High resolution frequency counters

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28 May 2008

Outline

1. Digital hardware
2. Basic counters
3. Microwave counters
4. Interpolation
   - time-interval amplifier
   - frequency vernier
   - time-to-voltage converter
   - multi-tap delay line
5. Basic statistics
6. Advanced statistics

home page http://rubiola.org
1. How to compare the input signal \((f, T, \tau)\) to the frequency standard
2. How to avoid degradation of the standard's metrological performance
1 – Digital hardware
**BASIC CELL**

\[ N \text{ pulses} \quad N = \frac{f}{T} \]

Count the number \( N \) of pulses \((\pm 1)\)

\[ N = \frac{f}{T} \]

Reference \( \frac{f}{T} \)

Measurement \( \frac{T}{f} \)

Resolution \[ \frac{S_T}{f} = \frac{1}{2N} \]

\[ \frac{S_f}{f} = \frac{1}{N} \]

The resolution is limited by the measurement time \( T \), and by the maximum switching frequency

200 MHz typ. \( \rightarrow \) resolution 5 ms

**Higher Resolution \( \rightarrow \) Interpolation**
TRIGGER

\[ V \xrightarrow{\text{ANALOG}} \xrightarrow{\text{PULSE}} \]

TRIVIAL FUNCTIONS
- 50 \Omega, AC/DC
- level, slope

TRIGGER \rightarrow fast device
- high slope

\[ \text{fluctuation} \quad \delta v = \frac{\delta V}{dv/\delta t} \]

HYSTERESIS
\( \text{threshold} \)

IN \rightarrow OUT

\( \text{first crossing of a random process is a difficult mathematical problem} \)
The SR flip-flop is a dual-button switch.

The D-type flip-flop samples the input D during the rising edge of the clock, and copies D → Q.
COUNTER

Counts the number of rising edges of the clock

E → enable
R → Reset

DIVIDER

Counts modulo M from 0 to M-1

A counter and a divider are almost the same thing, often a counter has a sufficiently large number of bits, for it does not overflow.
2 – Basic counters
TIME INTERVAL COUNTER

\[ \text{start} \quad \text{TRIG} \quad S \quad Q \quad \text{counter} \quad E \quad \text{stop} \quad \text{TRIG} \quad R \quad T_x \]

Start
Stop

E

\[ \text{error} \ 0/-1 \quad \text{error} \ 0/+1 \]

\[ T_x = N_c T_c = N_c / f_c \]

Estimated \( T_x \)

Resolution <-> Quantization

\[ \delta N_c = 1 \quad (\pm 1) \]

\[ \frac{\delta T_x}{T_x} = \frac{1}{N_c} \quad \leftrightarrow \quad \delta T_x = T_c \]

Use the highest possible \( f_c \)
INTERPOLATION PROBLEM

\[ T_x = N_c T_c + T_a - T_b \]

Measure \( N_c T_c, \) \( T_a, \) and \( T_b \)

RESOLUTION \( \delta T_x = \delta T_a + \delta T_b \) \( \text{or} \) \( \sqrt{(\delta T_a)^2 + (\delta T_b)^2} \)

ACCURACY \( \delta T_a \rightarrow \frac{\delta T_x}{T_x} = \frac{\delta T_a}{T_a} \) because \( T_a \ll T_x \)

relative accuracy \( \delta T_a/T_a \) is not (very) critical
FREQUENCY COUNTER

\[ N_x T_x = N_c T_c \]

Estimate \( f_x = 1/T_x \) \( \Rightarrow f_x = \frac{N_x}{N_c} f_c \)

QUANTIZATION \( \leftrightarrow \) \( \delta N_x = 1 \)

\[ \frac{\delta f_x}{f_x} = \frac{1}{N_x} \quad \leftrightarrow \quad \frac{\delta f_x}{f_x} = \frac{1}{f_x T_m} \]

- Poor resolution at low \( f_x \)
- Don't even think to interpolate the period \( T_x \) (variable in a wide range)
PERIOD COUNTER

\[
\frac{1}{f_x} \xrightarrow{\text{TRIG}} \frac{1}{f_x} \xrightarrow{\text{COUNTER}} \text{COUNTER} \xrightarrow{T_m = N_c T_x} \text{(reference gate time)}
\]

Set \( f_x \) according to the desired measurement time \( T_m \)

\[ N_x T_x = N_c T_c \]

 estimate \( f_x \) \[ f_x = \frac{N_x}{N_c} f_c \]

