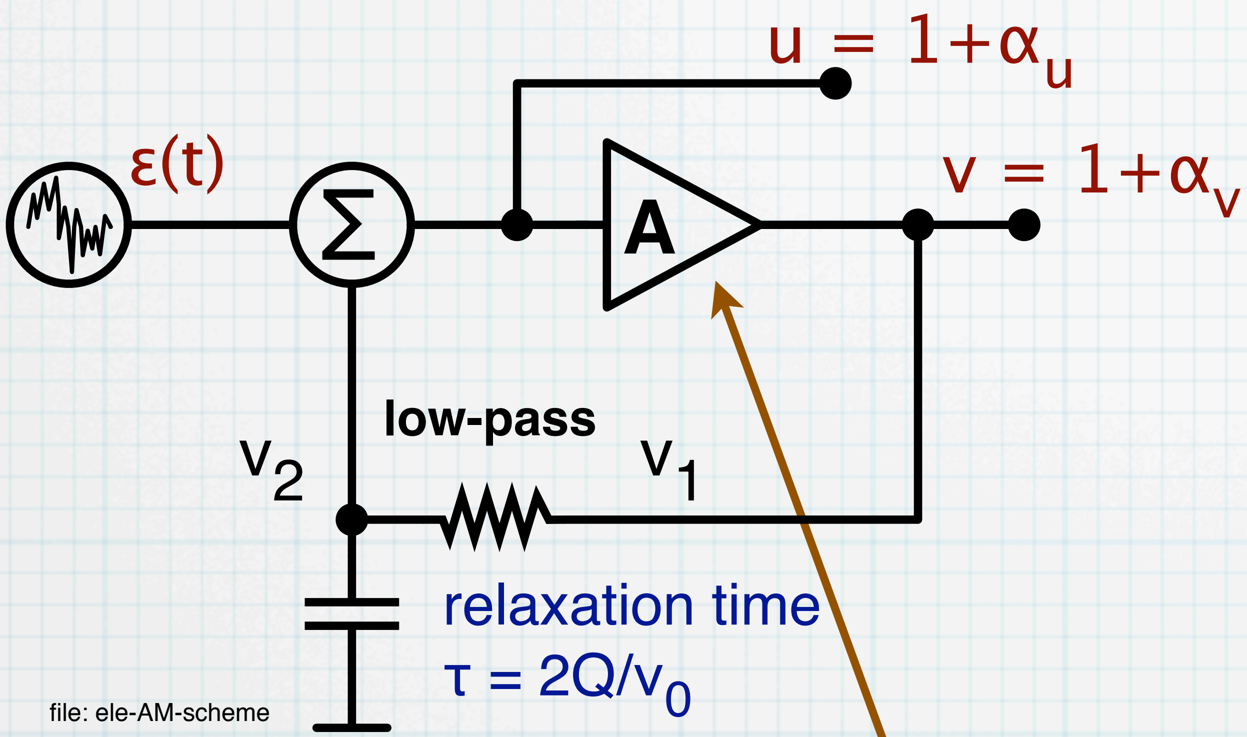




E. Rubiola, Phase noise and frequency stability in oscillators, Cambridge 2008

Low-pass model of amplitude (1)

