The AM noise mechanism in oscillators

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***Abstract*— The oscillator, inherently, turns the phase noise of its internal components into frequency noise, which results into a multiplication by $1/f^2$ in the phase-noise power spectral density. This phenomenon is known as the *Leeson effect*. This article extends the Leeson effect to the analysis of amplitude noise. This is done by analyzing the slow-varying complex envelope, after freezing the carrier. In the case of amplitude noise, the classical analysis based on the frequency-diomain transfer function is possible only after solving and linearizing the complete differential equation that describes the oscillator.**

I. Introduction

The oscillator noise, which in the absence of environmental or aging effect is cyclostationary, is best described as a baseband process, after freezing the periodic oscillation. The polar coordinate representation of the limit cycle splits the model of the oscillator into two subsystems, in which all signals are the amplitude and the phase of the main system, respectively. Putting things simply, these two subsystems are (almost) decoupled and all the nonlinearity goes to amplitude. This occurs because amplitude nonlinearity is necessary for the oscillation to be stationary. Conversely the phase, which ultimately is time, is not stretchable.

The baseband equivalent of a resonator, either for phase or amplitude is a lowpass filter whose time constant is equal to the resonator's relaxation time. Hence, the phase model of the oscillator consists of an amplifier of gain exactly equal to one and the lowpass filter in the feedback path, as extensively discussed in [1]. The amplitude model is a nonlinear amplifier, whose gain is equal to one at the stationary amplitude and decreases with power, and the lowpass filter in the feedback path. In the baseband representation both AM and PM perturbations map into additive noise, even in the case of flicker and other parametric noises. This model gives a new perspective on the classical van der Pol oscillator.

The elementary theory of nonlinear differential equations tells us that nonlinearity stretches the feedback time constant. Asymptotically, the time constant is split into two constants, one at the oscillator startup and one in stationary conditions. If the gain varies linearly with amplitude, which is always true for small perturbations, the oscillator can be solved in closed form.

It is well known that the instability of the resonator natural frequency contribute to the oscillator fluctuations, which in some cases turns out to be the most important source of frequency fluctuations. Nothing is known about the amplitude fluctuation of the resonator. That said, the resonator instability will not considered here. Standing upon our earlier works [1], [2], this article presents an unified approach to AM and PM noise in oscillators by analyzing the mechanism with which the noise of the oscillator internal components is transferred to the output.

After the pioneering work of D. B. Leeson [3], a number of different analyses were published. Hajimiri and Lee [4], [5], [6] proposed a model based on the "impulse-sensitivity function" (ISF), which emphasizes that the impulse has the largest effect on phase noise if it occurs at the zero-crossing of the carrier. This model, mainly oriented to the description of phase noise in CMOS circuits, is extended in [7]. Demir & al. proposed a theory based on the stochastic calculus [8], in which they introduce a decomposition of phase and amplitude noise through a projection onto the periodic time-varying eigenvectors (the Floquet eigenvectors), by which they analyze the oscillator phase noise as a stochastic diffusion problem. This theory was extended to the case of $1/f$ noise [9], and inspired our work on the phase noise in opto-electronic oscillators[10]. Though the article cited make use of sophisticated mathematics, as compared to the simple methods we propose, they give no or tiny attention to amplitude noise. Finally, this article provides a unified theory for amplitude and phase noise. This theory turns out to be particularly suitable to high stability oscillators, based on high quality-factor quartz resonators, microwave whispering gallery resonators, etc. Work is in progress to extend our method to the opto-electronic oscillator.

II. Basics

The simplest form of oscillator is a resonator with an amplifier of gain A in closed loop that compensates for the resonator loss[1] $1/\beta$. Stationary oscillation takes place at the angular[2] frequency ω_0 that verify $A\beta = 1$, which is known as the Barkhausen condition. The actual oscillator can be represented with the scheme of Fig. 1, which includes a gain compression mechanism and noise. The gain compression is necessary for the amplitude not to decay or diverge. Without loss of generality we can normalize the loop elements so that $A = 1$ and $\beta = 1$ at the oscillation frequency $\omega = \omega_0$ and at the nominal output amplitude $v = 1$. The equivalent

[1] The quantity β is the resonator gain, so that $1/\beta$ is the resonator loss.