QUANTIZATION \[ \delta f_x = \frac{1}{N_c} \]

Choose \( f_x \) as the highest frequency for the available technology (or a round no. just below it)

fixed \( f_c \) \[ \delta f_x = \frac{1}{f_c T_m} \]
Practical measurement

nominal time $T_{\text{nom}} = N'_c T_c$

measurement time $T_m = N_x T_x \geq T_{\text{nom}}$

measurement equation: $N_x T_x = N'_c T_c$

or $N_x T_x = (N'_c \pm 1) T_c$, including quantization uncertainty
3 – Microwave counters
- a prescaler is a n-bit binary divider $\div 2^n$
- GaAs dividers work up to $\approx 20$ GHz
- reciprocal counter $\Rightarrow$ there is no resolution reduction
- Most microwave counters use the prescaler
Transfer-oscillator counter

- The transfer oscillator is a PLL
- Harmonics generation takes place inside the mixer
- Harmonics locking condition: \( N f_{vco} = f_x \)
- Frequency modulation \( \Delta f \) is used to identify \( N \) (a rather complex scheme, \( \times N \Rightarrow \Delta f \rightarrow N\Delta f \) )
Heterodyne counter

- Down-conversion: $f_b = |f_x - N f_c|$
- $f_b$ is in the range of a classical counter (100-200 MHz max)
- no resolution reduction in the case of a classical frequency counter (no need of reciprocal counter)
- Old scheme, nowadays used only in some special cases (frequency metrology)
4 – Interpolation
DUAL-SLOPE VOLTMEETER

\[
\text{charge conservation: } \frac{V_x}{RC} T_1 + \frac{V_2}{RC} T_2 = 0
\]

Estimate \( V_x \):

\[
V_x = -\frac{T_2}{T_1} V_2
\]
**TIME INTERVAL AMPLIFIER**

- **Clock**
- **PULSE GEN**
- **Vx**
- **Up-Down Counter**

**Equation:**
\[ \frac{T_a'}{T_a} = \frac{I_1}{I_2} \gg 1 \]

- **Amplify** Ta to increase resolution
- **Up/Down Counter** to measure Ta - Tb
MINOR TECHNICAL DETAIL

$E$  

clock  

$T_a$  

SOLUTION

$E$  

$T_a$  

$T_{a+c}$

$Ck$

$V_{ox}$

CHARGE  

DISCHARGE ALSO $T_b$

The current pulse is integrated over $T_{a+c}$ ($\to T_c$) once $T_c$ is added.

Measuring $T_a-T_b$, the added $T_c$ rubs out.

$T_a$ (and $T_b$) may be a very short time.

- ARBITRATION
- ERROR due to the rising edges.
EXAMPLE: NANOFAST 536 B (Smithsonian \textit{Astro. lab.})

Main clock \( f_c = 10 \text{ MHz} \) \( \rightarrow \) \( \delta T = T_c = 100 \text{ ms} \)

Time Interval amplifier \( \frac{T_1}{T_2} = 4000 \)

\( T_2' \in \{200 \mu s, 400 \mu s\} \)

aux. clock 20 MHz for the measurement of \( T_2' \)

\( \delta T_2' = T_c = 50 \mu s \) \( (1/20 \text{ MHz}) \)

\( \delta T_2 = \frac{T_2}{T_1} T_1' \)

\( \delta T_2 = \frac{1}{4000} \times 50 \mu s = 12.5 \mu s \)

The Nanofast 536 B counter is (was?) a part of the Mark IV system for Very Long Baseline Interferometry (VLBI). Early TTL technology.

Note: a pulse propagates in a cable at \( c' = \frac{2}{3} c \)

\( \delta T_2 \) is equivalent to a length of 2.5 mm
VERNIER CALIPER

\[ f_c = 10 \]

Main scale (fixed)

Aux. scale

\[ f_a = 10 \cdot \frac{10}{5} \]

\( \left( \frac{10 \text{ ticks}}{9 \text{ mm}} \right) \)

Read: 8.6

Main \[ \underline{\text{-----}} \] Aux.
FREQUENCY VERNIER

Note that $f_u < f_c$ here

- Main clock $f_c$ produces $N_u$ pulses
- Our clock $f_u$ is synched

Early coincidence occurs after $N_u$ pulses

$$T_a + N_u T_c = N_u T_v \rightarrow T_o = N_u [T_v - T_c]$$

$$T_v = \frac{M+1}{M} T_c$$

$$T_a = N_u \frac{1}{M} T_c$$

Resolution $\frac{f_{Hv}}{N_u} \geq 2 \Rightarrow \delta T_a = \frac{1}{M} T_c$
Synchronized Oscillator

Circuit Diagram:
- Gates and connections are depicted with labels for inputs and outputs.