First we need to relate the system restoring time τ_r to the relaxation time τ



simple feedback theory

$$u = \epsilon + v_2$$

$$v_2 = \frac{1}{\tau}\int (v_1 - v_2)\, dt$$

$$v_2 = u - \epsilon$$

$$v_1 = v = Au$$

$$u = \epsilon + \frac{1}{\tau}\int (A-1)u + \epsilon\, dt$$

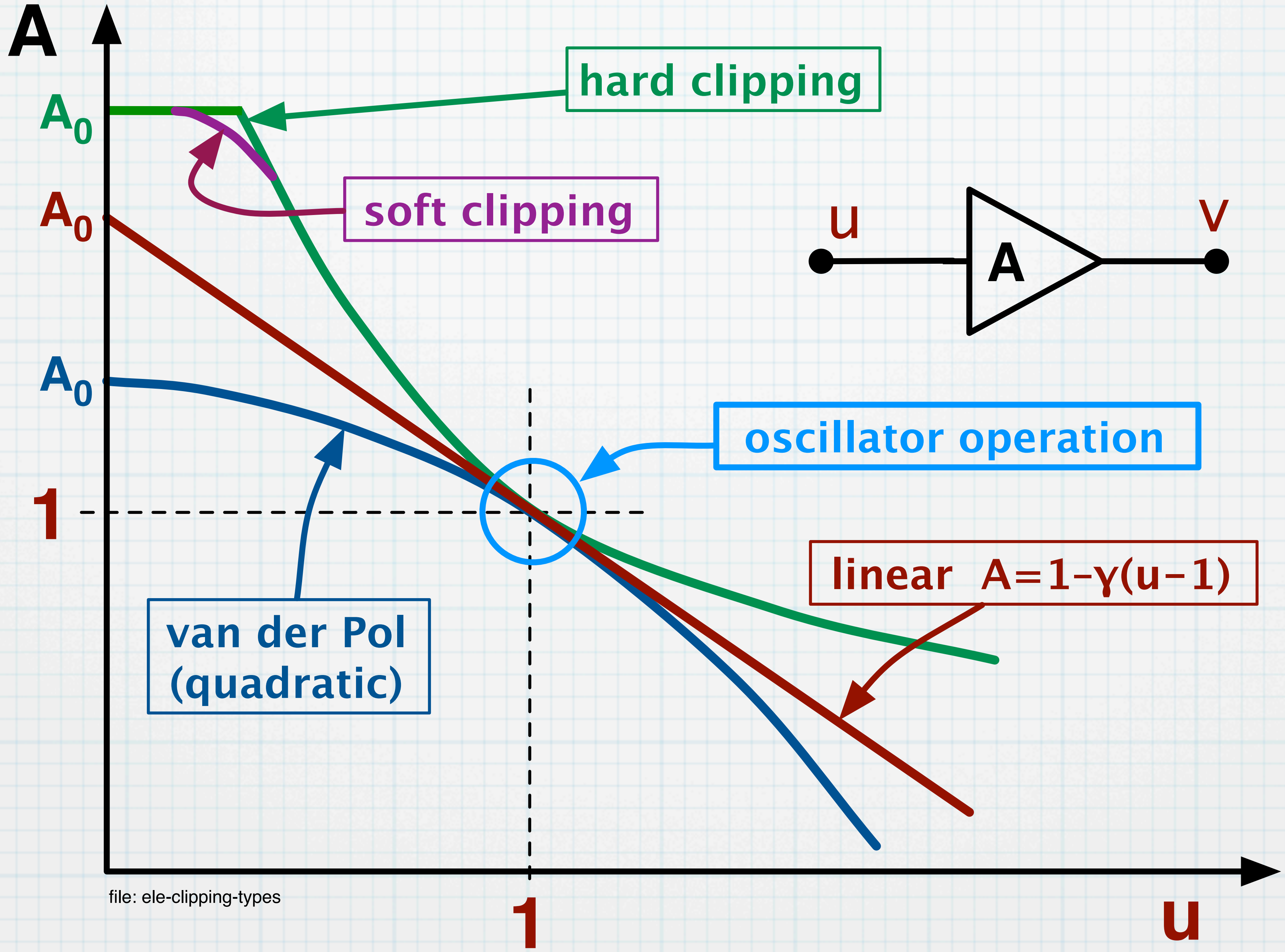
differential equation

$$\dot{u} - \frac{1}{\tau}\left(A - 1\right) u = \frac{1}{\tau}\epsilon + \dot{\epsilon}$$

Gain compression is necessary for the oscillation amplitude to be stable

The Laplace / Heaviside formalism cannot be used because the amplifier is non-linear

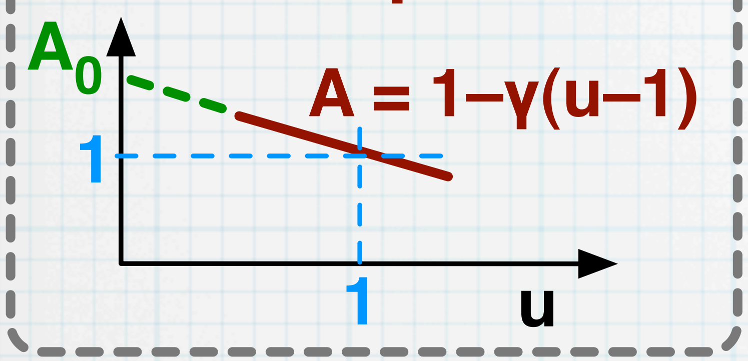
Common types of gain saturation



Gain compression is necessary for the oscillation amplitude to be stable

Low-pass model of amplitude (2)

Gain compression

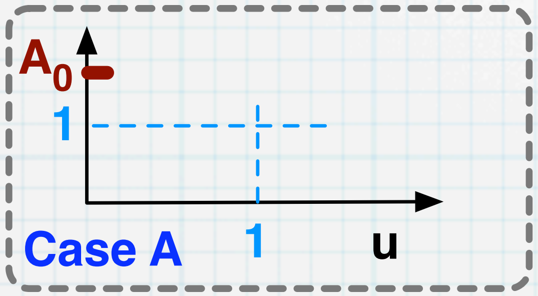


homogeneous differential equation

$$\dot{u} - \frac{1}{\tau}\,(A-1)\,u = 0$$

Three asymptotic cases

At low RF amplitude, let the gain be an arbitrary value denoted with A_0

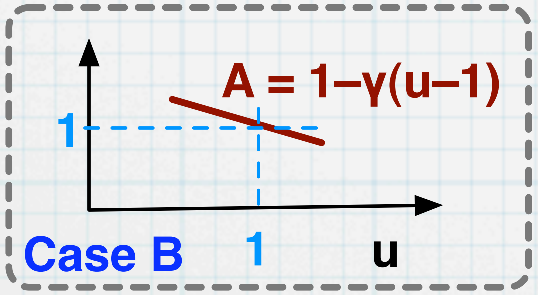


Startup: u→0, A → A_0 > 1

$$\dot{u} - \frac{1}{\tau}\,(A_0-1)\,u = 0 \quad\Rightarrow\quad \boxed{u = C_1 e^{(A_0-1)\,t/\tau}}$$

rising exponential

For small fluctuation of the stationary RF amplitude, the gain amplitude, the gain varies linearly with V

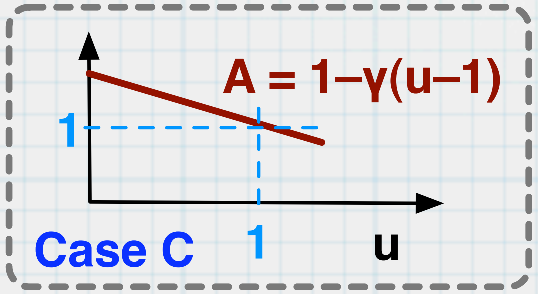


Regime: u→1, A = 1 – γ (u – 1)

$$\dot{u} + \frac{\gamma}{\tau}\,(u-1)\,u = 0 \quad\Rightarrow\quad \boxed{u = C_2 e^{-\gamma\,t/\tau}}$$

restoring time constant $\tau_r = \tau/\gamma$

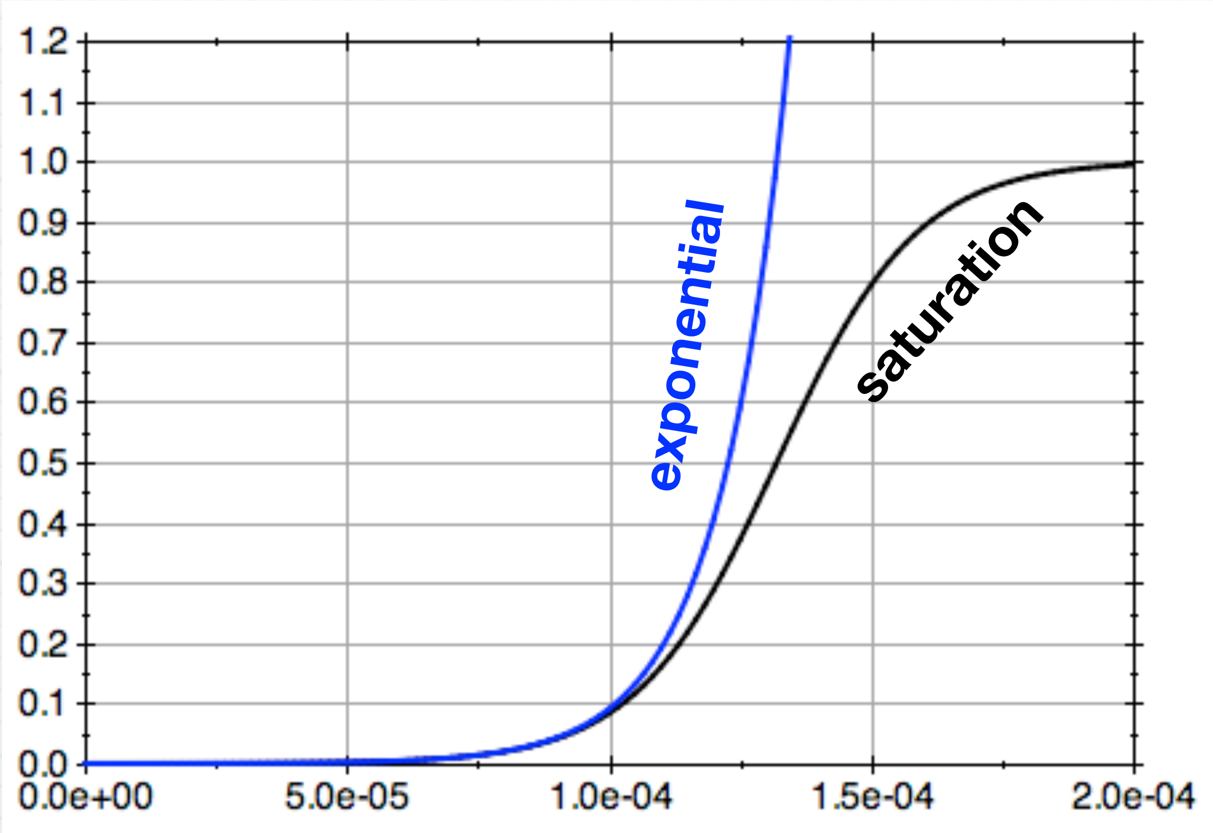
Simplification: the gain varies linearly with V in all the input range