[2] We use interchangeably ω as a shorthand for $2\pi\nu$ for the carrier frequency, and as a shorthand for $2\pi f$ for the offset (Fourier) frequency, making the meaning clear with appropriate subscripts when needed but omitting the word 'angular.'

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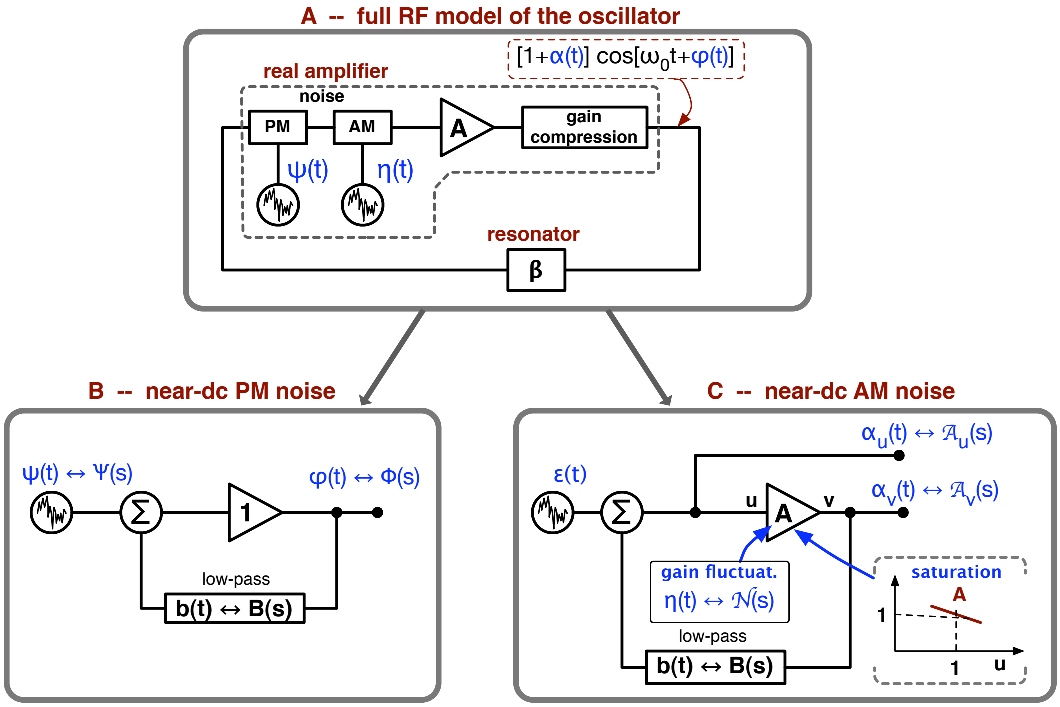


Fig. 1. Feedback oscillator and its decomposition in PM and AM models.

phase noise and amplitude noise at the amplifier input originate frequency and amplitude fluctuations of the oscillator.

The oscillator is characterized by two time constants, the inverse of the angular frequency ω_0 and the resonator relaxation time τ. We assume that, as it occurs in most practical cases, $1/\omega_0$ and τ differ by at least two decades. In such cases the oscillator behavior can be mathematically described in terms of the slow-varying amplitude and phase as they were decoupled from the oscillation. From a physical standpoint, the resonator eliminates all the harmonics multiple of ω_0 present at the amplifier output, hence the only practical effect of the nonlinearity is to reduce the gain at the fundamental frequency ω_0. Additionally, the resonator attenuates all the signals of frequency not exactly equal to ω_0, so that after a few roundtrips only a quasi-sinusoidal signal is permitted in the loop.

In the slow-varying-signal representation the oscillator splits into two subsystems, one for the phase and one for the amplitude, as shown in the lower part of Fig. 1. Since phase represents time, which cannot be stretched[3], all the nonlinearity goes in the amplitude subsystem. Noise is introduced in the loop. Interestingly, in this representation phase noise, and amplitude noise as well, is additive even if it is of parametric nature. Additionally, the amplifier gain is allowed to fluctuate.

