The Magic of Cross Correlation in Measurements from DC to Optics

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I. INTRODUCTION

Measuring a device under test (DUT) with an instrument, the observed spectrum contains both the DUT noise and the background noise of the instrument. The basic idea of the cross-spectrum measurement is that we can measure the DUT simultaneously with two equal instruments. With good experimental skill and a pinch of good luck, the DUT noise and the noise of the two instruments are statistically independent. So, statistics *must* help in extracting the DUT noise spectrum.

We can formalize the two-channel measurement with the model of Fig. 1, denoting with $a(t)$ and $b(t)$ the background of the two instruments, and with $c(t)$ the common noise. By definition, $a(t)$, $b(t)$ and $c(t)$ are statistically independent. Thus, the observed signals are

$$\begin{aligned} x(t) &= c(t) + a(t) \\ y(t) &= c(t) + b(t) \,. \end{aligned}$$

Thanks to the Wiener-Khintchine theorem for stationary ergodic processes, the measured cross spectrum $S_{yx}(f)$ converges to $S_c(f)$, which is what we want to measure.

Requiring that the noise processes be stationary and ergodic is not a stringent constraint in the laboratory practice. In fact, in statistics the words 'stationary' and 'ergodic' are the equivalent of 'repeatable' and 'reproducible' in experimental physics.

As a remarkable property the cross-spectrum method, the background noise is determined by the thermal homogeneity, instead of the absolute temperature of the instrument. Hence, the instrument sensitivity is surprisingly high.

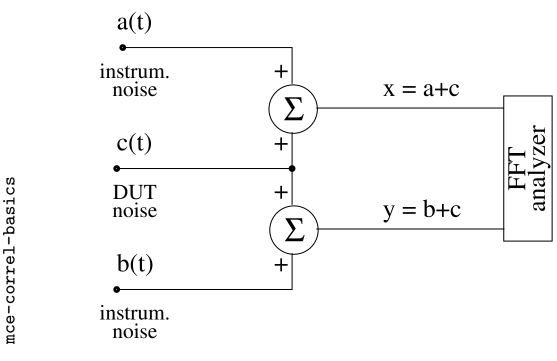

Figure 1: Basics of the cross-spectrum method.

All this article is about how and why the cross-spectrum converges, and about how this can be used in actual measurements. The basic facts of statistics, well established and available in classical textbooks [1, 2], are adapted to this specific experimental technique.

Though I might be inclined to use phase and amplitude noise as the favorite examples, the cross-spectrum method is of far more general interest. For example, an extreme application is to compare shot noise to thermal noise, in order to replace the practical scale of temperature with fundamental constants. Tracking the cross-spectrum method back in the literature, we found applications to radio-astronomy and radiometry [3], and to the phase-noise measurement of Hydrogen masers [4], before the method became popular in phase noise measurements.

II. MATHEMATICAL BACKGROUND

This Section recalls some basic mathematical facts. To make the concepts accessible to the reader, we use the symbols the most common in the general literature. Unfortunately, conflicts with some symbols, like x, y, f, etc., are inevitable. They will be solved when appropriate.

A. Power spectra

For a generic random processes $x(t)$, the PSD (power spectral density) is defined as

$$S_x(f) = \mathcal{F}\left\{R_x(t')\right\} = \mathcal{F}\left\{\mathbb{E}\left\{x(t)\,x(t+t')\right\}\right\} \;, \quad (1)$$

where $\mathcal{F}\{\ \}$ is the Fourier transform operator, R the autocorrelation function, and $\mathbb{E}$ the mathematical expectation. According to the usual notation of statistics, we should use the boldface symbol $[\mathbf{x}(t)]$ for the random process, and the normal typeface $[x(t)]$ for the realization. Since we deal only with ergodic stationary processes, we simplify the notation using the same symbol for the process and for the realization. To simplify the notation, we use the upper case for the Fourier transform, and the left-right arrow for the transform inverse-transform pair, thus $x(t) \leftrightarrow X(\imath f)$ Thanks to the Wiener-Khintchine theorem for ergodic and stationary processes (i.e., for reproducible ad repeatable experiments) the PSD is

$$\mathbb{E}\left\{S_x(f)\right\} = \mathbb{E}\left\{|X(\imath f)|^2\right\} \quad \text{(W-K theorem)}. \quad (2)$$

Notice that in $\mathbb{E}\{S_x(f)\}$ the expectation is often implied in the definition (1), while we have made it explicit because we need to emphasize the difference

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between expectation and average. In experiments, the expectation is replaced with the average on a suitable number of measured spectra

$$\langle S_x(f)\rangle_m = \left\langle |X(\imath f)|^2 \right\rangle_m \quad \text{(avg, } m \text{ spectra)} \ . \qquad (3)$$

The cross PSD of two generic random processes $x(t)$ and $y(t)$ is defined as

$$S_{yx}(f) = \mathcal{F}\left\{R_{yx}(t')\right\} \ , \qquad (4)$$

which is equivalent to

$$\mathbb{E}\left\{S_{yx}(f)\right\} = \mathbb{E}\left\{Y(\imath f)X^*(\imath f)\right\} \qquad (5)$$

for ergodic and stationary processes, and measured as

$$\langle S_{yx}(f)\rangle_m = \langle Y(\imath f)X^*(\imath f)\rangle_m \qquad (6)$$

The superscript '$*$' means complex conjugate.