Linear gain: A = 1 – γ (u – 1)

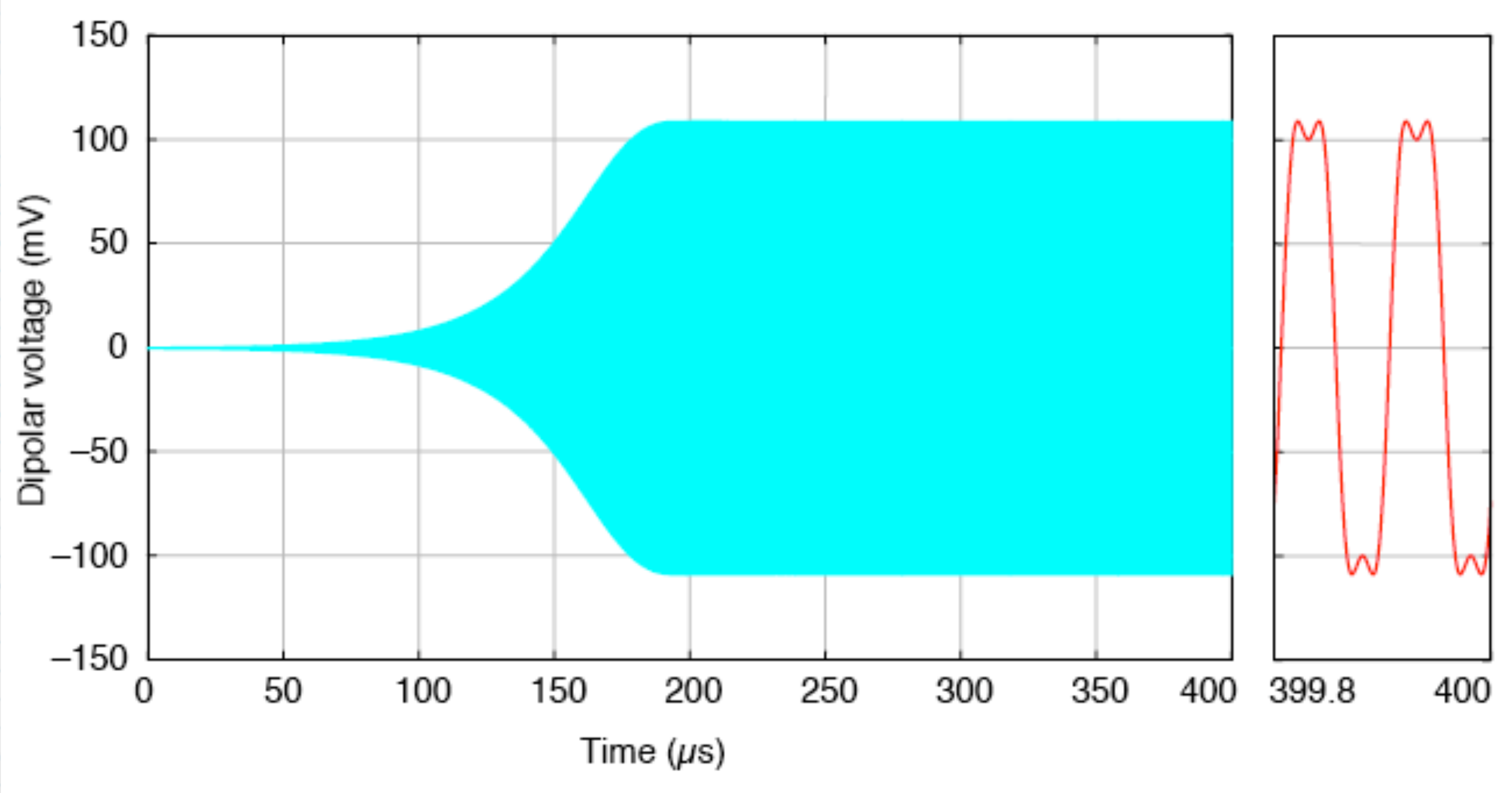
$$u = \frac{1}{\left(\frac{1}{u(0)} - 1\right) e^{-\gamma t/\tau} + 1}$$

Startup – analysis vs. simulation



analytical solution,
A = 1–γ(u–1)