We are interested in the mechanism that governs the noise propagation of the internal components to the oscillator output. Virtually all oscillators are stable enough for the noise to be a small perturbation to the stationary oscillations, and consequently for a linear model to be accurate for any practical purpose. Linearization gives access to the Laplace-Heaviside formalism. The response[4] $y(t)$ to the input $x(t)$ is therefore given by

$$y(t) = x(t) * h(t) \quad \leftrightarrow \quad Y(s) = X(s)H(s)$$

where $h(t)$ is the *impulse response*, i.e., the response to the Dirac $\delta(t)$ function, $H(s)$ is the *transfer function*, the symbol '$*$' is the convolution operator, the double arrow '$\leftrightarrow$' stands for Laplace transform inverse-transform pair, and $s = \sigma + j\omega$ is the Laplace complex variable. Given the input power spectral density $S_x(f)$, the output power spectral density is given by

$$S_y(f) = |H(jf)|^2 \, S_x(f) \ .$$

Finding a representation suitable to linearization and the transfer function of the oscillator parts rises some difficulties, which will be solved in the next Sections.

III. Resonator low-pass model

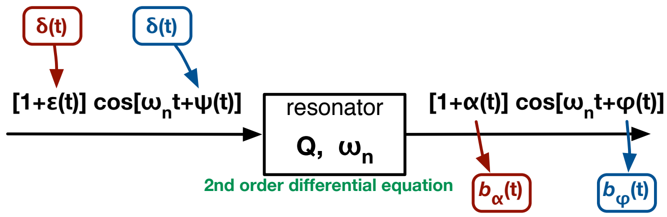
The resonator is governed, or locally well approximated by the differential equation

$$\ddot{x} + \frac{\omega_n}{Q}\dot{x} + \omega_n^2 = \xi(t) \ ,$$

where ω_n is the natural frequency, Q is the quality factor, and $\xi(t)$ the external force. After normalization, the resonator responds to a sinusoid at the exact resonant frequency ($\omega_0 = \omega_n$) with a sinusoid of the same phase and amplitude.

[3]This is no longer true in extreme nonlinear oscillators, like the femtosecond laser, which are out of our scope.

[4]Here $x(t)$ and $y(t)$ are generic functions of time, thus *not* the phase time and the fractional frequency fluctuation commonly used in the oscillator literature.



ele-resonator-delta-response

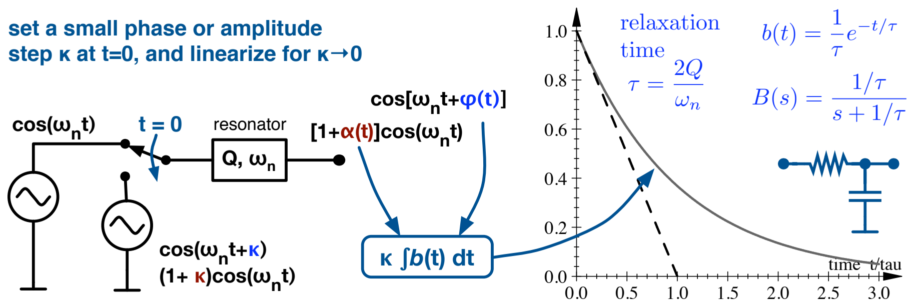


Fig. 2. AM and PM response of a resonator.

Let us analyze the impulse response of the resonator phase and amplitude at $\omega_0 = \omega_n$. Feeding $\delta(t)$ in the input phase, the resonator responds with the phase $\mathrm{b}_\varphi(t)$

$$\begin{array}{ll} \text{input} & \cos[\omega_0 t + \delta(t)] \\ \text{output} & \cos[\omega_0 t + \mathrm{b}_\varphi(t)] \end{array}$$

Similarly, feeding $\delta(t)$ in the input amplitude, the resonator responds with the amplitude $\mathrm{b}_\alpha(t)$

$$\begin{array}{ll} \text{input} & [1 + \delta(t)] \cos \omega_0 t \\ \text{output} & [1 + \mathrm{b}_\alpha(t)] \cos \omega_0 t \end{array}$$