The two-sided frequency analysis is preferred in theoretical issues, while the experimentalist often prefers the single-sided representation. Though we use the one-sided representation in all figures, in most of this article we do not need the distinction between one-sided and two-sided representation. In most practical measurements the frequency is a discrete variable because the Fast Fourier Transform (FFT) replaces the traditional Fourier transform. Thus, X, Y, etc. can be seen as

1. an array of values, where the running index is the integer frequency,
2. at a given frequency, a time series formed by the sequence of the measured spectra,

and of course a sequence of arrays.

B. Rayleigh distribution

Let $x_1(t)$ and $x_2(t)$ two independent random functions with gaussian distribution, zero mean and equal variance σ, and

$$y = \sqrt{x_1^2 + x_2^2} \qquad (7)$$

a new random function. The latter has Rayleigh probability density function (PDF)

$$f(y) = \frac{y}{\sigma^2}\, e^{-y^2/2\sigma^2}, \quad y > 0 \qquad \text{Rayleigh PDF} \qquad (8)$$

$$\mathbb{E}\{y\} = \sqrt{\pi/2}\,\sigma \qquad \text{average} \qquad (9)$$

$$\mathbb{E}\{y^2\} = 2\sigma^2 \qquad \text{2nd moment} \qquad (10)$$

$$\mathbb{E}\{|y - \mathbb{E}\{y\}|^2\} = \frac{4-\pi}{2}\sigma^2 \qquad \text{variance} \qquad (11)$$

The functions $x_1(t)$ and $x_2(t)$ can be interpreted as the random amplitude of two orthogonal vectors, or the real and imaginary part of a complex random function. Following this interpretation, $y(t)$ is the absolute value of the vector sum.

Table I: Relevant values for the Rayleigh distribution with $\sigma = 1/2$.

quantity q		value	$10\log(q)$, dB
average	$\sqrt{\frac{\pi}{4}}$	0.886	-0.525
deviation	$\sqrt{1-\frac{\pi}{4}}$	0.463	-3.34
$\frac{\text{dev}}{\text{avg}}$	$\sqrt{\frac{4}{\pi}-1}$	0.523	-2.8
$\frac{\text{avg+dev}}{\text{avg}}$	$1+\sqrt{\frac{4}{\pi}-1}$	1.523	$+1.83$
$\frac{\text{avg}-\text{dev}}{\text{avg}}$	$1-\sqrt{\frac{4}{\pi}-1}$	0.477	-3.21

A case of interest is that of $\sigma^2 = 1/2m$, which yields

$$\mathbb{E}\{y\} = \sqrt{\pi/4m} \qquad \text{average} \qquad (12)$$

$$\mathbb{E}\{y^2\} = 1/m \qquad \text{2nd moment} \qquad (13)$$

$$\mathbb{E}\{|y - \mathbb{E}\{y\}|^2\} = \left(1 - \tfrac{\pi}{4}\right)\tfrac{1}{m} \qquad \text{variance,} \qquad (14)$$

and

$$\sqrt{\mathbb{E}\{|y - \mathbb{E}\{y\}|^2\}} = \sqrt{\left(1 - \tfrac{\pi}{4}\right)\tfrac{1}{m}} \qquad \text{deviation} \qquad (15)$$

$$\frac{\sqrt{\mathbb{E}\{|y - \mathbb{E}\{y\}|^2\}}}{\mathbb{E}\{y\}} = \sqrt{\tfrac{4}{\pi} - 1} \qquad \text{dev/avg} \qquad (16)$$

Table I reports some useful numerical values related to the $\sigma^2 = 1/2$ Rayleigh distribution.

C. Useful statistical properties

The following properties will be used to understand the cross-spectrum method. We report only the results, omitting the proof, and without attempting to state more general forms, if any.

1. *Sum of zero-mean gaussian random functions*

Let $x_1(t)$ and $x_2(t)$ two random functions with gaussian distribution, zero mean and variance σ_1^2 and σ_2^2. The sum $y(t) = x_1(t)+x_2(t)$ is a random function with gaussian distribution, zero mean and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$.

2. *Product of zero-mean gaussian random functions*

Let $x_1(t)$ and $x_2(t)$ two random functions with gaussian distribution, zero mean and variance σ_1^2 and σ_2^2. The product $y(t) = x_1(t)\,x_2(t)$ is a random function with gaussian distribution, zero mean and variance $\sigma^2 = \sigma_1^2\,\sigma_2^2$.

3. *Fourier transform of a gaussian random function*

Let $x(t)$ a random process with gaussian distribution and white spectrum. Its Fourier transform

$X(\imath f)$ is a random white gaussian process. This means that

1. At a given frequency f, the real part $X'(\imath f) = \Re\{X(\imath f)\}$ is a random variable with gaussian distribution. Also the imaginary part $X''(\imath f) = \Im\{X(\imath f)\}$.
2. $X'(\imath f)$ and $X''(\imath f)$, are a pair of statistically independent random functions with equal variance.
3. Given two frequencies f_1 and f_2 (or two separate frequency slots), $X'(\imath f_1)$, $X'(\imath f_2)$, $X''(\imath f_1)$, and $X''(\imath f_2)$ are statistically independent.