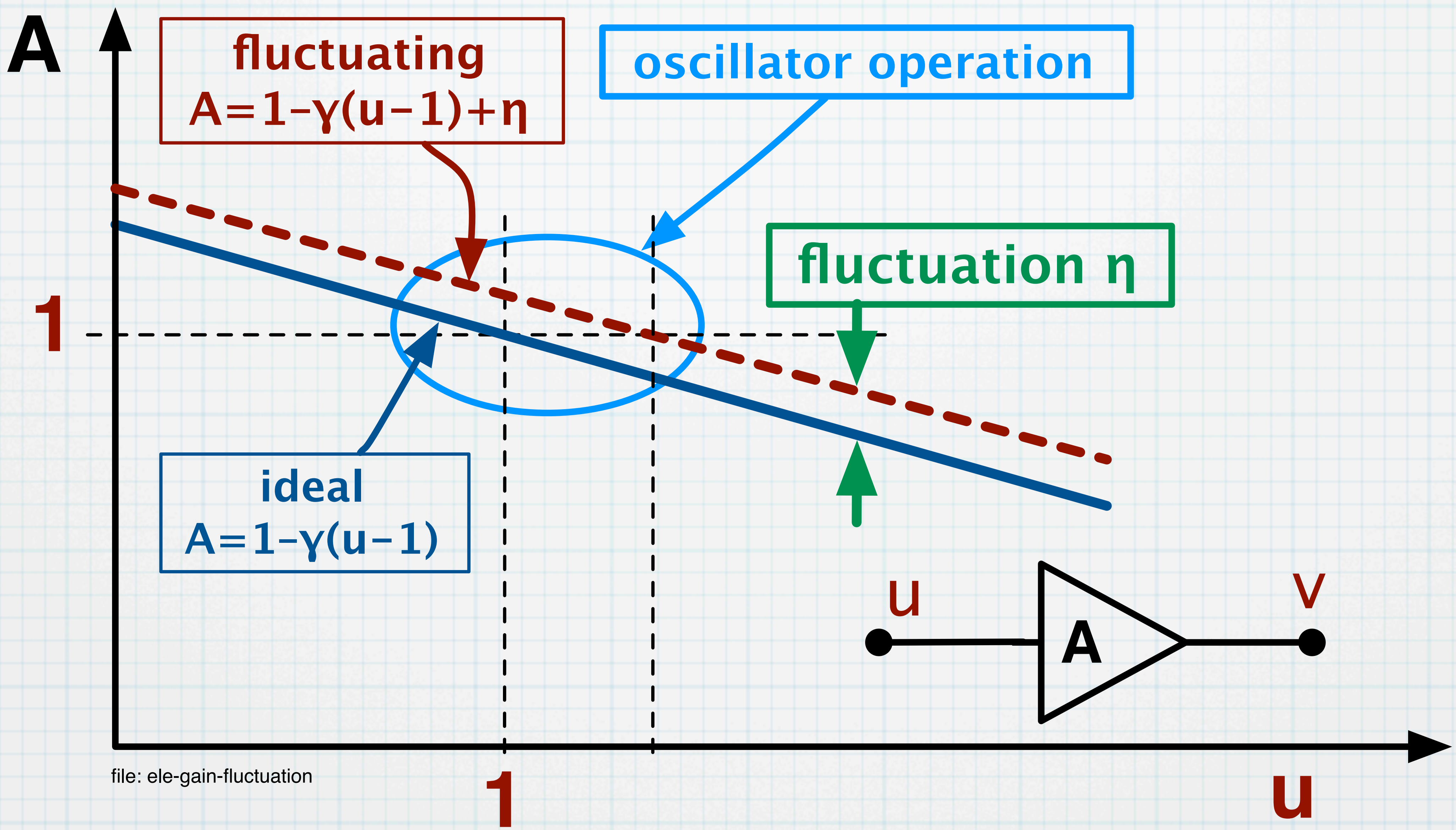
10 MHz oscillator
L = 1 mH
R = 125 Ω
Q ~ 503



van der Pol oscillator
simulated by RB

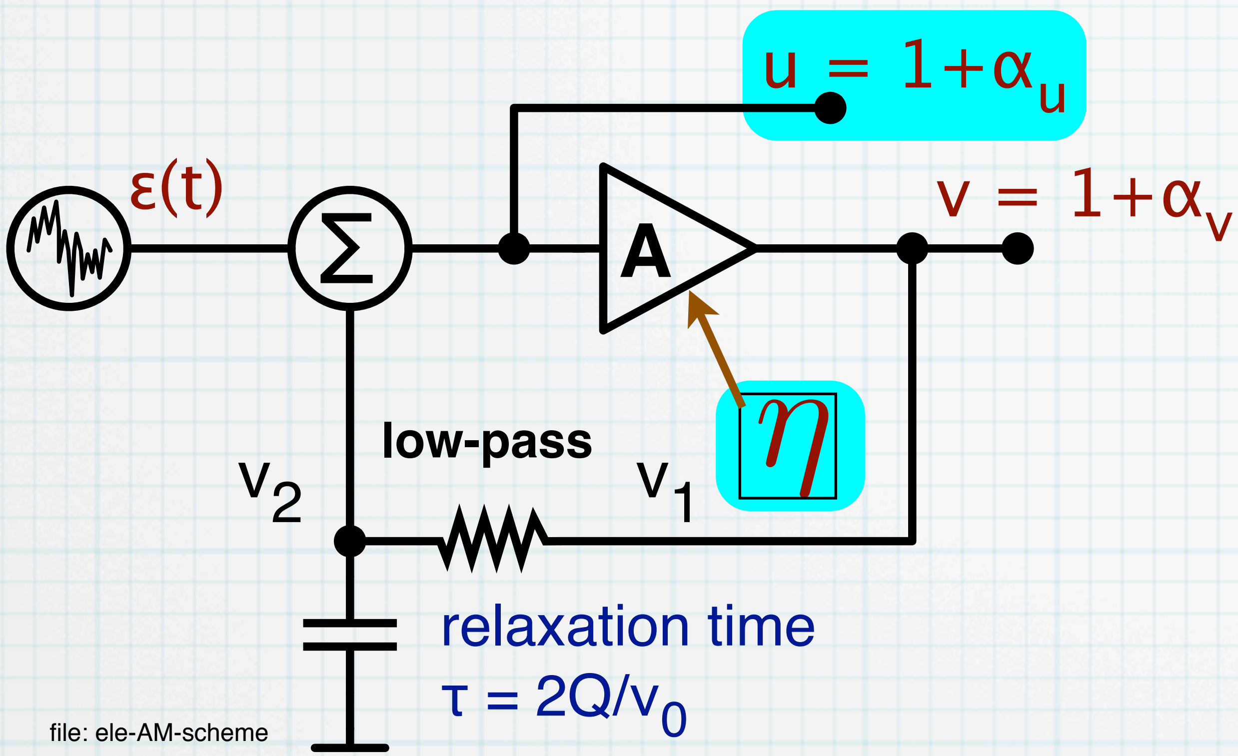
Rising exponential.
We find the same time constant –τ/γ