It turns out that the resonator's impulse response is the same for amplitude and phase

$$\mathrm{b}_\alpha(t) = \mathrm{b}_\varphi(t) = \mathrm{b}(t) \ ,$$

hence the subscript will be omitted. The proof for the phase response is discussed extensively in [1], for only the guidelines are given here. The proof for the amplitude response is almost the same. Referring to Fig. 2, the response is calculated replacing the perturbation with the Heaviside function

$$\mathrm{u}(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

and using the property of linear systems that the response to $\mathrm{u}(t)$ is $\int \mathrm{b}(t)\, dt$. The system is linearized by using a small step $\kappa \mathrm{u}(t)$ instead of $\mathrm{u}(t)$. After some boring, yet simple mathematics we get

$$\mathrm{b}(t) = \frac{1}{\tau} e^{-s\tau} \quad \leftrightarrow \quad \mathrm{B}(s) = \frac{1/\tau}{s + 1/\tau} \tag{1}$$

where

$$\tau = \frac{2Q}{\omega_n} \qquad \text{(relaxation time)}$$

ele-PM-scheme

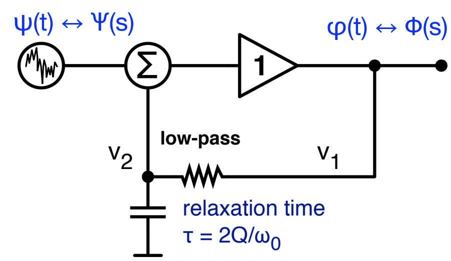


Fig. 3. Phase-noise model of the feedback oscillator.

is the resonator relaxation time. The inverse of τ is known as the Leeson (cutoff) frequency of the resonator

$$\omega_L = \frac{1}{\tau} = \frac{\omega_n}{2Q} \qquad f_L = \frac{1}{2\pi\tau} = \frac{\nu_n}{2Q} \ .$$

Note that Equation (1) is that of a simple RC low-pass filter, which we will use in all block diagrams.

IV. The Leeson effect

Figure 3 shows the phase-noise model of the oscillator. In this figure, all signals are the phase fluctuation of the oscillator sinusoidal signal. Here, the resonator turns into a lowpass filter of time constant τ, as explained in Section III. A noise-free amplifier has a gain exactly equal to one because the amplifier repeats the phase of the input signal. The real amplifier introduces? the random phase $\psi(t)$, which in this representation is additive noise, regardless of the physical origin. For the sake of simplicity, we put in $\psi(t)$ all the phase-noise sources.

We define the phase-noise transfer function as

$$\mathrm{H}(s) = \frac{\Phi(s)}{\Psi(s)} \ .$$

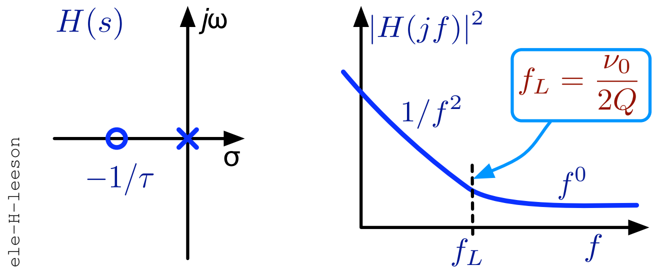


Fig. 4. Phase-noise transfer function.

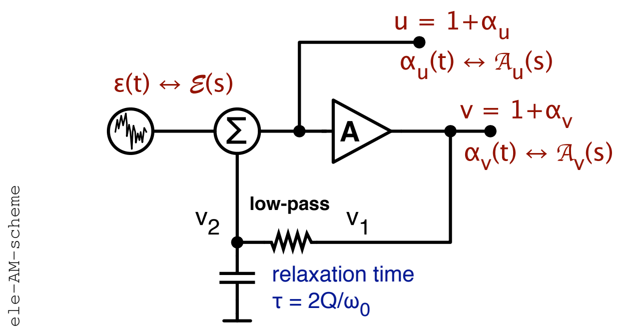


Fig. 5. Amplitude-noise model of the feedback oscillator.