Interestingly, the absolute value

$$|X| = \sqrt{(X')^2 + (X'')^2}$$

has Rayleigh distribution.

III. THE CROSS-SPECTRUM METHOD

Recalling the definitions of Section I, we denote with $a(t)$ and $b(t)$ the background of the two instruments, with $c(t)$ the common noise, and with A, B and C their Fourier transform. The variable $\imath f$ is implied. By definition, $a(t)$, $b(t)$ and $c(t)$ are statistically independent. We also assume that they are ergodic and stationary. The two output of the two instruments are

$$x(t) = c(t) + a(t) \quad \leftrightarrow \quad X = C + A \tag{17}$$
$$y(t) = c(t) + b(t) \quad \leftrightarrow \quad Y = C + B\,. \tag{18}$$

First, we observe that the cross-spectrum S_{yx} converges to $S_c(f)$. In fact, using the Wiener-Khintchine theorem for ergodic stationary processes

$$\begin{aligned}\mathbb{E}\{S_{yx}\} &= \mathbb{E}\{YX^*\}\\ &= \mathbb{E}\{[C+A]\times[C+B]^*\}\\ &= \mathbb{E}\{CC^*\} + \mathbb{E}\{CB^*\} + \\ &\qquad + \mathbb{E}\{AC^*\} + \mathbb{E}\{AB^*\}\\ &= S_c \end{aligned}\tag{19}$$

because the statistical independence gives $\mathbb{E}\{CB^*\} = 0$, $\mathbb{E}\{AC^*\} = 0$, and $\mathbb{E}\{AB^*\} = 0$.

Second, we replace the expectation with the average on m measured spectra

$$\begin{aligned}\langle S_{yx}\rangle_m &= \langle YX^*\rangle_m\\ &= \langle\{[C+A]\times[C+B]^*\rangle_m\\ &= \langle CC^* + CB^* + AC^* + AB^*\rangle_m\\ &= \langle CC^*\rangle_m + \langle CB^*\rangle_m + \langle AC^*\rangle_m + \langle AB^*\rangle_m\\ &= S_c + O(\sqrt{1/m})\,, \end{aligned}\tag{20}$$

where $O(\,)$ means 'order of.' Owing to statistical independence, the cross terms decrease proportionally to $\sqrt{1/m}$. The convergence law will be further discussed in Section IV. In most practical cases, the displayed quantity is

$$|\,\langle S_{yx}\rangle_m\,| = |\,\langle YX^*\rangle_m\,| \qquad \text{(displayed)}. \tag{21}$$

A. Statistical limit

With no DUT noise, it holds that $c = 0$, hence $S_c = 0$. Maintaining the hypothesis of statistical independence of the two channels, we notice that the number of averaged spectra sets a statistical limit to the measurement. In Eq. (20), only the cross terms remain, which decrease proportionally to $1/\sqrt{m}$. Thus, the statistical limit is

$$S_{yx} = \langle AB^*\rangle_m \tag{22}$$
$$\approx \sqrt{\frac{1}{m}\,S_y\,S_x} \qquad \text{(stat. limit)}\,. \tag{23}$$

Accordingly, a 5 dB improvement on the single-channel noise costs a factor of 10 in averaging, thus in measurement time.

B. Hardware limit

Breaking the hypothesis of the statistical independence of the two channels, we interpret c as the *correlated noise* of the instrument. This can be due for example to the crosstalk between the two channels, or to environmental fluctuations (ac magnetic fields, temperature, etc.) acting simultaneously on the two channels. Thus, still at *zero DUT noise*, yet with unlimited number of averaged spectra, we get the hardware limit of the instrument sensitivity

$$\mathbb{E}\{S_{yx}\} = \mathbb{E}\{S_c\} \qquad \text{(hardware limit)}. \tag{24}$$

C. Regular DUT measurement

The accurate measurement of a regular DUT requires that

1. The number m is large enough for the statistical limit to be negligible
2. The hardware background noise is negligible as compared to the DUT noise

In this conditions, the average cross spectrum converges to the expectation of the DUT noise

$$\langle S_{yx}\rangle_m = \mathbb{E}\{S_c\} \qquad \text{(DUT meas.)}. \tag{25}$$

This is the regular use of the instrument.

IV. CONVERGENCE LAW

We study the convergence law of the displayed quantity $|\,\langle S_{yx}\rangle\,|_m$ in the case of white gaussian noise. Though somewhat simplistic, this case gives indications about the general behavior of the instrument. We further restrict our attention to the case of two equal instruments, in the absence of the DUT ($C = 0$). Thus, X and Y have the same statistical

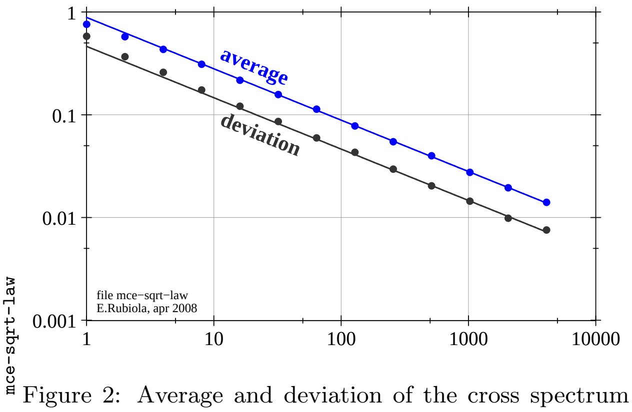


Figure 2: Average and deviation of the cross spectrum $|\langle S_{yx}\rangle_m|$, as s function of the number of averages. The two processes are statistically independent, white, and gaussian distributed. Solid line: theoretical law; dots: simulated data.