Coincidence Detector

Waveform Diagram:
- Waveforms labeled as 'Early' and 'Late' with detected coincidence at certain points.
EXAMPLE: HP 5370A

\[ f_c = 200 \text{ MHz} \rightarrow \delta T_x = 5 \text{ ms} \]

(ECL Technology)

\[ M = 256 \rightarrow \delta T_a = \delta T_b = \frac{1}{256} \times 5 \text{ ms} = 19.5 \text{ ps} \]

(taken at 199.22 MHz)

It takes a max. of 257 cycles of \( f_c \) for the two clocks to coincide.

conversion time:

\[ 257 \times 5 \text{ ms} = 1.285 \text{ ms} \]

Light speed in cable \( \times 0.67c \)

\[ \delta T_c \leftrightarrow 8L = 4 \text{ mm} \]

(length)
The method is equivalent to the TF amplifier. Successive approx. conversion is faster than dual-slope.
EXAMPLE STANFORD SR 620

\[ f_c = 90 \text{ MHz} \]
\[ T_c = 1.1 \text{ ms} \]

Phase-locked to the 40 MHz reference.

ECL Technology

12-bit converter, 1 bit lost because of the extra \( T_c \)

11 bits

\[ \delta T_c = \frac{11 \text{ bits}}{2^{11}} = 5.4 \text{ ps} \]
Interpolation by sampling delayed copies of the clock or of the stop signal

The resolution is determined by the delay $\tau$, instead of by the toggling speed of the flip-flops.
Sampling circuits

(a) START  \[ \tau \]
STOP (Latching)

(b) START  \[ \tau \]
STOP (Sampling)

(c) START  \[ (Clock) \]
STOP
Ring Oscillator
used in PLL circuits for clock-frequency multiplication
5 – Basic statistics
ACCURACY AND PRECISION

ERROR BUDGET

Random errors (resolution)

- quantization
- buffer fluctuations
- noise
- short-term stability of IC

bias (systematic errors)

- channel asymmetry (TI)
- trigger level (TI)
- reference level (IC)

- calibration
  - gain
  - temperature

Old Hewlett Packard application notes
TRIGGER

The measured frequency/period is (almost) independent of the trigger threshold, and on slow fluctuations.

Start

Stop

Accuracy and stability of the trigger thresholds are critical parameters in time-interval measurements.

FILTER

\[ t_f = \frac{8v}{\text{d}v/\text{d}t} \quad \text{Voltage noise} \]

\[ \text{slew rate} \]
Quantization uncertainty

\[ \frac{1}{T_c} = \frac{T_c^2}{12} \]

\[ \sigma^2 = \frac{T_c^2}{12} \]

Example: 100 MHz clock

\[ T_x = 10 \text{ ns} \]

\[ \sigma = 2.9 \text{ ns} \]
DON'T BLAME THE TRIGGER

\[ 50 \Omega \rightarrow \text{INPUT ATTENU} \rightarrow -20 \text{ dB} \rightarrow \text{OVERLOAD PROTECTION} \rightarrow F = 1 \text{ dB} \]

\( B \approx 1 \text{ GHz} \)

Noise figure bandwidth

**THERMAL NOISE**

\[ \frac{V_{\text{NTR}}}{V_{\text{TH}^2}} \rightarrow 0.9 \text{ mV/VHz} \]

Boltzmann constant: \( k = 1.38 \times 10^{-23} \)

Temperature: \( T = 300 \text{ K} \)

Resistor: \( R = 50 \Omega \)

**THERMAL NOISE INTEGRATED OVER THE BANDWIDTH**

\[ \sqrt{B} \]

(HP 5370)

\( B = 225 \text{ MHz} \) \[ \rightarrow \] \( V_m = 13.5 \mu\text{V} \)

(SR 620)

\( B = 1.3 \text{ GHz} \) \[ \rightarrow \] \( V_m = 32.5 \mu\text{V} \)

Account for the loss: \( l = 20 \text{ dB} \)

(multiply by 10)

and for the noise figure: \( F = 1 \text{ dB} \)