Applying the elementary feedback theory to the circuit of Fig. 3 we find

$$\mathrm{H}(s) = \frac{1}{1+\mathrm{B}(s)} ,$$

where $\mathrm{B}(s)$ is the resonator transfer function (1), and therefore

$$\mathrm{H}(s) = \frac{1+s\tau}{s\tau} \qquad \text{(Fig. 4)} . \tag{2}$$

This is the Leeson effect, by which the oscillator integrates the slow phase fluctuation, turning it into frequency fluctuation. The phase-noise transfer function is plotted in Fig. 4.

V. Low-pass model of the oscillator amplitude

Figure 5 shows the low-pass model that describes the oscillator amplitude. Since the gain A depends on amplitude, the Laplace/Heaviside formalism cannot be used directly. We first need to

A. Differential equation

Cutting the feedback loop at the amplifier input, we get

$$u = \epsilon + v_2 ,$$

where v_2 results from the lowpass filter

$$v_2 = \frac{1}{\tau}\int (v_1 - v_2)\, dt .$$

Replacing $v_1 = Au$ and $v_2 = u - \epsilon$ and combining the above equations, we get

$$\dot{u} - \frac{1}{\tau}(A-1)\,u = \frac{1}{\tau}\epsilon + \dot{\epsilon} \qquad \text{(general equation)} . \tag{3}$$

Notice that (3) is general because A is still unspecified.

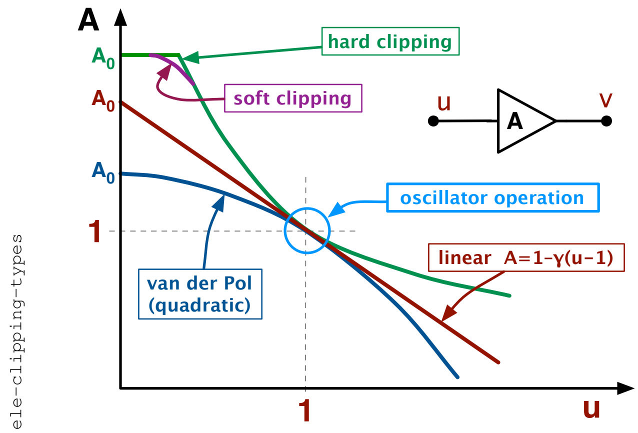


Fig. 6. Most common types of gain saturation. The quantities u and v are the rms amplitude at the carrier frequency.

B. Gain saturation

In large signal conditions, all amplifiers have some kind of nonlinearity that limits the maximum output power. Neglecting the band limitation, when a sinusoidal signal $U_{\text{rf}}(t) = U_1\cos(\omega_0 t)$ at the input of an amplifier, the saturated output can be written as the Fourier series $V_{\text{rf}}(t) = \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t)$. The $n = 1$ term is the fundamental and the $n > 1$ terms are the harmonics generated by the nonlinearity. The effect of the band limitation is to change (in most cases to reduce) the amplitudes V_n and to introduce a phase in each sinusoidal term. In a linear amplifier only the fundamental is present at the output, thus $V_n = 0\ \forall n \geq 2$.

In the case of the oscillator, the resonator allows only the fundamental to be feed back to the input, for the harmonics can be neglected. Hence, the oscillation amplitude is described using the slow-varying signals u and v instead of the instantaneous peak amplitudes U_1 and V_1. The amplifier gain is

$$A = \frac{v}{u} ,$$

which of course is equivalent to $A = V_1/U_1$.

Figure 6 shows the gain saturation types most frequently encountered and described underneath. The plot is normalized for $A = 1$ when the input signal is $u = 1$, where it can be linearized as

$$A = 1 - \gamma(u-1) . \tag{4}$$

Of course $u = 1$ is the oscillation regime we refer to. In all cases the small-signal gain is denoted with A_0. Notice that A is referred to the *input* amplitude. It is necessary that

$$0 \geq \frac{dA}{du} > -1 , \tag{5}$$

thus

$$0 \leq \gamma < 1 . \tag{6}$$

The first condition means that A can only decrease increasing the input. The second condition is equivalent to state that output amplitude cannot decreases when the input is increased.