Gain fluctuations – definition



Gain compression is necessary for the oscillation amplitude to be stable

Gain fluctuations – output is u



$$\dot{u} = \frac{1}{\tau}(A-1)u$$ **non-linear equation**

$$A = 1 - \gamma(u-1) + \eta$$

$$\dot{u} + \frac{\gamma}{\tau}(u-1)u = \frac{\eta}{\tau}u$$ **linearization for low noise**

$$\dot{u} \to \dot{\alpha}_u, \quad (u-1) \to \alpha_u, \quad u \to 1, \quad u \to 1$$

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \frac{1}{\tau}\eta$$ **linearized equation**

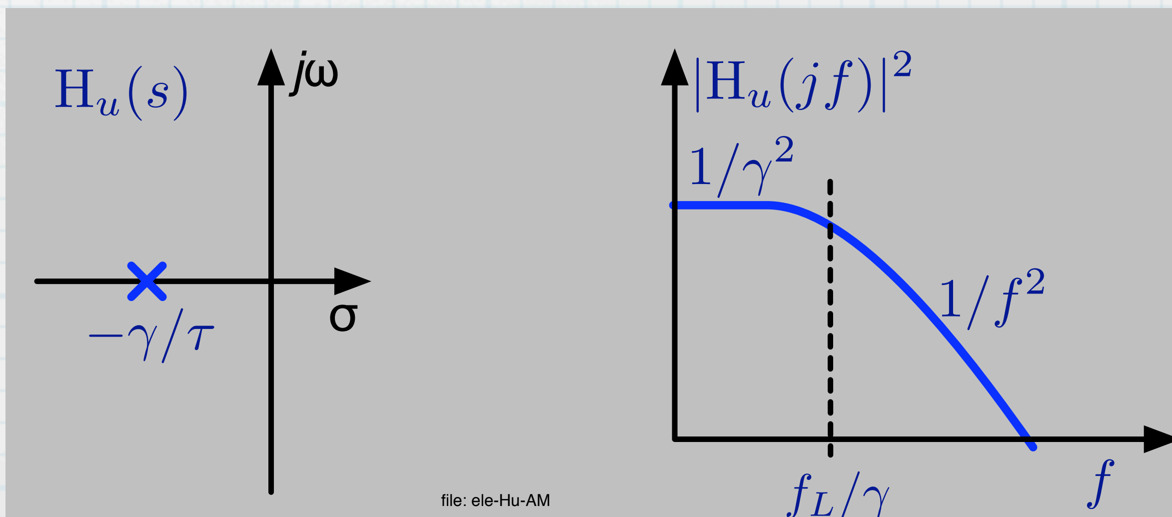
$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \frac{1}{\tau}\mathcal{N}(s)$$ **Laplace transform**

Linearize for low noise and use the Laplace transforms

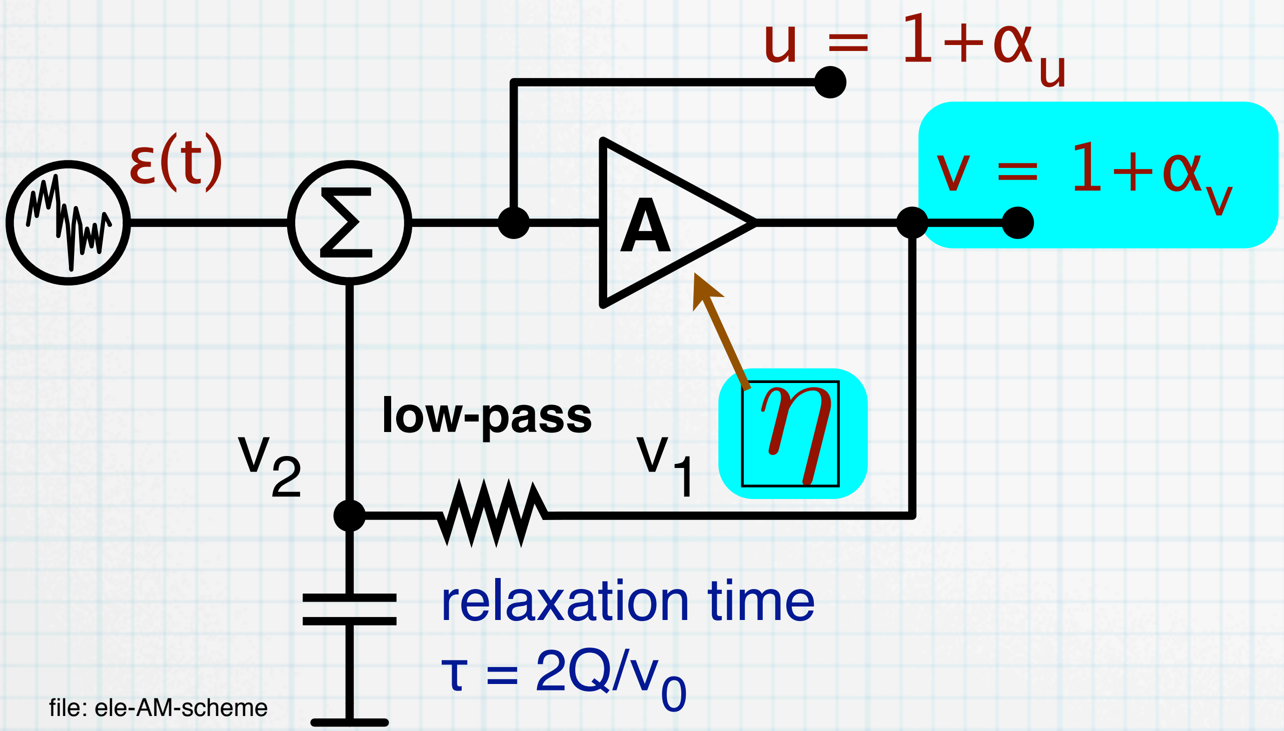
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \eta(t) \leftrightarrow \mathcal{N}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}_u(s) = \frac{1/\tau}{s+\gamma/\tau}$$ **result**