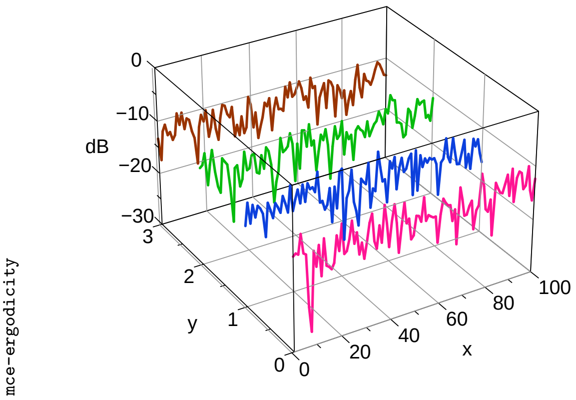


Figure 3: Sequence of cross spectra $|\langle S_{yx}\rangle_{32}|$. The axes are x: frequency, y: sequence, z: $|\langle S_{yx}\rangle_{32}|$, dB.

properties, the same of A and B. We normalize the variance assuming that

$$\begin{aligned}\text{var}\{X\} &= 1\\ \text{var}\{Y\} &= 1\end{aligned}$$

in one Hz bandwidth, or in the bandwidth of interest. Expanding $X = X' + \imath X''$ and $Y = Y' + \imath Y''$, we notice that X', X'', Y', and Y'' are gaussian processes with zero mean, that

$$\begin{aligned}\text{var}\{X'\} &= 1/2 \qquad & \text{var}\{X''\} &= 1/2\\ \text{var}\{Y'\} &= 1/2 \qquad & \text{var}\{Y''\} &= 1/2\ .\end{aligned}$$

Dropping the subscript m, the displayed cross spectrum is

$$\begin{aligned}&|\langle S_{yx}\rangle| =\\ &= \sqrt{[\langle Y'X' + Y''X''\rangle]^2 + [\langle Y''X' - Y'X''\rangle]^2} \qquad (26)\end{aligned}$$

We notice that the terms $Y'X'$, $Y''X''$, $Y''X'$, and $Y'X''$ are zero-mean gaussian independent random processes, and that

$$\begin{aligned}\text{var}\{Y'X'\} &= 1/4 \qquad & \text{var}\{Y''X''\} &= 1/4\\ \text{var}\{Y''X'\} &= 1/4 \qquad & \text{var}\{Y'X''\} &= 1/4\ .\end{aligned}$$

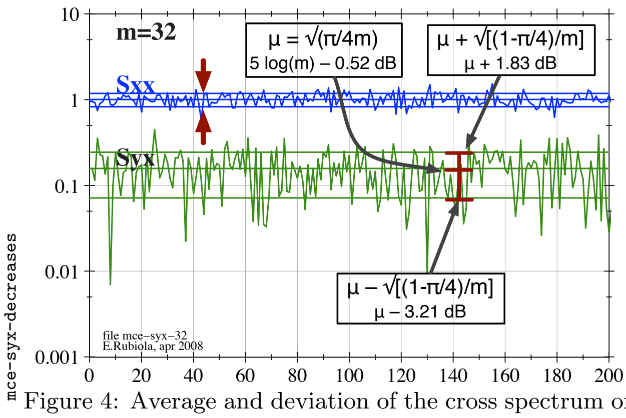


Figure 4: Average and deviation of the cross spectrum of statistically independent white gaussian processes.

The terms $\langle Y'X' + Y''X''\rangle_m$ and $\langle Y''X' - Y'X''\rangle_m$ are zero-mean gaussian independent random processes with variance

$$\begin{aligned}\text{var}\left\{\langle Y'X' + Y''X''\rangle_m\right\} &= 1/2m\\ \text{var}\left\{\langle Y''X' + Y'X''\rangle_m\right\} &= 1/2m\ .\end{aligned}$$

Thus, the displayed cross-spectrum (26) can be rewritten as

$$\begin{aligned}|\langle S_{yx}\rangle_m| &= \sqrt{(Z')^2 + (Z'')^2} \qquad (27)\\ Z' &= \langle Y'X' + Y''X''\rangle_m\\ Z'' &= \langle Y''X' + Y'X''\rangle_m\end{aligned}$$

with Z' and Z'' independent random functions with

$$\begin{aligned}&\mathbb{E}\{Z'\} = 0 && \text{average}\\ &\mathbb{E}\{Z''\} = 0\\ &\mathbb{E}\{|Z' - \mathbb{E}\{Z'\}|^2\} = 1/2m && \text{variance}\\ &\mathbb{E}\{|Z'' - \mathbb{E}\{Z''\}|^2\} = 1/2m\end{aligned}$$

which makes $|\langle S_{yx}\rangle_m|$ match to Eq. (7). Hence, $|\langle S_{yx}\rangle_m|$ is Rayleigh distributed, thus it has all the properties listed in Section II B.