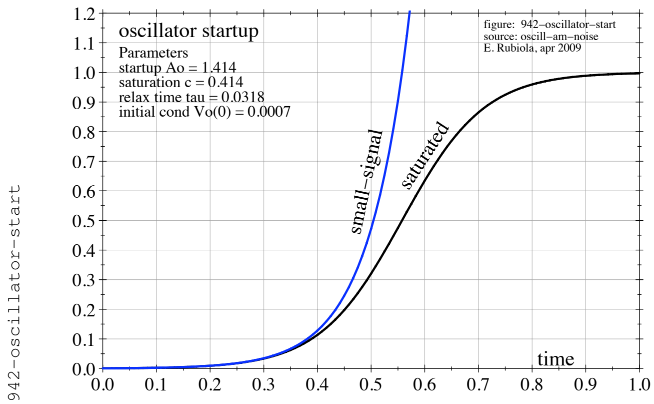


Fig. 7. Oscillator startup.

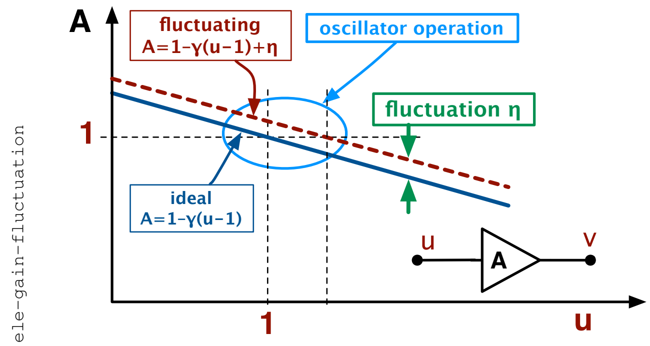


Fig. 8. Parametric fluctuation of the amplifier gain.

1) Quadratic (van der Pol): In the classical van der Pol oscillator [], the amplifier input-output function is defined as $y = x - x^3$. Feeding the signal $U_{\text{rf}}(t) = U_1 \cos(\omega_0 t)$ in such amplifier and taking only the fundamental frequency, the output is $V_{\text{rf}}(t) = U_1 \left[1 - \frac{3}{4} U_1^2\right] \cos(\omega_0 t)$. Accordingly, the gain becomes $A = 1 - \frac{3}{4} U_1^2$, which is a 'cap' parabola.

2) Hard clipping: In small-signal condition the gain is A_0, independent of the signal level. Increasing the input level the output is clipped when it hits a threshold, where the sinusoid progressively turns into a square wave. The asymptotic amplitude of the fundamental is $4/\pi$ (2.1 dB) higher than the threshold. This behavior is often encountered in amplifiers linearized by a strong feedback, as most circuits based on operational amplifiers. Of course the feedback is no longer effective when the output is expected to exceed the supply voltage.

3) Soft clipping: With moderate feedback, the output clipping starts gradually when the output approaches the dynamic-range boundary. This behavior is typical of microwave amplifiers. The knee of the gain curve occurs approximately at the 1 dB compression power.

4) Linear gain compression: The gain is described by $A = 1 - \gamma(u - 1)$ in the whole dynamic range. Though this model seems no more than an academic exercise, it provides useful results in a simple and compact form.

C. Oscillation start

Assuming that a strong switch-on transient is not present, oscillation start from noise. This means that u is initially close to zero and that A takes the small-signal value A_0, locally constant for $u \to 0$ (Fig. 6). Thus the homogeneous equation associated to (3) becomes

$$\dot{u} - \frac{1}{\tau}\left(A_0 - 1\right) u = 0 \ .$$

The solution is (Fig. 7, the blue curve labeled 'small signal')

$$u = C e^{(A_0 - 1)\, t/\tau} \ .$$

Simulations based on actual circuits (Colpitts and other oscillators) fit exactly the theoretical expectation.