(multiply by 1.12)

\( B = 225 \text{ MHz} \) \[ \rightarrow \] \( V_m = 150 \mu\text{V} \)

\( B = 1.3 \text{ GHz} \) \[ \rightarrow \] \( V_m = 355 \mu\text{V} \)
classical reciprocal counter (1)

period measurement (count the clock pulses) is preferred to frequency measurement (count the input pulses) because:

- it provides higher resolution in a given measurement time \( \tau \) (the clock frequency can be close to the maximum switching speed)
- interpolation (\( M \) is rational instead of integer) can be used to reduce the quantization (interpolators only work at a fixed frequency, thus at the clock freq.)
classical reciprocal counter (2)

\[ \tau = N T \]

measurement time

\[ \frac{1}{\tau} \]

weight

period \( T_{00} \)

\[ w_\Pi(t) = \begin{cases} 
1/\tau & 0 < t < \tau \\
0 & \text{elsewhere} 
\end{cases} \]

weight

\[ \mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Pi(t) \, dt \]

\( \Pi \) estimator

\[ w_\Pi(t) = \begin{cases} 
1/\tau & 0 < t < \tau \\
0 & \text{elsewhere} 
\end{cases} \]

weight

\[ \int_{-\infty}^{+\infty} w_\Pi(t) \, dt = 1 \]

normalization

variance

\[ \sigma_y^2 = \frac{2\sigma_x^2}{\tau^2} \]

classical variance
The enhanced-resolution counter is used to measure phase time $x$ (i.e., time jitter) with varying measurement times $\tau$. The variance of the measurement $\nu$ over $n$ measurements is given by:

$$\mathbb{E}\{\nu\} = \frac{1}{n} \sum_{i=0}^{n-1} \bar{\nu}_i \quad \bar{\nu}_i = N/\tau_i$$

The $\Lambda$ estimator is defined as:

$$\mathbb{E}\{\nu\} = \int_{-\infty}^{+\infty} \nu(t) w_\Lambda(t) \, dt$$

The weight function $w_\Lambda(t)$ is given by:

$$w_\Lambda(t) = \begin{cases} t/\tau & 0 < t < \tau \\ 2 - t/\tau & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases}$$

The normalization condition is:

$$\int_{-\infty}^{+\infty} w_\Lambda(t) \, dt = 1$$

For white noise, the autocorrelation function is a narrow pulse, about the inverse of the bandwidth. The variance is divided by $n$:

$$\sigma_y^2 = \frac{1}{n} \frac{2 \sigma_x^2}{\tau^2}$$

The limit $\tau_0 \to 0$ of the weight function is shown in the diagram.
actual formulae look like this

\[(\Pi) \quad \sigma_y = \frac{1}{\tau} \sqrt{2(\delta t)^2_{\text{trigger}}} + 2(\delta t)^2_{\text{interpolator}}\]

\[(\Lambda) \quad \sigma_y = \frac{1}{\tau \sqrt{n}} \sqrt{2(\delta t)^2_{\text{trigger}}} + 2(\delta t)^2_{\text{interpolator}}\]

\[n = \begin{cases} 
\nu_0 \tau & \nu_0 \leq \nu_I \\
\nu_I \tau & \nu_0 > \nu_I 
\end{cases}\]
## Understanding Technical Information

### Classical Reciprocal Counter

The slope of the classical variance tells the whole story.

\[
1/\tau^2 \quad \Rightarrow \quad \Pi \text{ estimator (classical reciprocal)}
\]

\[
1/\tau^3 \quad \Rightarrow \quad \Lambda \text{ estimator (enhanced-resolution)}
\]

### Enhanced-Resolution Counter

- **Low frequency:** full speed

  \[
  \sigma_y^2 = \frac{1}{n} \frac{2\sigma_x^2}{\tau^2}
  \]

  \[
  \tau_0 = T \quad \Rightarrow \quad n = \nu_{00}\tau
  \]

- **High frequency:** housekeeping takes time

  \[
  \sigma_y^2 = \frac{1}{\nu_{00}} \frac{2\sigma_x^2}{\tau^3}
  \]

  \[
  \tau_0 = DT \text{ with } D > 1 \quad \Rightarrow \quad n = \nu_{00}\tau
  \]