Gain fluctuations – output is v



$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_u(s)=\frac{1}{\tau}\mathcal{N}(s)$$ **starting equation**

$$\mathcal{A}_u(s)=\frac{\mathcal{A}_v(s)-\mathcal{N}(s)}{1-\gamma}$$

$$\left(s+\frac{\gamma}{\tau}\right)\mathcal{A}_v(s)=\left(s+\frac{1}{\tau}\right)\mathcal{N}(s)$$

$$\mathrm{H}(s)=\frac{\mathcal{A}_v(s)}{\mathcal{N}(s)}$$ **definition**

$$\mathrm{H}(s)=\frac{s+1/\tau}{s+\gamma/\tau}$$ **result**

boring algebra relates α_v to α_u

$$v=Au$$

$$A=-\gamma(u-1)+1+\eta$$

$$v=[-\gamma(u-1)+1+\eta]\,u$$

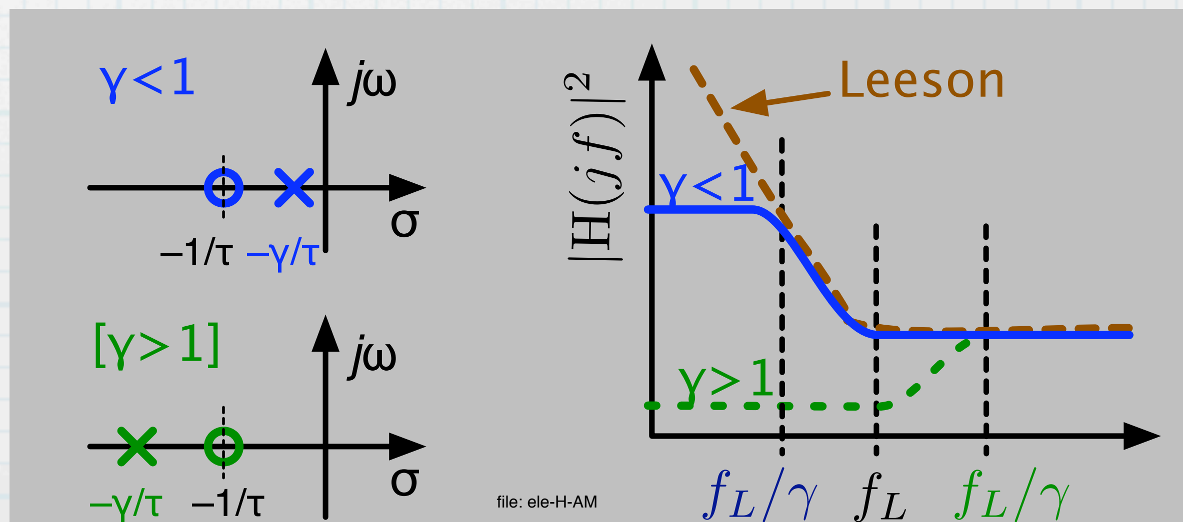
$$v=[-\gamma\alpha_u+1+\eta]\,[1+\alpha_u]$$

$$\cancel{1}+\alpha_v=\cancel{1}+\eta-\gamma\alpha_u+\alpha_u-\cancel{\alpha_u\eta}-\cancel{\gamma\alpha_u^2}$$

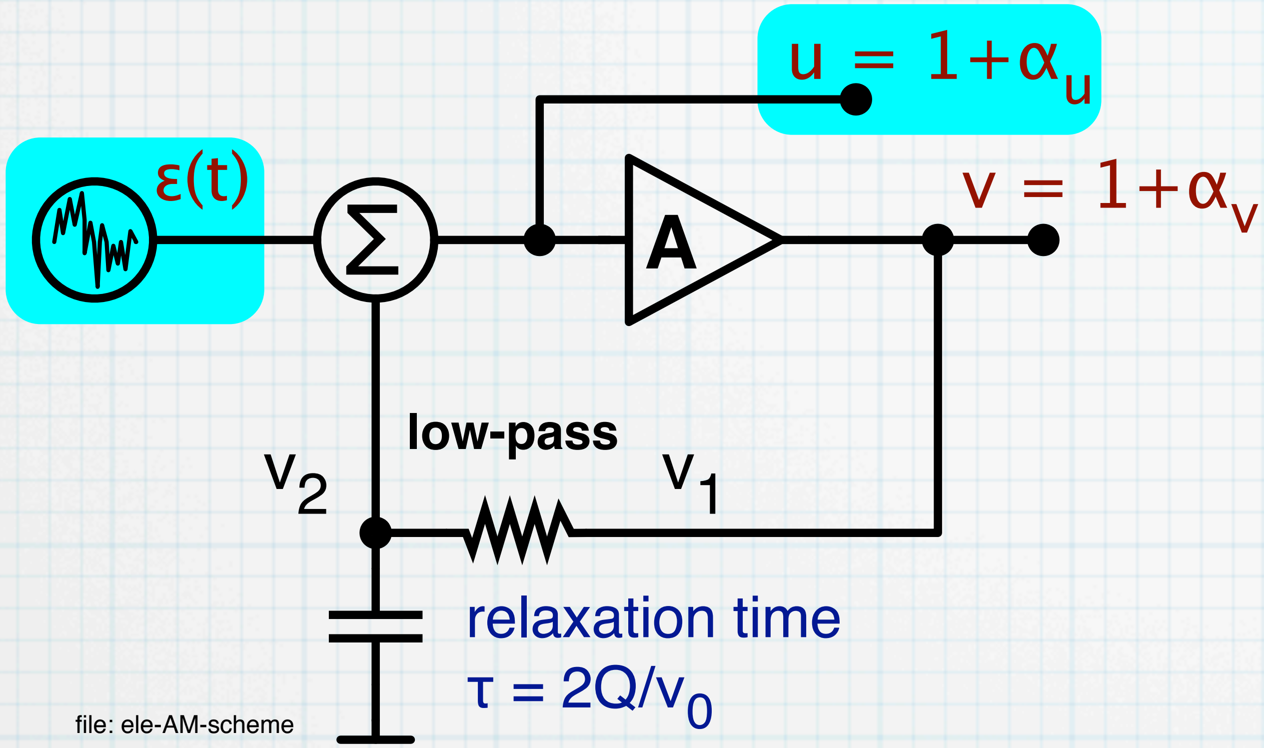
linearization for low noise

$$\alpha_v=(1-\gamma)\alpha_u+\eta$$

$$\alpha_u=\frac{\alpha_v-\eta}{1-\gamma}$$



Additive noise – output is u



$$\dot{u} = \frac{1}{\tau}(A-1)u + \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

$$A = 1 - \gamma(u-1)$$

non-linear equation

$$\dot{u} + \frac{\gamma}{\tau}\underbrace{(u-1)}_{\alpha_u}\underbrace{u}_{1} = \dot{\epsilon} + \frac{1}{\tau}\epsilon \qquad \dot{u} \to \dot{\alpha}_u$$

lineariz. for low noise

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

linearized equation

$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \left(s + \frac{1}{\tau}\right)\mathcal{E}(s)$$