D. Stationary oscillation

In the absence of noise and perturbations the oscillation takes the value $u = 1$, where the gain is $A = 1$ (Barkhausen condition). If a small perturbation is introduced, u is still close to 1 and the gain is $A = 1 - \gamma(u - 1)$. The homogeneous form of (3) becomes

$$\dot{u} + \frac{\gamma}{\tau}\left(u - 1\right) u = 0 \ ,$$

which yields

$$u = C e^{-\gamma t/\tau} \ . \tag{7}$$

In simple words, the oscillator responds to an amplitude perturbation with a decaying exponential whose time constant

$$\tau_r = \frac{\tau}{\gamma} \qquad \text{(restoring time)}$$

is the oscillator restoring time.

E. Simplified oscillator model

Under the assumption that the gain follows the linear approximation $A = 1 - \gamma(u - 1)$ in the whole range, the solution of the homogeneous form of (3) is (Fig. 7, the black curve labeled 'saturated')

$$u = \frac{1}{\left(\frac{1}{u(0)} - 1\right) e^{-\gamma t/\tau} + 1}$$

Of course, the gain of actual amplifiers can only be linearized in a narrow region around $u = 1$. Hence case is only of academic interest. Nonetheless, simulations show that the oscillator startup is similar to the solution obtained here.

VI. Extension of the Leeson Effect to AM Noise

We analyze the oscillator (Fig. 5) in the presence of noise around the stationary oscillation $u = 1$, where the amplitude is

$$\text{ampli input} \qquad u = 1 + \alpha_u \tag{8}$$

$$\text{ampli output} \qquad v = 1 + \alpha_v \ . \tag{9}$$

This is done by letting A fluctuate (Fig. 8)

$$A = 1 - \gamma(u - 1) + \eta \tag{10}$$

$$\eta(t) \leftrightarrow \mathcal{N}(s) \ .$$

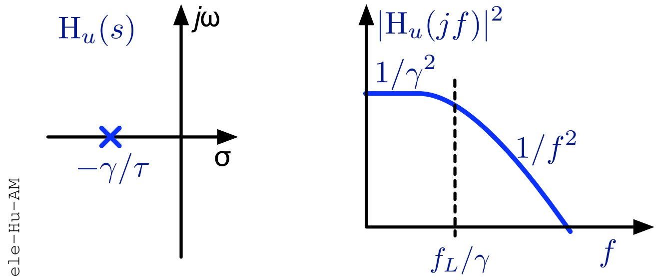


Fig. 9. Amplitude-noise transfer function (amplifier input).

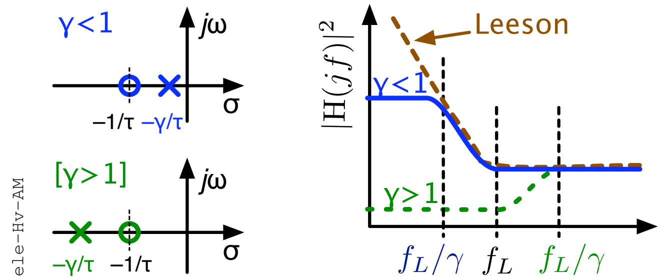


Fig. 10. Amplitude-noise transfer function (oscillator output).

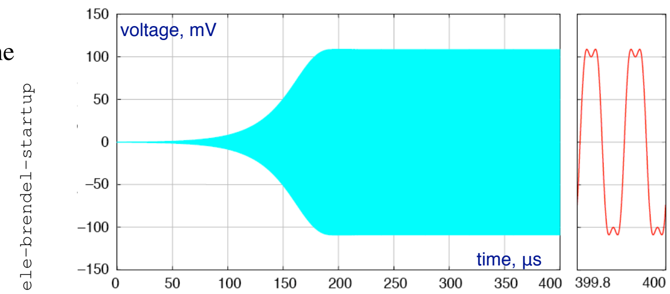


Fig. 11. Simulated oscillator startup.