Figure 2 shows the average and the deviation of the cross spectrum as a function of m. We observe that, increasing m,

the average decreases following the law

$$\sqrt{\pi/4m} \qquad \text{[Eq. (12)]}$$

the deviation follows the law

$$\sqrt{(1-\pi/4)/m} \qquad \text{[Eq. (15)]}$$

the dev/avg ratio is constant vs. m [Eq. (16)]

V. THE CROSS SPECTRUM OBSERVED ON THE SCREEN OF THE FFT ANALYZER

A. Ergodicity

A running measurement gives a sequence of spectra $|\langle S_{yx}\rangle_m|_i$ of running index i, as shown in Fig. 3. For

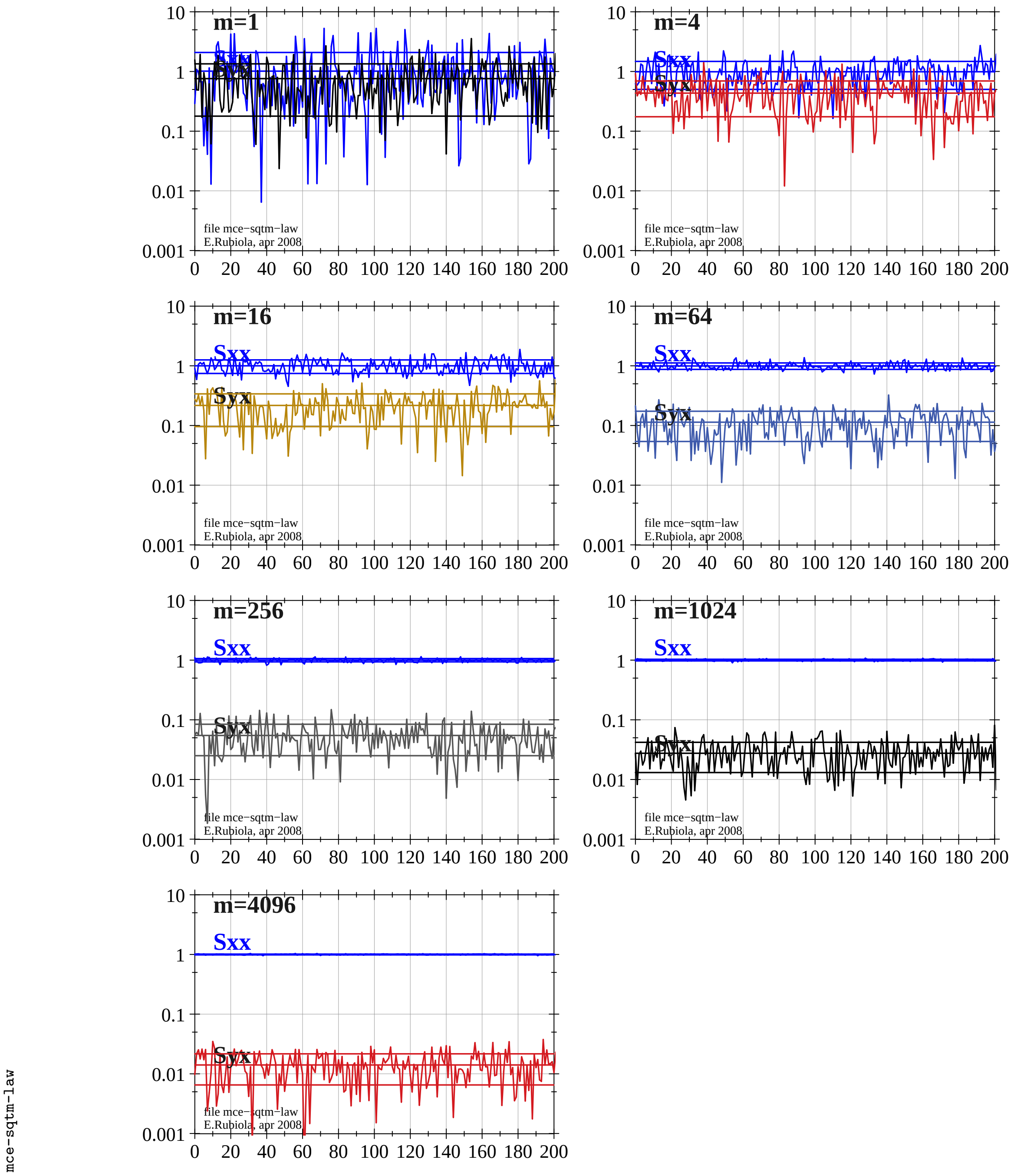


Figure 5: The single-channel spectrum $\langle S_{xx}\rangle_m$ shrinks proportionally to $1/\sqrt{m}$, while the cross spectrum $\langle S_{yx}\rangle_m$ decreases proportionally to $1/\sqrt{m}$. The processes x and y are statistically independent, white and gaussian distributed.

a given frequency f_0, the sequence of the values

$$|\,\langle S_{yx}(f_0)\rangle_m\,|_i$$

is a time series. We are still on gaussian white noise. Since $|\,\langle S_{yx}(f_1)\rangle_m\,|_i$ and $|\,\langle S_{yx}(f_2)\rangle_m\,|_i$ are statistically independent random variables $\forall f_1 \neq f_2$, sweeping the frequency axis gives access to the statistical ensemble.