Look for formulae and plots in the instruction manual.
Stanford SRS-620

\[
\text{RMS resolution (in Hz)} = \frac{\text{frequency}}{\text{gate time}} \sqrt{(25 \text{ ps})^2 + \left[ \left( \text{short term stability} \right) \times (\text{gate time}) \right]^2 + 2 \times \left[ \text{trigger jitter} \right]^2}
\]

RMS resolution \( \sigma_\nu = \nu_0 \sigma_y \)

frequency \( \nu_0 \)

gate time \( \tau \)

---

Agilent 53132A

\[
\text{RMS resolution} = \left( \text{frequency or period} \right) \times \left[ \frac{4 \times \sqrt{(t_{\text{res}})^2 + 2 \times (\text{trigger error})^2}}{(\text{gate time}) \times \sqrt{\text{no. of samples}}} + \frac{t_{\text{jitter}}}{\text{gate time}} \right]
\]

t_{\text{res}} = 225 \text{ ps}

t_{\text{jitter}} = 3 \text{ ps}

number of samples = \( \begin{cases} (\text{gate time}) \times (\text{frequency}) & \text{for } f < 200 \text{ kHz} \\ (\text{gate time}) \times 2 \times 10^5 & \text{for } f \geq 200 \text{ kHz} \end{cases} \)

RMS resolution \( \sigma_\nu = \nu_0 \sigma_y \) or \( \sigma_T = T_0 \sigma_y \)

frequency \( \nu_0 \)

period \( T_0 \)

gate time \( \tau \)

no. of samples \( n = \begin{cases} \nu_0 \tau & \nu_0 < 200 \text{ kHz} \\ \tau \times 2 \times 10^5 & \nu_0 \geq 200 \text{ kHz} \end{cases} \)
5 – Advanced statistics
Allan variance

definition
\[ \sigma^2_y(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \bar{y}_{k+1} - \bar{y}_k \right]^2 \right\} \]

\[ \sigma^2_y(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \frac{1}{\tau} \int_{(k+1)\tau}^{(k+2)\tau} y(t) \, dt - \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} y(t) \, dt \right]^2 \right\} \]

wavelet-like variance
\[ \sigma^2_y(\tau) = \mathbb{E} \left\{ \left[ \int_{-\infty}^{+\infty} y(t) w_A(t) \, dt \right]^2 \right\} \]

\[ w_A = \begin{cases} \frac{-1}{\sqrt{2}\tau} & 0 < t < \tau \\ \frac{1}{\sqrt{2}\tau} & \tau < t < 2\tau \\ 0 & \text{elsewhere} \end{cases} \]

energy
\[ E\{w_A\} = \int_{-\infty}^{+\infty} w_A^2(t) \, dt = \frac{1}{\tau} \]

the Allan variance differs from a wavelet variance in the normalization on power, instead of on energy
modified Allan variance

definition
\[ \text{mod } \sigma^2_y(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{1}{\tau} \int_{(i+n)\tau_0}^{(i+2n)\tau_0} y(t) \, dt - \frac{1}{\tau} \int_{i\tau_0}^{(i+n)\tau_0} y(t) \, dt \right) \right]^2 \right\} \]

with \( \tau = n\tau_0 \).

wavelet-like variance
\[ \text{mod } \sigma^2_y(\tau) = \mathbb{E} \left\{ \left[ \int_{-\infty}^{+\infty} y(t) w_M(t) \, dt \right]^2 \right\} \]

\[ w_M = \begin{cases} 
-\frac{1}{\sqrt{2}\tau^2} t & 0 < t < \tau \\
\frac{1}{\sqrt{2}\tau^2} (2t - 3) & \tau < t < 2\tau \\
-\frac{1}{\sqrt{2}\tau^2} (t - 3) & 2\tau < t < 3\tau \\
0 & \text{elsewhere}
\end{cases} \]

energy
\[ E\{w_M\} = \int_{-\infty}^{+\infty} w_M^2(t) \, dt = \frac{1}{2\tau} \]

compare the energy
\[ E\{w_M\} = \frac{1}{2} E\{w_A\} \]

this explains why the mod Allan variance is always lower than the Allan variance
spectra and variances