After linearizing the system for low noise, we search for the transfer functions

$$\mathrm{H}_u(s) = \frac{\mathcal{A}_u(s)}{\mathcal{N}(s)} \qquad \text{and} \qquad \mathrm{H}_v(s) = \frac{\mathcal{A}_v(s)}{\mathcal{N}(s)} \;,$$

where

$$\alpha_u(t) \leftrightarrow \mathcal{A}_u(s) \qquad \text{and} \qquad \alpha_v(t) \leftrightarrow \mathcal{A}_v(s) \;,$$

we search

A. Amplifier input

By replacing $A = 1 - \gamma(u-1) + \eta$ in the homogeneous form of (3) [i.e., $\dot{u} = \frac{1}{\tau}(A-1)u$], we get

$$\dot{u} + \frac{\gamma}{\tau}(u-1)u = \frac{\eta}{\tau}u \;.$$

We notice that $\dot{u} = \dot{\alpha}_u$ and $u - 1 = \alpha_u$, thus

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u u = \frac{\eta}{\tau}u \;.$$

Since α_u and η are small fluctuations, we linearize the above using $u \simeq 1$

$$\dot{\alpha}_u + \frac{\gamma}{\tau}\alpha_u = \frac{1}{\tau}\eta \;.$$

The Laplace transform

$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_u(s) = \frac{1}{\tau}\mathcal{N}(s) \tag{11}$$

gives the transfer function (Fig. 9)

$$\mathrm{H}_u(s) = \frac{1/\tau}{s + \gamma/\tau} \;. \tag{12}$$

B. Amplifier output

We first need to relate α_v to α_u. This is done by replacing $A = -\gamma(u-1) + 1 + \eta$ in $v = Au$

$$v = \left[-\gamma(u-1) + 1 + \eta\right] u \;,$$

expanding $v = 1 + \alpha_v$ and $u = 1 + \alpha_v$

$$1 + \alpha_v = 1 + \eta - \gamma\alpha_u + \alpha_u - \alpha_u\eta - \gamma\alpha_u^2 \;,$$

neglecting the terms $\alpha_u\eta$ and α^2

$$\alpha_v = (1-\gamma)\alpha_u + \eta \;,$$

and inverting

$$\alpha_u = \frac{\alpha_v - \eta}{1-\gamma} \qquad \leftrightarrow \qquad \mathcal{A}_u(s) = \frac{\mathcal{A}_v(s) - \mathcal{N}(s)}{1-\gamma} \;.$$

Then, by replacing the above $\mathcal{A}_u(s)$ in Eq. (11) we get

$$\left(s + \frac{\gamma}{\tau}\right)\mathcal{A}_v(s) = \left(s + \frac{1}{\tau}\right)\mathcal{N}(s) \;,$$

and finally

$$\mathrm{H}_v(s) = \frac{s + 1/\tau}{s + \gamma/\tau} \;. \tag{13}$$

The transfer function $\mathrm{H}_v(s)$ is shown in Fig. 10 Notice that the case $\gamma > 1$ (dotted green) is not allowed by the condition (5)–(6) about the real amplifier.

VII. Simulations

A number of computer simulations were done independently by one of us (RB), well before the approach presented here was developed [11], [12]. This led to the preliminary work published in [13].

Figure 11 show the oscillator startup. The left-hand side of the envelope, until $t \approx 100$ μs, fits well the theoretical prediction (7).

Figure 12 shows the close-in noises pectrum of a van der Pol quartz oscillator. The blue curve (phase noise) fits the Leeson effect as described by (2), while the red curve (amplitude noise) is in a close agreement with Eq. (13).

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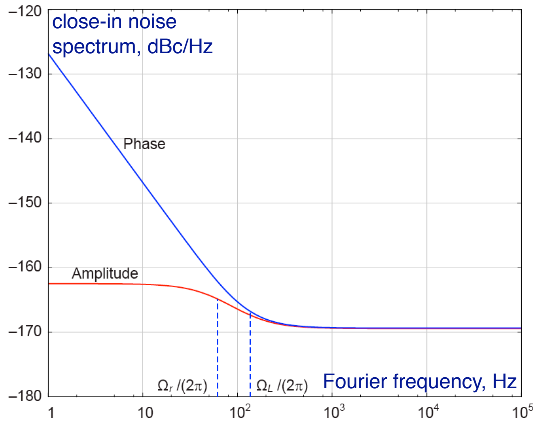

Fig. 12. Simulated van der Pol oscillator.

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