Ergodicity allows to interchange time statistics and ensemble statistics, thus i and f. The important consequence is that the average and the deviation calculated on the frequency axis give the average and the deviation at which the time series converges. This property enables to *detect when the cross spectrum leaves the $\sqrt{1/m}$ law and converges to the DUT noise.*

B. Sequence of spectra as a function of m

Looking at the single-channel noise spectrum and the cross spectrum on the screen of the FFT analyzer as the average number m increases, we observe the following facts (Fig. 4).

a. ***The single-channel noise*** shrinks proportionally to $\sqrt{1/m}$. This is the noise of one instru-

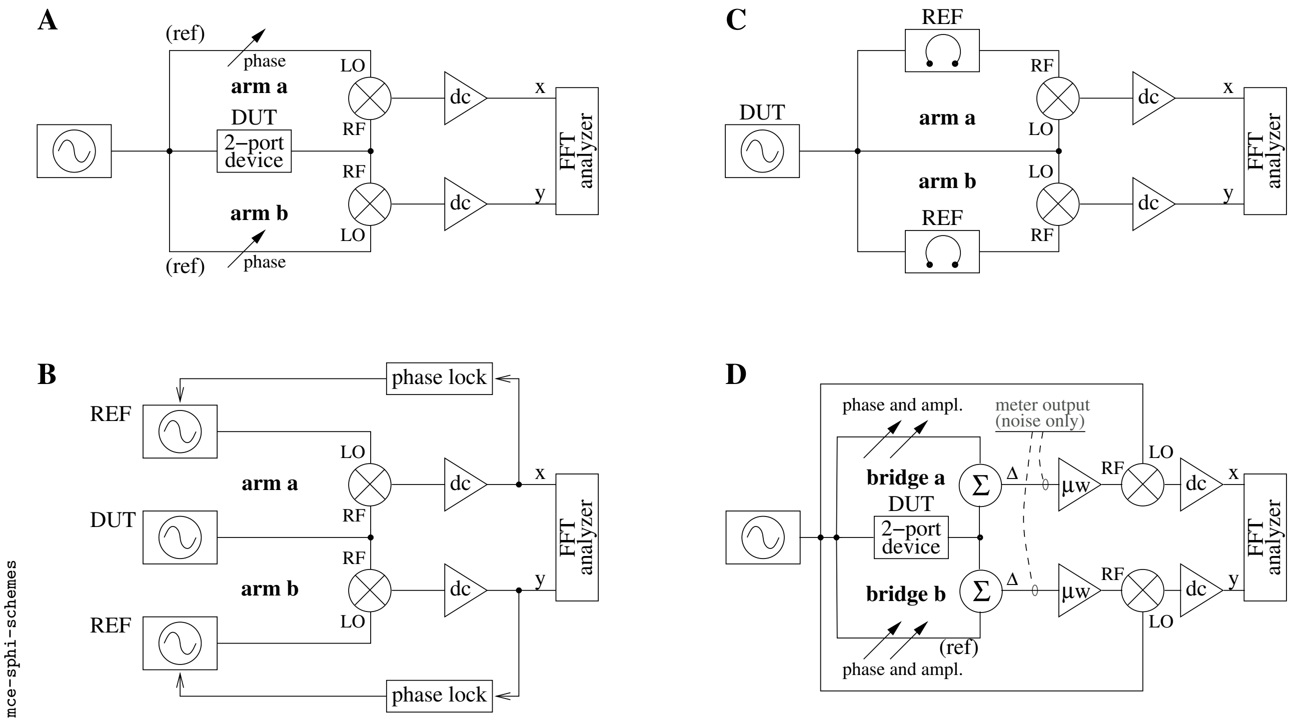


Figure 6: Basics schemes for the measurement of phase noise.

ment, measured with increasing precision, yet *not rejected*.

b. ***In the absence of DUT noise,*** the expectation of the displayed cross spectrum is

$$\mathbb{E}\{|\left\langle S_{yx}\right\rangle_m|\} = S_x\sqrt{\pi/4m} \qquad \text{average} \tag{28}$$

The deviation of the displayed cross spectrum is

$$\sqrt{\mathbb{E}\{|\left\langle S_{yx}\right\rangle_m| - \mathbb{E}\{|\left\langle S_{yx}\right\rangle_m|\}|^2\}} = \\ = S_x\sqrt{\left(1-\tfrac{\pi}{4}\right)\tfrac{1}{m}} \qquad \text{deviation} \tag{29}$$

The ratio between deviation and expectation is

$$\frac{\sqrt{\mathbb{E}\{|\left\langle S_{yx}\right\rangle_m| - \mathbb{E}\{|\left\langle S_{yx}\right\rangle_m|\}|^2\}}}{\mathbb{E}\{|\left\langle S_{yx}\right\rangle_m|\}} = \\ = \sqrt{\left(\tfrac{4}{\pi}-1\right)\tfrac{1}{m}} = 0.523 \qquad \text{dev/avg,} \tag{30}$$

independent of m and of S_x. In logarithmic scale, the cross spectrum appears as a strip decreasing as $5\log(m)$ dB, yet *of constant thickness* of approximately 5 dB (dev/avg). This is seen in Fig. 5.

c. ***Introducing the DUT noise,*** the cross spectrum converges to the DUT noise S_c. When this happens, at a sufficiently large value of m, the cross spectrum no longer decreases. But the variance still does. The observed behavior is that the track on the *logarithmic scale* of the FFT analyzer

- no longer decreases,
- shrinks proportionally to $\sqrt{1/m}$ (as S_x does in Fig. 4).