Laplace transform

Linearize for low noise and use the Laplace transforms

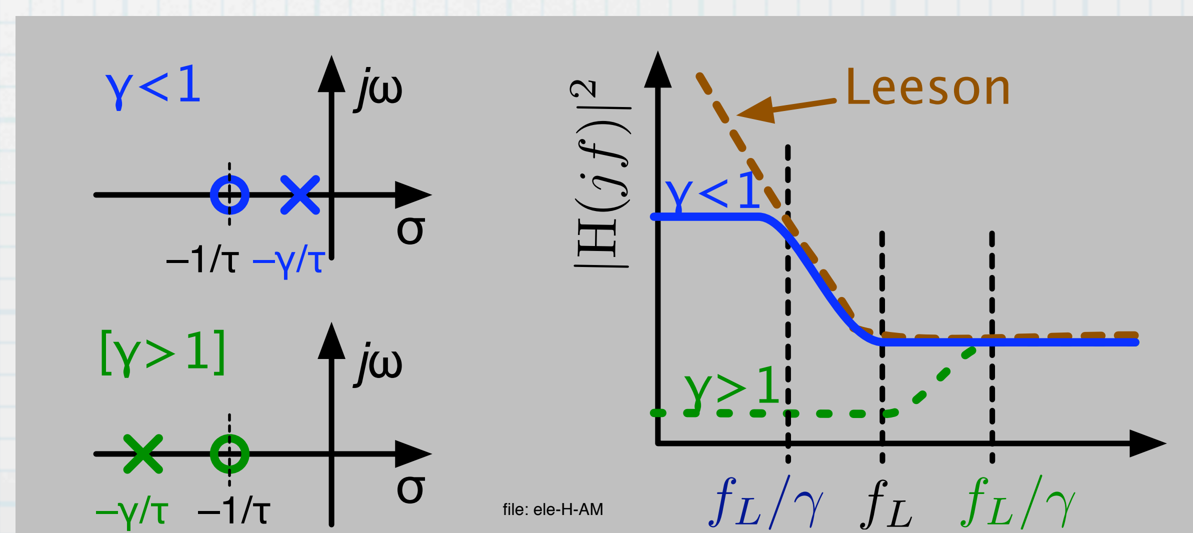
$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \quad \text{and} \quad \epsilon(t) \leftrightarrow \mathcal{E}(s)$$

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{E}(s)}$$

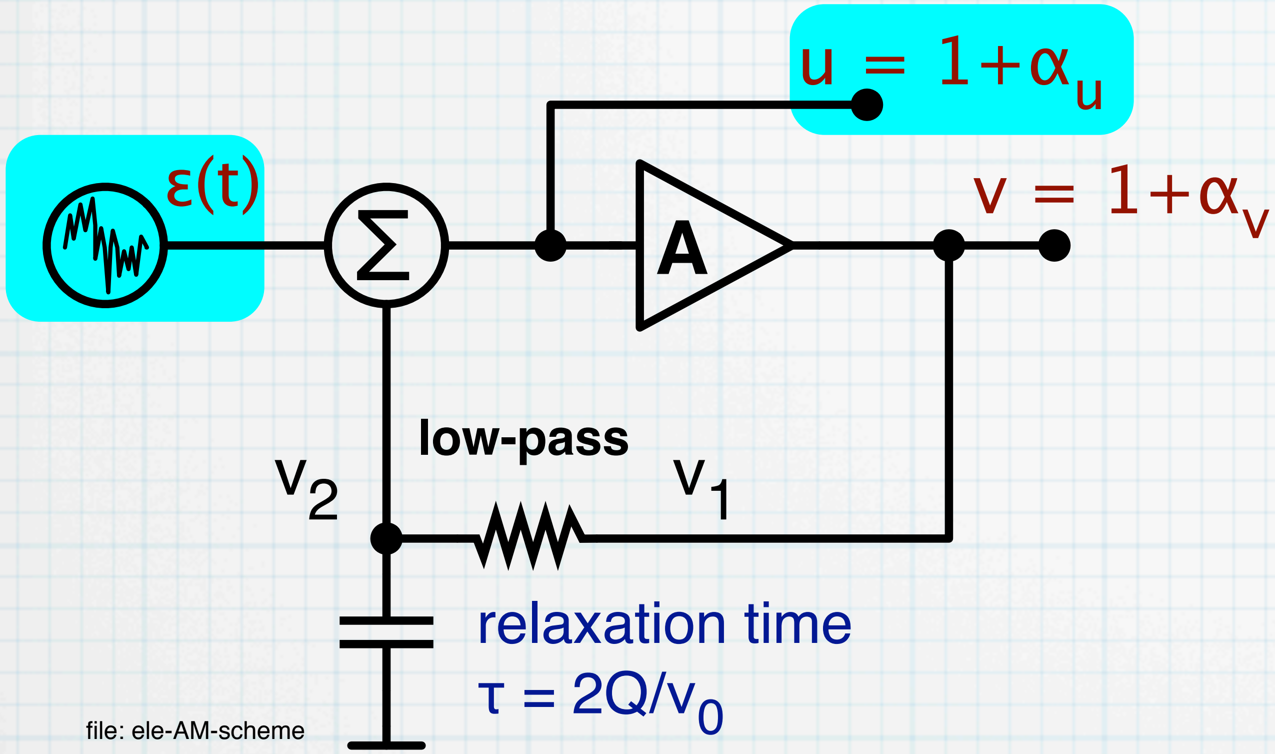
definition

$$\mathrm{H}_u(s) = \frac{s + 1/\tau}{s + \gamma/\tau}$$

result



Additive noise – output is v



$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

linearized equation

$$\alpha_u = \alpha_v/(1-\gamma)$$

$$\frac{1}{1-\gamma}\left(\dot{\alpha}_v + \frac{\gamma}{\tau}\alpha_v\right) = \dot{\epsilon} + \frac{1}{\tau}\epsilon$$

$$\frac{1}{1-\gamma}\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_v(s) = \left(s + \frac{1}{\tau}\right)\mathcal{E}(s)$$