<table>
<thead>
<tr>
<th>Noise Type</th>
<th>( S_y(f) )</th>
<th>Allan ( (\sigma_A^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>White PM</td>
<td>( h_2 f^2 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{3 f_H^2}{4 \pi^2} h_2 \tau^{-2} )</td>
<td>( \frac{3}{8 \pi^2} h_2 \tau^{-3} )</td>
</tr>
<tr>
<td>Flicker PM</td>
<td>( h_1 f )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{1.038+3 \ln(2 \pi f_H \tau)}{4 \pi^2} h_1 \tau^{-2} )</td>
<td>( \frac{3 \ln\left(\frac{256}{32}\right)}{8 \pi^2} \frac{3 \ln\left(\frac{256}{32}\right)}{3.37} h_1 \tau^{-2} )</td>
</tr>
<tr>
<td>White FM</td>
<td>( h_0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} h_0 \tau^{-1} )</td>
<td>( \frac{1}{4} h_0 \tau^{-1} )</td>
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<tr>
<td>Flicker FM</td>
<td>( h_{-1} f^{-1} )</td>
<td></td>
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<tr>
<td></td>
<td>( 2 \ln(2) h_{-1} )</td>
<td>( 2 \ln\left(\frac{33}{16}\right) )</td>
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<tr>
<td>Random Walk FM</td>
<td>( h_{-2} f^{-2} )</td>
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<td></td>
<td>( \frac{2}{3} \pi^2 h_{-2} \tau )</td>
<td>( \frac{11}{20} \pi^2 h_{-2} \tau )</td>
</tr>
<tr>
<td>Frequency Drift ( (\dot{y} = D_y) )</td>
<td>-</td>
<td>( \frac{1}{2} D_y^2 \tau^2 )</td>
</tr>
</tbody>
</table>

\( \nu_{00} \) is replaced with \( \nu_0 \) for consistency with the general literature

\( f_H \) is the high cutoff frequency, needed for the noise power to be finite

Π estimator $\rightarrow$ Allan variance

given a series of contiguous non-overlapped measures

\[
\sigma_y^2(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \bar{y}_{k+1} - \bar{y}_k \right]^2 \right\}
\]
by feeding a series of \( \Lambda \)-estimates of frequency in the formula of the Allan variance

\[
\sigma^2_y(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \overline{y}_{k+1} - \overline{y}_k \right]^2 \right\}
\]

as they were \( \Pi \)-estimates

one gets exactly the modified Allan variance!

\[
\text{mod } \sigma^2_y(\tau) = \mathbb{E} \left\{ \frac{1}{2} \left[ \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{1}{\tau} \int_{(i+n)\tau_0}^{(i+2n)\tau_0} y(t) \, dt - \frac{1}{\tau} \int_{i\tau_0}^{(i+n)\tau_0} y(t) \, dt \right) \right]^2 \right\}
\]

with \( \tau = n\tau_0 \).
There is a mistake in one of my articles: I believed that in the case of the Agilent counters the contiguous measures were overlapped. They are not.
by feeding a series of $\Lambda$-estimates of frequency in the formula of the Allan variance

$$\sigma_y^2(\tau) = E\left\{\frac{1}{2} \left[ \bar{y}_{k+1} - \bar{y}_k \right]^2 \right\}$$

as they were $\Pi$-estimates

one gets the triangular variance!

Conclusions

- The multi-tap delay-line interpolator is simple with modern FPGAs
- In frequency measurements, the Λ (triangular) estimator provides higher resolution
- The Λ estimator can not be used in single-event time-interval measurements
- Mistakes are around the corner if the counter inside is not well understood
- Some of the reported ideas are suitable to education laboratories and classroom works (I used a bicycle and milestones to demonstrate the Λ estimator)

Thanks to J. Dick (JPL), C. Greenhall (JPL), D. Howe (NIST) and M. Oxborrow (NPL) for discussions

To know more:
1 - rubiola.org, slides and articles
3 - Rev. of Sci. Instrum. vol. 76 no. 5, art.no. 054703, May 2005.

home page http://rubiola.org
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IEEE Trans. Nucl. Sci. 31(1) 167  (Feb. 1984)

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Construction manual, USA (unknown date)
FREQUENCY VERNIER

E.D. Peterson, R. P. Valentik, Time Interval measuring system employing digital means for coarse ambiguity resolution.
U.S. patent 3,218,553, Nov. 16, 1965

Hewlett Packard (now Agilent Technologies)
HP 5370A Operating and Service Manual

Hewlett Packard - Fundamentals of electronic counters, HP AN 100, 1997

Elindorado counter, I could not track the reference

TIME-TO-VOLTAGE CONVERTER

Stamford SR 620 operating manual.
Note: the full sheen, included, is difficult to interpret. No "theory of operation" chapter is present.