This reasoning can be reversed. When the cross spectrum stops decreasing and shrinks, the single-channel random background is rejected and the instrument measures the DUT noise (or the hardware limit, which is higher). This fact is of paramount importance in some measurements, where we can not remove the DUT.

VI. EXPERIMENTAL TECHNIQUES

A. PM noise

Figure 6 shows some of the most popular schemes for the measurement of the phase noise of an oscillator. The mixer is a saturated phase-to-voltage converter in A-C, and a synchronous down-converter in D. In all cases correlation rejects the noise of the two mixers. The background noise turns out to be limited by the thermal homogeneity, instead of the absolute temperature referred to the carrier power. This property was understood only after working on the scheme D [5], when the other schemes were well known.

The scheme A [6] is suitable to the measurement of low-noise two-port devices, mainly passive devices showing small group delay, so that the noise of the reference oscillator can be rejected.

The scheme B consists of two separate PLLs that measure the oscillator under test. Correlation rejects the noise of the two reference oscillators. In this way, it is possible to measure an oscillator by comparing it to a pair of synthesizers, even if the noise of the synthesizers is higher than that of the oscillator. This fact is relevant to the development of the oscillator technology, when manufacturing makes it difficult to have the oscillator at the round frequency of the available standards.

The scheme C derives from A after introducing a delay in the independent arms [7]. It can be implemented using a pair of resonators, or a pair a delay lines. The use of the optical-fiber delay line is the most promising solution because the delay line can be adapted to the arbitrary frequency of the oscillator under test, while a resonator can not [8]. Correlation removes the fluctuations of the delay line [9, 10].

The scheme D is based on a bridge that nulls the carrier before amplification and synchronous detection of the noise sidebands. This scheme derives from the pioneering work [11]. At that time, the mixer was used to down convert the fluctuation of the null at the output of a magic Tee. Amplification [12] and correlation [5] were introduced afterwards.

With modern RF/microwave components, isolation between the two channels is no longer a problem. The hardware sensitivity is limited environmental effects, like temperature fluctuations and low-frequency magnetic fields, and by the AM noise. The latter is taken in through the sensitivity of the mixer offset to the input power. Only partial solutions are available [13].

B. Intermezzo

In the measurement of PM noise it is more or less possible to test the background of a single-channel instrument by removing the DUT. This happens because a phase reference is always available, deriving the two inputs of the phase detector from a single oscillator. On the other hand, the correlation schemes are complex, and sometimes difficult to operate. The obvious conclusion is that the experimentalist prefers the single-channel measurements, and uses the correlation schemes only when the sensitivity of the former is insufficient.

Conversely, the measurement of AM noise relies upon the power detector, which does not work without the source. Thus, we can not asses the single-channel background noise *of the instrument* by removing the device under test. One can object that even in the case of PM noise we can not measure an oscillator in single-channel mode if we do not have a low-noise reference oscillator. The difference is that in the case of PM noise we can at least validate the instrument, while in the case of AM noise we can not.

Another difference between AM and PM is that the phase detector is always more or less sensitive to AM noise [13], while the amplitude detector is not sensitive to phase noise. In correlation systems, this fact makes the channel separation simple to achieve and to test.

The conclusion is that the cross-spectrum measurement is inherently simpler with AM noise than with PM noise.

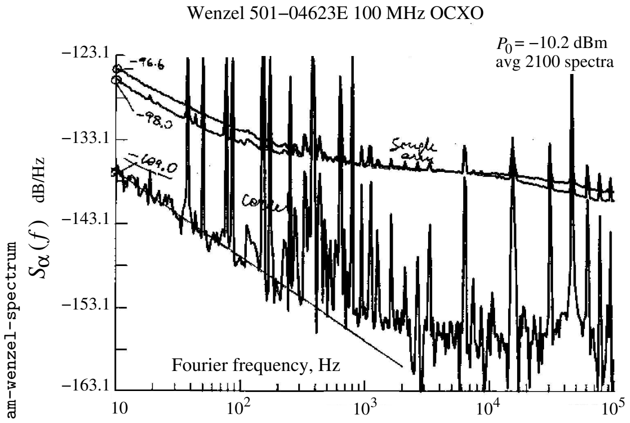


Figure 7: Example of cross spectrum measurement (amplitude noise of an oven-controlled quartz oscillator), taken from [14].

C. AM noise

Figure 8 shows some schemes for the cross spectrum measurement of AM noise, taken from [14].

In A, two Schottky-diode or tunnel-diode passive power detectors measure the power fluctuations of the source under test. Isolation between channels is guaranteed by the isolation of the power splitter (18–20 dB) and by the fact that the power detectors do not send noise back to the input. Correlation rejects the single-channel noise.

As an example, Fig. 7 shows the measurement of a quartz oscillator. Converting the $1/f$ noise of this oscillator into stability of the fractional amplitude α, we get $\sigma_\alpha(\tau) = 4.3\times10^{-7}$ (Allan deviation, constant vs. the measurement time τ), which is the lowest AM noise measured in our laboratory. Correlation rejects the single-channel noise by more than 10 dB.