Laplace transform

$$\mathrm{H}(s) = \frac{\mathcal{A}_v(s)}{\mathcal{E}(s)}$$

definition

$$\mathrm{H}(s) = (1-\gamma)\,\frac{s+1/\tau}{s+\gamma/\tau}$$

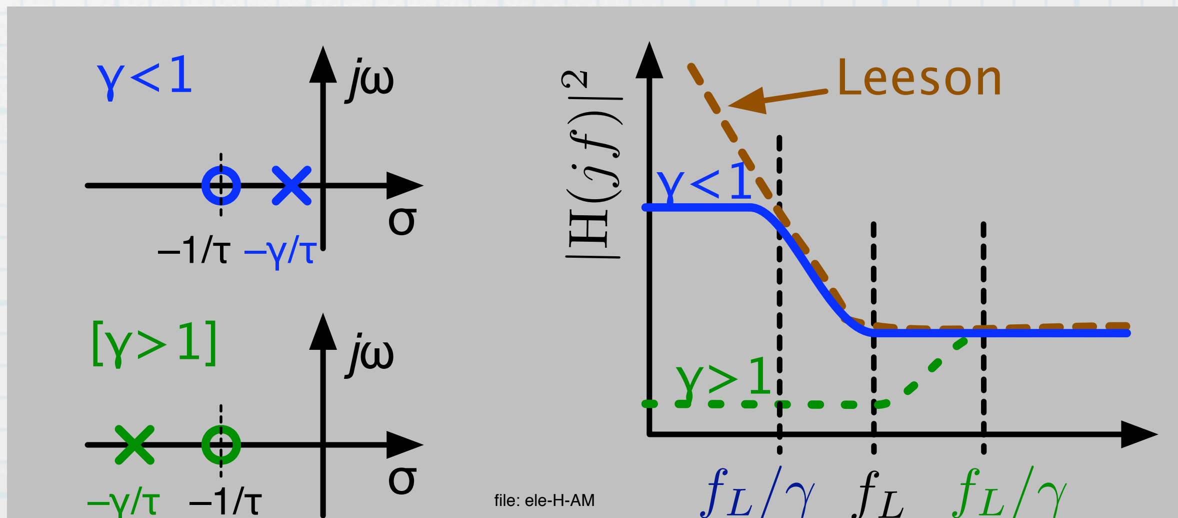
result

boring algebra relates α' to α

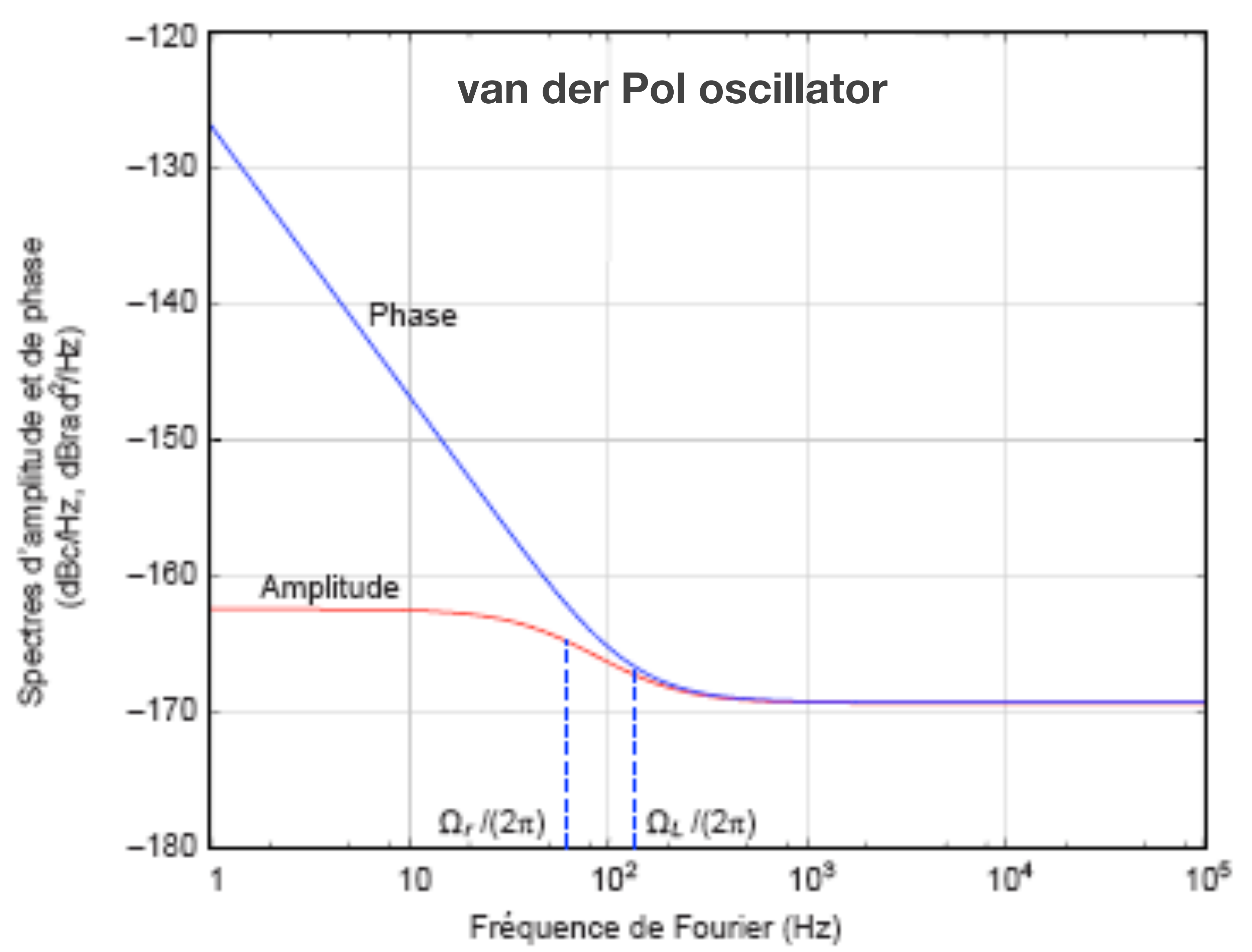
$$v = Au$$
$$A = 1 - \gamma(u-1)$$
$$v = [1-\gamma(u-1)]\,u$$
$$1+\alpha_v = [1-\gamma\alpha_u]\,[1+\alpha_u]$$
$$\cancel{1}+\alpha_v = \cancel{1}+\alpha_u - \gamma\alpha_u - \cancel{\gamma\alpha_u^2}$$

linearization for low noise

$$\alpha_v = (1-\gamma)\alpha_u$$
$$\alpha_u = \frac{\alpha_v}{1-\gamma}$$



Simulation (RB)



Analytic model and numeric simulation yield same time constants and slopes

Conclusions

- **Well-established framework, fully tested with PM noise**
- **Extension of the Leeson effect to the oscillator AM noise**
- **Simple analytical model and theory**
- **The analytical model (ER) is in agreement with the simulations (RB), developed independently**
- More emphasis is given to the analytical model *only* because
  - final formulae are easier to implement
  - the talk is given by ER
- **AM-FM coupling via Miller effect**
  (notice that other types of coupling exist)
- **Experiments starting soon. The sapphire oscillator gives easy access to (almost) all parameters**

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