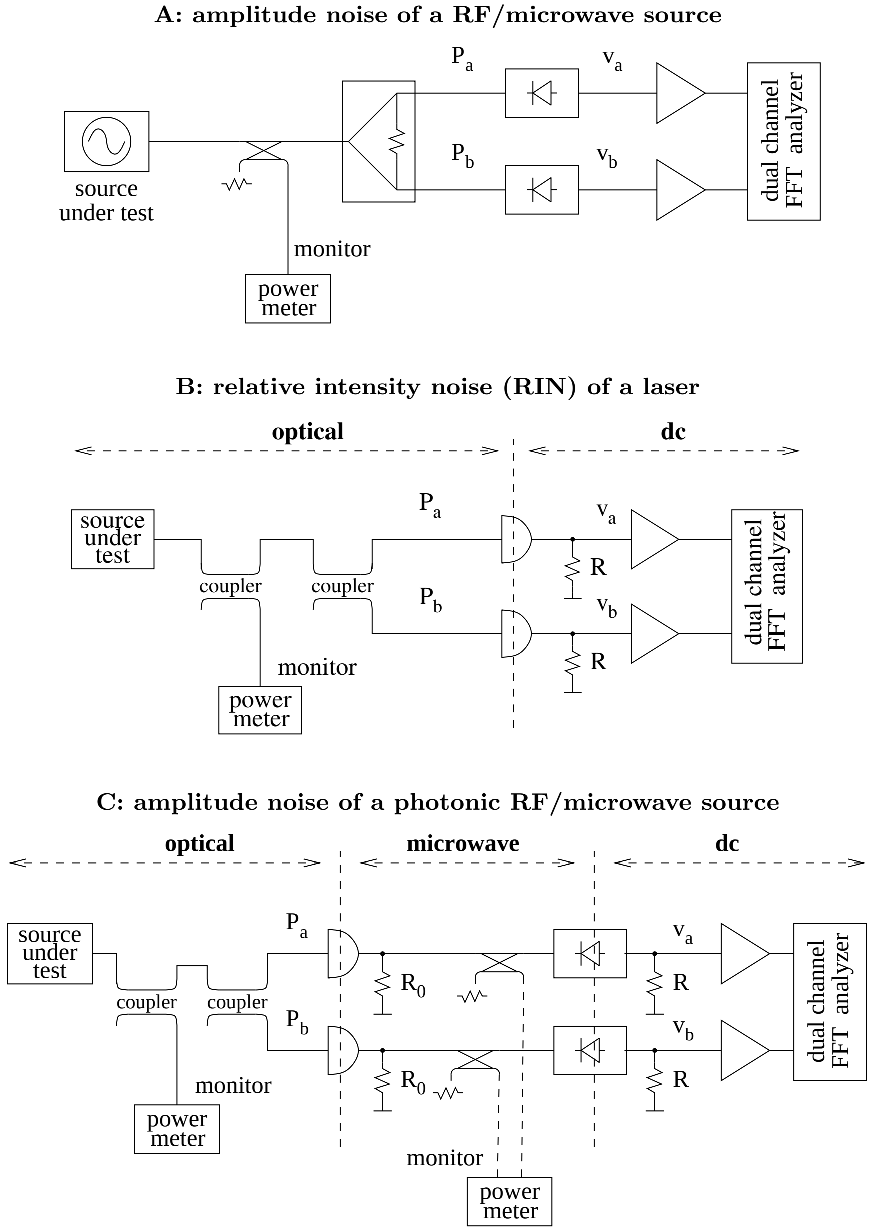
Figure 8 is the obvious adaptation of the scheme A to the measurement of the laser relative intensity noise (RIN). We start using it routinely. The scheme C, presently under study, is intended to measure the AM noise of the microwave-modulated light at the output of the new generation of opto-electronic oscillators based on optical fibers [15] and on whispering-gallery optical resonators.

VII. OTHER APPLICATIONS

After [3], the the cross-spectrum method is now routinely used in radio-astronomy and in radiometry.

The measurement of the low $1/f$ voltage fluctuations is an important diagnostic tool in semiconductor technology. The field-effect transistors are suitable to this task because of the low bias current at the input. In fact, the bias current flowing into the sample turns into a fully correlated voltage through the Ohm law. Additionally, the electrode capacitance may limit the instrument sensitivity. The reader can refer to [16] for a detailed treatise.

In metallurgy, the cross spectrum method has been used for the measurement of electromigration in thin



am-correl-scheme

mce-am-optical

am-mwave-photonic

Figure 8: Basics schemes for the measurement of amplitude noise (from [14]).

metal films through the $1/f$ fluctuation of the conductor resistance. This is relevant in microprocessor technology because the high current density in metal connexions can limit the life of the component and make it unreliable. For this reason, Aluminum is no longer used. The high sensitivity is based on the idea that with white gaussian noise X' and X'' (real and imaginary part) are statistically independent. Synchronously detecting the signal with two orthogonal references, it is therefore possible to reject the amplifier noise even if a single amplifier is shared by the two channel [17]. Adapting this idea to RF and microwaves is straightforward [18]. Unfortunately, we still have no application for this.

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